PAPER 3

FOUNDATION COURSE

STUDY MATERIAL

QUANTITATIVE APTITUDE

Board of Studies (Academic)
The Institute of Chartered Accountants of India
(Set up by an Act of Parliament)

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This Study Material has been prepared by the faculty of the Board of Studies (Academic). The objective of the Study Material is to provide teaching material to the students to enable them to obtain knowledge in the subject. In case students need any clarifications or have any suggestions for further improvement of the material contained herein, they may write to the Director of Studies.

All care has been taken to provide interpretations and discussions in a manner useful for the students. However, the Study Material has not been specifically discussed by the Council of the Institute or any of its Committees and the views expressed herein may not be taken to necessarily represent the views of the Council or any of its Committees.

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BEFORE WE BEGIN

The contents of the study material for Foundation have been designed and developed by the Board of Studies (Academic), ICAI with an objective to synchronize the syllabus with the International Education Standards (IESs) of IFAC (International Federation of Accountants) to instill and enhance the necessary pre-requisites for becoming a well-rounded, competent and globally competitive Accounting Professional.

This study material also lays emphasis on NEP 2020 initiatives like conceptual clarity rather than rote learning and new pedagogical and curriculum restructuring based on the use of technology while teaching.

The requirements of "IES 1 Entry Level Requirements" have been kept in mind while developing the different chapters of study material.

The subject "Quantitative Aptitude" has been designed specifically for the students who are aiming pursue CA course, keeping in view the relevance of subjects after they become full-fledged professional. Mathematics and Statistics applications are very important for the students of Chartered Accountancy Course as professional work in future will demand quantitative and analytical skills. Through this section, students will be able to understand the basic mathematical and statistical tools and apply the same in business, finance and economics situations. Logical Reasoning has been included to test analytical and mental ability skills which will help them in honing their interpretative skills while pursuing and thereafter CA course.

Through these chapters of Quantitate Aptitude, students will be equipped with the knowledge to absorb various concepts of other subjects of the chartered accountancy course like accounting, auditing and assurance, financial management, cost and management accounting, strategic cost management, etc.

The Study material Quantitative Aptitude is divided into three parts, the first part of the study material (Chapters 1-8) covers basic application mathematical techniques like ratio, proportion, indices, logarithms, equations and linear inequalities, time value of money, permutations and combinations, sequence and series, sets, relations and basic applications of differential and integral calculus in economics and business. The second part of the study material (Chapters 9-12) covers Logical Reasoning and the third part (Chapters 13-18) of the basic principles of statistical techniques and measurement thereof.

The entire study material Quantitative Aptitude has been written in a simple and easy to understand language. Every concept has been explained with the help of solved examples. A number of illustrations have been incorporated in each chapter to explain various concepts and related computational techniques dealt within each chapter. The diagrams have been drawn neatly in a such way that the students have the complete understanding of the problem by perusing them. This entire paper is tested on multiple choice questions or objective type of questions pattern only. Keeping in view the examination pattern, a reasonably good question bank has been included in the study material which will help the students to prepare for the Foundation examination.

Happy Reading and Best Wishes!

SYLLABUS

PAPER – 3: QUANTITATIVE APTITUDE

(One paper – Two hours – 100 Marks)

Objectives:

(a) To develop an understanding of the basic mathematical and statistical tools and their application in Business, Finance and Economics.

(b) To develop logical reasoning skills and apply the same in simple problem solving.

Contents:

PART – A BUSINESS MATHEMATICS (40 MARKS)

1. Ratio and Proportion, Indices and Logarithms

Ratio and Proportion (Business Applications), Laws of Indices, Exponents and Logarithms and Anti Logarithms.

2. Equations

Simultaneous linear equations up to three variables, Quadratic and Cubic equations in one variable.

3. Linear Inequalities with Objective Functions and Optimization w r t objective function

4. Mathematics of Finance

(i) Simple Interest

(ii) Compound interest

(iii) Depreciation

(iv) Effective Rate of Interest

(v) Present Value

(vi) Net Present Value

(vii) Future Value (viii) Perpetuity (ix) Annuities

(x) Sinking Funds

(xi) Valuation of Bonds

(xii)Calculating of EMI

(xiii) Calculations of Returns:

(a) Nominal Rate of Return

(b) Effective Rate of Return

(c) Compound Annual Growth Rate (CAGR)

5. Permutations and Combinations

Basic concepts of Permutations and Combinations: Introduction, the Factorial, Permutations, results, Circular Permutations, Permutations with restrictions, Combinations with standard results.

6. Sequence and Series

Introduction Sequences, Series, Arithmetic and Geometric progression, Relationship between AM and GM and Sum of n terms of special series.

7. Sets, Relations and Functions and Basics of Limits and Continuity functions

8. Basic applications of Differential and Integral calculus in Business and Economics (Excluding the trigonometric applications)

PART – B: LOGICAL REASONING (20 MARKS)

1. **Number series, Coding and Decoding and odd man out**
2. **Direction Tests**
3. **Seating Arrangements**
4. **Blood Relations**

PART – C: STATISTICS (40 MARKS)

1. Unit: I Statistical Description of Data

Statistical Representation of Data, Diagrammatic representation of data, Frequency distribution, Graphical representation of Frequency Distribution – Histogram, Frequency Polygon, Ogive, Pie-chart.

Unit: II Sampling: Basic principles of sampling theory, comparison between sample survey and complete enumeration, some important terms associated sampling types of sampling, sampling and non-sampling errors.

2. Measures of Central tendency and Dispersion

Measures of Central Tendency and Dispersion: Mean Median, Mode, Mean Deviation, Quartiles and Quartile Deviation, Standard Deviation, Co-efficient of Variation, Coefficient of Quartile Deviation.

3. Probability

Probability: Independent and dependent events; mutually exclusive events Total and Compound Probability and Mathematical Expectation.

4. Theoretical Distributions

Theoretical Distributions: Binomial Distribution, Poisson distribution – Basic application and Normal Distribution – Basic applications.

5. Correlation and Regression

Correlation and Regression: Scatter diagram, Karl Pearson's Coefficient of Correlation Rank Correlation Regression lines, Regression equations, Regression coefficients.

6. Index Numbers

Uses of Index Numbers, Problems involved in construction of Index Numbers, Methods of construction of Index Numbers. BSE SENSEX and NSE.

CONTENTS

PART – A : BUSINESS MATHEMATICS

Chapter-7: Sets, Relations and Functions, Basics of Limits and Continuity functions

Chapter-8: Basic applications of Differential and Integral calculus in Business and Economics

(A) Differential Calculus

Unit: II Sampling

Chapter-14: Measures of Central Tendency and Dispersion

Unit I : Measures of Central Tendency

Unit II: Dispersion

Chapter- 18: Index Numbers

CHAPTER B

RATIO AND PROPORTION, INDICES, LOGARITHMS

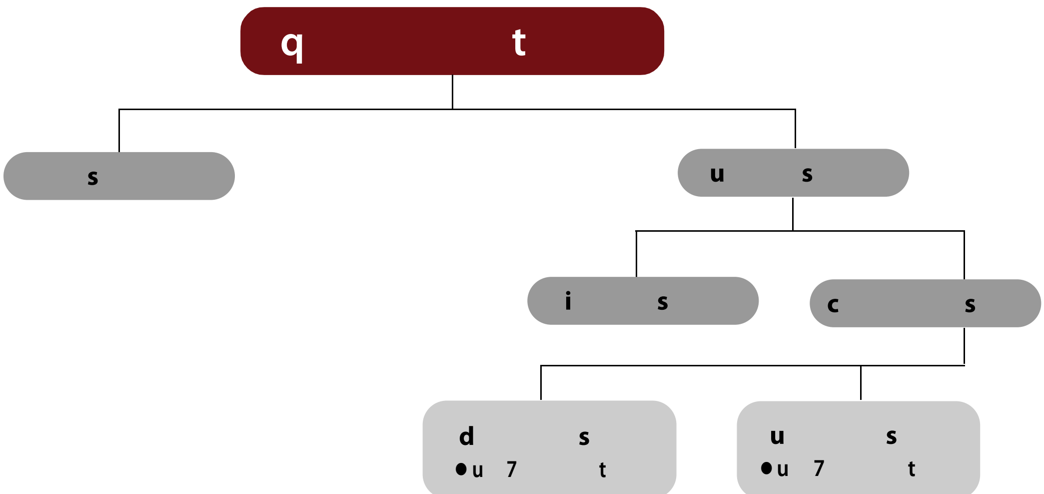
UNIT I: RATIO

LEARNING OBJECTIVES

After reading this unit a student will learn –

- How to compute and compare two ratios;
- Effect of increase or decrease of a quantity on the ratio;
- The concept and application of different types of ratios.

UNIT OVERVIEW



We use ratio in many ways in practical fields. For example, it is given that a certain sum of money is divided into three parts in the also given ratio. If first part is given then we can find out total amount and the other two parts.

In the case when ratio of boys and girls in a school is given and the total number of student the number of boys in the school is given, we can find out the number of girls of that school by using ratios.

1.1 RATIO

A ratio is a comparison of the sizes of two or more quantities of the same kind by division.

If a and b are two quantities of the same kind (in same units), then the fraction a/b is called the ratio of a to b. It is written as a : b. Thus, the ratio of a to b = a/b or a : b. The quantities a and b are called the **terms** of the ratio, a is called the **first term or antecedent** and b is called the **second term or consequent**.

For example, in the ratio 5 : 6, 5 & 6 are called terms of the ratio. 5 is called the first term and 6 is called second term.

1.1.2 Remarks

- **Both terms of a ratio can be multiplied or divided by the same (non–zero) number.**
- **Usually a ratio is expressed in lowest terms (or simplest form).**

Illustration I:

12 : 16 = 12/16 = (3 × 4)/(4 × 4) = 3/4 = 3 : 4

- **The order of the terms in a ratio is important.**

Illustration II:

3 : 4 is not same as 4 : 3.

- **Ratio exists only between quantities of the same kind.**

Illustration III:

(i) There is no ratio between number of students in a class and the salary of a teacher.

(ii) There is no ratio between the weight of one child and the age of another child.

- **Quantities to be compared (by division) must be in the same units**.

Illustration IV:

(i) Ratio between 150 gm and 2 kg = Ratio between 150 gm and 2000 gm
= 150/2000 = 3/40 = 3 : 40

(ii) Ratio between 25 minutes and 45 seconds = Ratio between (25 × 60) sec. and 45 sec.
= 1500/45 = 100/3 = 100 : 3

Illustration V:

(i) Ratio between 3 kg & 5 kg = 3/5

- **To compare two ratios, convert them into equivalent like fractions.**

Illustration VI: To find which ratio is greater _____________

$$2\frac{1}{3} : 3\frac{1}{3} \; ; 3.6 : 4.8$$

Solution: $2\frac{1}{3} : 3\frac{1}{3}$ = 7/3 : 10/3 = 7 : 10 = 7/10

3.6 : 4.8 = 3.6/4.8 = 36/48 = 3/4

L.C.M of 10 and 4 is 20.

So, 7/10 = (7 × 2)/(10 × 2) = 14/20

And 3/4 = (3 × 5)/(4 × 5) = 15/20

As 15 > 14 so, 15/20 > 14/20 i. e. 3/4 > 7/10

Hence, 3.6 : 4.8 is greater ratio.

- **If a quantity increases or decreases in the ratio a : b then new quantity = b/a of the original quantity/a**

The fraction by which the original quantity is multiplied to get a new quantity is called the factor multiplying ratio.

Illustration VII: Rounaq weighs 56.7 kg. If he reduces his weight in the ratio 7 : 6, find his new weight.

Solution: Original weight of Rounaq = 56.7 kg

He reduces his weight in the ratio 7 : 6

His new weight = $\frac{6}{7}$ × 56.7 = 6 × 8.1 = 48.6 kg

Applications:

Example 1: Simplify the ratio 1/3 : 1/8 : 1/6

Solution: L.C.M. of 3, 8 and 6 is 24.

1/3 : 1/8 : 1/6 = 1 × 24/3 : 1 × 24/8 : 1 × 24/6

= 8 : 3 : 4

Example 2: The ratio of the number of boys to the number of girls in a school of 720 students is 3 : 5. If 18 new girls are admitted in the school, find how many new boys may be admitted so that the ratio of the number of boys to the number of girls may change to 2 : 3.

Solution: The ratio of the number of boys to the number of girls = 3 : 5

Sum of the ratios	=	3 + 5	=	8
So, the number of boys in the school	=	(3 × 720)/8	=	270
And the number of girls in the school	=	(5 × 720)/8	=	450

Let the number of new boys admitted be x, then the number of boys become (270 + x).

After admitting 18 new girls, the number of girls become 450 + 18 = 468

According to given description of the problem, (270 + x)/468 = 2/3

or, 3 (270 + x) = 2 x 468
or, 810 + 3x = 936 or, 3x = 126 or, x = 42.

Hence the number of new boys admitted = 42.

1.1.3 Inverse Ratio

One ratio is the inverse of another if their product is 1. Thus a : b is the inverse of b : a and vice-versa.

Some Properties of Ratios:

1. A ratio a : b is said to be of greater inequality if a>b and of lesser inequality if a<b.
2. The ratio compounded of the two ratios a : b and c : d is ac : bd.

   For example compound ratio of 3 : 4 and 5 : 7 is 15 : 28.

   Compound ratio of 2 : 3, 5 : 7 and 4 : 9 is 40 : 189.
3. A ratio compounded of itself is called its duplicate ratio.

   Thus $a^2 : b^2$ is the duplicate ratio of a : b. Similarly, the triplicate ratio of a : b is $a^3 : b^3$.

   For example, duplicate ratio of 2 : 3 is 4 : 9. Triplicate ratio of 2 : 3 is 8 : 27.
4. The sub-duplicate ratio of a : b is $\sqrt{a} : \sqrt{b}$ and the sub-triplicate ratio of a : b is $\sqrt[3]{a} : \sqrt[3]{b}$.

   For example sub-duplicate ratio of 4 : 9 is $\sqrt{4} : \sqrt{9} = 2 : 3$

   And sub-triplicate ratio of 8 : 27 is $\sqrt[3]{8} : \sqrt[3]{27} = 2 : 3$.
5. If the ratio of two similar quantities can be expressed as a rational numbers, the quantities are said to be commensurable; otherwise, they are said to be incommensurable. $\sqrt{3} : \sqrt{2}$ cannot be expressed as the ratio of two integers and therefore, $\sqrt{3}$ and $\sqrt{2}$ are incommensurable quantities.
6. Continued Ratio is the relation (or comparison) between the magnitudes of three or more quantities of the same kind. The continued ratio of three similar quantities a, b, c is written as a : b : c.

Applications:

Illustration I: The continued ratio of ₹ 200, ₹ 400 and ₹ 600 is ₹ 200 : ₹ 400 : ₹ 600 = 1 : 2 : 3.

Example 1: The monthly incomes of two persons are in the ratio 4 : 5 and their monthly expenditures are in the ratio 7 : 9. If each saves ₹ 50 per month, find their monthly incomes.

Solution: Let the monthly incomes of two persons be ₹ 4x and ₹ 5x so that the ratio is ₹ 4x : ₹ 5x = 4 : 5. If each saves ₹ 50 per month, then the expenditures of two persons are ₹ (4x – 50) and ₹ (5x – 50).

$$\frac{4x-50}{5x-50}=\frac{7}{9} \text{ or } 36x-450=35x-350$$

or, 36x – 35x = 450 – 350, or, x = 100

Hence, the monthly incomes of the two persons are ₹ 4 × 100 and ₹ 5 × 100 i.e. ₹ 400 and ₹ 500.

Example 2 : The ratio of the prices of two houses was 16 : 23. Two years later when the price of the first has increased by 10% and that of the second by ₹ 477, the ratio of the prices becomes 11 : 20. Find the original prices of the two houses.

Solution: Let the original prices of two houses be ₹ 16x and ₹ 23x respectively. Then by the given conditions,

$$\frac{16x + 10\%\text{ of }16x}{23x + 477} = \frac{11}{20}$$

or, $\frac{16x + 1.6x}{23x + 477} = \frac{11}{20}$, or, 320x + 32x = 253x + 5247

or, 352x – 253x = 5247, or, 99x = 5247; $\therefore$ x = 53

Hence, the original prices of two houses are ₹ 16 × 53 and ₹ 23 × 53 i.e. ₹ 848 and ₹ 1,219.

Example 3: Find in what ratio will the total wages of the workers of a factory be increased or decreased if there be a reduction in the number of workers in the ratio 15 : 11 and an increment in their wages in the ratio 22 : 25.

Solution: Let x be the original number of workers and ₹ y the (average) wages per workers. Then the total wages before changes = ₹ xy.

After reduction, the number of workers = (11x)/15

After increment, the (average) wages per workers = ₹ (25y)/22

$\therefore$ The total wages after changes = $(\frac{11}{15}x) \times (₹\ \frac{25}{22}y) = ₹\ \frac{5xy}{6}$

Thus, the total wages of workers get decreased from ₹ xy to ₹ 5xy/6

Hence, the required ratio in which the total wages decrease is $xy : \frac{5xy}{6} = 6:5$.

EXERCISE 1(A)

Choose the most appropriate option (a) (b) (c) or (d).

1. The inverse ratio of 11 : 15 is
   (a) 15 : 11 (b) $\sqrt{11} : \sqrt{15}$ (c) 121 : 225 (d) none of these
2. The ratio of two quantities is 3 : 4. If the antecedent is 15, the consequent is
   (a) 16 (b) 60 (c) 22 (d) 20
3. The ratio of the quantities is 5 : 7. If the consequent of its inverse ratio is 5, the antecedent is
   (a) 5 (b) $\sqrt{5}$ (c) 7 (d) none of these

4. The ratio compounded of 2 : 3, 9 : 4, 5 : 6 and 8 : 10 is
(a) 1 : 1 (b) 1 : 5 (c) 3 : 8 (d) none of these

5. The duplicate ratio of 3 : 4 is
(a) $\sqrt{3}$: 2 (b) 4 : 3 (c) 9 : 16 (d) none of these

6. The sub-duplicate ratio of 25 : 36 is
(a) 6 : 5 (b) 36 : 25 (c) 50 : 72 (d) 5 : 6

7. The triplicate ratio of 2 : 3 is
(a) 8 : 27 (b) 6 : 9 (c) 3 : 2 (d) none of these

8. The sub-triplicate ratio of 8 : 27 is
(a) 27 : 8 (b) 24 : 81 (c) 2 : 3 (d) none of these

9. The ratio compounded of 4 : 9 and the duplicate ratio of 3 : 4 is
(a) 1 : 4 (b) 1 : 3 (c) 3 : 1 (d) none of these

10. The ratio compounded of 4 : 9, the duplicate ratio of 3 : 4, the triplicate ratio of 2 : 3 and 9 : 7 is
(a) 2 : 7 (b) 7 : 2 (c) 2 : 21 (d) none of these

11. The ratio compounded of 4 : 5, 9 : 7, the triplicate ratio of 3 : 4, and the triplicate ratio of 2 : 3 is
(a) 4 : 512 (b) 3 : 32 (c) 1 : 12 (d) none of these

12. If a : b = 3 : 4, the value of (2a+3b) : (3a+4b) is
(a) 54 : 25 (b) 8 : 25 (c) 17 : 24 (d) 18 : 25

13. Two numbers are in the ratio 2 : 3. If 4 be subtracted from each, they are in the ratio 3 : 5. The numbers are
(a) (16, 24) (b) (4, 6) (c) (2, 3) (d) none of these

14. The angles of a triangle are in ratio 2 : 7 : 11. The angles are
(a) $(20°, 70°, 90°)$ (b) $(30°, 70°, 80°)$ (c) $(18°, 63°, 99°)$ (d) none of these

15. Division of ₹ 324 between X and Y is in the ratio 11 : 7. X & Y would get Rupees
(a) (204, 120) (b) (200, 124) (c) (180, 144) (d) none of these

16. Anand earns ₹ 80 in 7 hours and Promode ₹ 90 in 12 hours. The ratio of their earnings is
(a) 32 : 21 (b) 23 : 12 (c) 8 : 9 (d) none of these

17. The ratio of two numbers is 7 : 10 and their difference is 105. The numbers are
(a) (200, 305) (b) (185, 290) (c) (245, 350) (d) none of these

18. P, Q and R are three cities. The ratio of average temperature between P and Q is 11 : 12 and that between P and R is 9 : 8. The ratio between the average temperature of Q and R is
(a) 22 : 27 (b) 27 : 22 (c) 32 : 33 (d) none of these

19. If x : y = 3 : 4, the value of $x^2y + xy^2 : x^3 + y^3$ is
(a) 13 : 12 (b) 12 : 13 (c) 21 : 31 (d) none of these

20. If p : q is the sub-duplicate ratio of $p-x^2 : q-x^2$ then x^2 is

(a) $\frac{p}{p+q}$ (b) $\frac{q}{p+q}$ (c) $\frac{pq}{p+q}$ (d) none of these

21. If 2s : 3t is the duplicate ratio of 2s – p : 3t – p then

(a) $p^2 = 6st$ (b) p = 6st (c) 2p = 3st (d) none of these

22. If p : q = 2 : 3 and x : y = 4 : 5, then the value of $5px + 3qy : 10px + 4qy$ is

(a) 71 : 82 (b) 27 : 28 (c) 17 : 28 (d) none of these

23. The number which when subtracted from each of the terms of the ratio 19 : 31 reducing it to 1 : 4 is

(a) 15 (b) 5 (c) 1 (d) none of these

24. Daily earnings of two persons are in the ratio 4:5 and their daily expenses are in the ratio 7 : 9. If each saves ₹ 50 per day, their daily earnings in ₹ are

(a) (40, 50) (b) (50, 40) (c) (400, 500) (d) none of these

25. The ratio between the speeds of two trains is 7 : 8. If the second train runs 400 kms. in 5 hours, the speed of the first train is

(a) 10 Km/hr (b) 50 Km/hr (c) 70 Km/hr (d) none of these

SUMMARY

- A ratio is a comparison of the sizes of two or more quantities of the same kind by division.
- If a and b are two quantities of the same kind (in same units), then the fraction a/b is called the ratio of a to b. It is written as a : b. Thus, the ratio of a to b = a/b or a : b.
- The quantities a and b are called the terms of the ratio, a is called the first term or antecedent and b is called the second term or consequent.
- The ratio compounded of the two ratios a : b and c : d is ac : bd.
- A ratio compounded of itself is called its duplicate ratio. $a^2 : b^2$ is the duplicate ratio of a : b. Similarly, the triplicate ratio of a : b is $a^3 : b^3$.
- For any ratio a : b, the inverse ratio is b : a
- The sub-duplicate ratio of a : b is a½ : b½ and the sub-triplicate ratio of a : b is $a^{1/3} : b^{1/3}$.
- Continued Ratio is the relation (or comparsion) between the magnitudes of three or more Quantities of the same kind. The continued ratio of three similar quantities a, b, c is written as a : b : c.

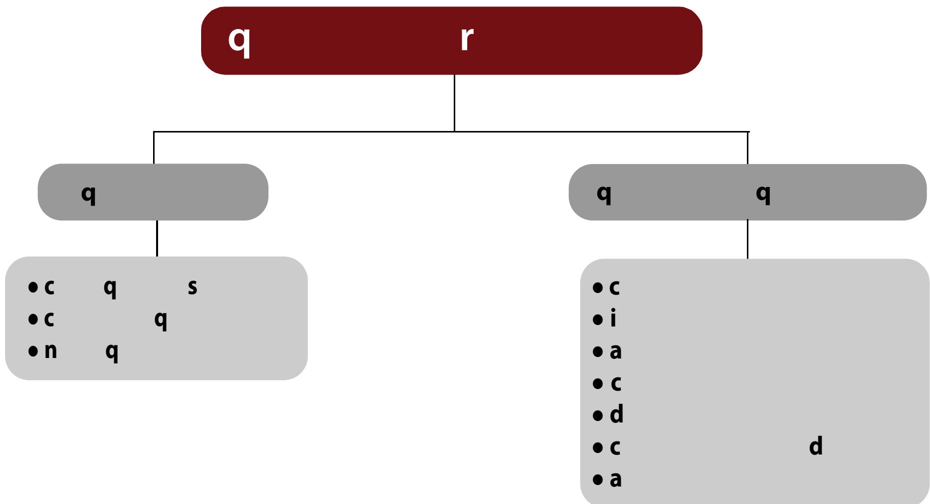
UNIT II: PROPORTIONS

LEARNING OBJECTIVES

After reading this unit a student will learn –

- What is proportion?
- Properties of proportion and how to use them.

UNIT OVERVIEW



1.2 PROPORTIONS

If the income of a man is increased in the given ratio and if the increase in his income is given then to find out his new income, in a Proportion problem is used.

Again if the ages of two men are in the given ratio and if the age of one man is given, we can find out the age of the another man by Proportion.

An equality of two ratios is called a **proportion**. Four quantities a, b, c, d are said to be in proportion if a : b = c : d (also written as a : b :: c : d) i.e. if a/b = c/d i.e. if ad = bc.

The quantities a, b, c, d are called **terms** of the proportion; a, b, c and d are called its first, second, third and fourth terms respectively. First and fourth terms are called **extremes** (or extreme terms). Second and third terms are called **means (or middle terms).**

If a : b = c : d then d is called fourth proportional.

If a : b = c : d are in proportion then a/b = c/d i.e. ad = bc

i.e. **product of extremes = product of means**.

This is called *cross product rule.*

Three quantities a, b, c of the same kind (in same units) are said to be in continuous proportion if a : b = b : c i.e. a/b = b/c i.e. $b^2 = ac$

If a, b, c are in continuous proportion, then the middle term b is called the mean proportional between a and c, a is the first proportional and c is the third proportional.

Thus, if b is mean proportional between a and c, then $b^2 = ac$ i.e. $b = \sqrt{ac}$.

When three or more numbers are so related that the ratio of the first to the second, the ratio of the second to the third, third to the fourth etc. are all equal, the numbers are said to be in **continued proportion.** We write it as

x/y = y/z = z/w = w/p = p/q = ... when

x, y, z, w, p and q are in continued proportion. If a ratio is equal to the reciprocal of the other, then either of them is in inverse (or reciprocal) proportion of the other. For example 5/4 is in inverse proportion of 4/5 and vice-versa.

Note: In a ratio a : b, both quantities must be of the same kind while in a proportion a : b = c : d, all the four quantities need not be of the same type. The first two quantities should be of the same kind and last two quantities should be of the same kind.

Applications :

Illustration I:

₹ 6 : ₹ 8 = 12 toffees : 16 toffees are in a proportion.

Here 1st two quantities are of same kind and last two are of same kind.

Example 1: The numbers 2.4, 3.2, 1.5, 2 are in proportion because these numbers satisfy the property the product of extremes = product of means.

Here 2.4 × 2 = 4.8 and 3.2 × 1.5 = 4.8

Example 2: Find the value of x if 10/3 : x : : 5/2 : 5/4.

Solution: 10/3 : x = 5/2 : 5/4

Using cross product rule, x × 5/2 = (10/3) × 5/4

Or, x = (10/3) × (5/4) × (2/5) = 5/3

Example 3: Find the fourth proportional to 2/3, 3/7, 4.

Solution: If the fourth proportional be x, then 2/3, 3/7, 4, x are in proportion.

Using cross product rule, (2/3) × x = (3 × 4)/7

or, x = (3 × 4 × 3)/(7 × 2) = 18/7.

Example 4: Find the third proportion to 2.4 kg, 9.6 kg.

Solution: Let the third proportion to 2.4 kg, 9.6 kg be x kg.

Then 2.4 kg, 9.6 kg and x kg are in continued proportion since $b^2 = ac$

So, 2.4/9.6 = 9.6/x or, x = (9.6 × 9.6)/2.4 = 38.4

Hence the third proportional is 38.4 kg.

Example 5: Find the mean proportion between 1.25 and 1.8.

Solution: Mean proportion between 1.25 and 1.8 is $\sqrt{(1.25\times1.8)} = \sqrt{2.25} = 1.5$.

1.2.1 Properties of Proportion

1. If a : b = c : d, then ad = bc

   Proof. $\frac{a}{b}=\frac{c}{d}$; $\therefore ad=bc$ (By cross-multiplication)

2. If a : b = c : d, then b : a = d : c (Invertendo)

   Proof. $\frac{a}{b}=\frac{c}{d}$ or $1\Big/\frac{a}{b}=1\Big/\frac{c}{d}$, or, $\frac{b}{a}=\frac{d}{c}$

   Hence, b : a = d : c.

3. If a : b = c : d, then a : c = b : d (Alternendo)

   Proof. $\frac{a}{b}=\frac{c}{d}$ or, $ad=bc$

   Dividing both sides by cd, we get

   $\frac{ad}{cd}=\frac{bc}{cd}$, or $\frac{a}{c}=\frac{b}{d}$, i.e. $a:c = b:d$.

4. If a : b = c : d, then a + b : b = c + d : d (Componendo)

   Proof. $\frac{a}{b}=\frac{c}{d}$, or, $\frac{a}{b}+1=\frac{c}{d}+1$

   or, $\frac{a+b}{b}=\frac{c+d}{d}$, i.e. $a+b:b=c+d:d$.

5. If a : b = c : d, then a – b : b = c – d : d (Dividendo)

   Proof. $\frac{a}{b}=\frac{c}{d}$, $\therefore \frac{a}{b}-1=\frac{c}{d}-1$

   $\frac{a-b}{b}=\frac{c-d}{d}$, i.e. $a-b:b=c-d:d$.

6. If a : b = c : d, then a + b : a – b = c + d : c – d (Componendo and Dividendo)

   Proof. $\frac{a}{b}=\frac{c}{d}$, or $\frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$ 1

   Again $\frac{a}{b}-1,=\frac{c}{d}-1$, or $\frac{a-b}{b}=\frac{c-d}{d}$ 2

Dividing (1) by (2) we get

$$\frac{a+b}{a-b}=\frac{c+d}{c-d},\quad \text{i.e.}\quad a+b:a-b=c+d:c-d$$

7. If a : b = c : d = e : f = ……………….……, then each of these ratios (Addendo) is equal (a + c + e + ……..) : (b + d + f + …….)

Proof. $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\text{...........(say)}\,k,$

$\therefore$ a = bk, c = dk, e = fk, ………….

Now $a+c+e\text{.......}=k(b+d+f)\text{...........}$ or $\frac{a+c+e\text{.....}}{b+d+f\text{.....}}=k$

Hence, (a + c + e + ……..) : (b + d + f + …….) is equal to each ratio

8. Subtrahendo : If a : b = c : d = e : f = ……………….……, then each of these ratios is equal (a – c – e – ……..) : (b – d – f – …….)

Example 1: If a : b = c : d = 2.5 : 1.5, what are the values of ad : bc and a + c : b + d?

Solution: We have $\frac{a}{b}=\frac{c}{d},=\frac{2.5}{1.5}$..........................(1)

From (1) ad = bc, or, $\frac{ad}{bc}=1$, i.e. ad : bc = 1 : 1

Again from (1) $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$ (By addendo property)

$\therefore \frac{a+c}{b+d}=\frac{2.5}{1.5}=\frac{25}{15}=\frac{5}{3},$ i.e. a + c : b + d = 5 : 3

Hence, the values of ad : bc and a + c : b + d are 1 : 1 and 5 : 3 respectively.

Example 2: If $\frac{a}{3}=\frac{b}{4}=\frac{c}{7}$, then prove that $\frac{a+b+c}{c}=2$

Solution: We have $\frac{a}{3}=\frac{b}{4}=\frac{c}{7}=\frac{a+b+c}{3+4+7}=\frac{a+b+c}{14}$ (By addendo property)

$\therefore \frac{a+b+c}{14}=\frac{c}{7}$ or $\frac{a+b+c}{c}=\frac{14}{7}=2$

Example 3: A dealer mixes tea costing ₹ 6.92 per kg. with tea costing ₹ 7.77 per kg and sells the mixture at ₹ 8.80 per kg and earns a profit of $17\frac{1}{2}\%$ on his sale price. In what proportion does he mix them?

Solution: Let us first find the cost price (C.P.) of the mixture. If S.P. is ₹ 100, profit is $17\frac{1}{2}$ Therefore C.P. = ₹ $(100 - 17\frac{1}{2})$ = ₹ $82\frac{1}{2}$ = ₹ $\frac{165}{2}$

If S.P. is ₹ 8.80, C.P. is (165 × 8.80)/(2 × 100) = ₹ 7.26

∴ C.P. of the mixture per kg = ₹ 7.26

2nd difference = Profit by selling 1 kg. of 2nd kind @ ₹ 7.26

= ₹ 7.77 – ₹ 7.26 = 51 Paise

1st difference = ₹ 7.26 – ₹ 6.92 = 34 Paise

We have to mix the two kinds in such a ratio that the amount of profit in the first case must balance the amount of loss in the second case.

Hence, the required ratio = (2nd diff) : (1st diff.) = 51 : 34 = 3 : 2.

EXERCISE 1(B)

Choose the most appropriate option (a) (b) (c) or (d).

1. The fourth proportional to 4, 6, 8 is
   (a) 12 (b) 32 (c) 48 (d) none of these
2. The third proportional to 12, 18 is
   (a) 24 (b) 27 (c) 36 (d) none of these
3. The mean proportional between 25, 81 is
   (a) 40 (b) 50 (c) 45 (d) none of these
4. The number which has the same ratio to 26 that 6 has to 13 is
   (a) 11 (b) 10 (c) 21 (d) none of these
5. The fourth proportional to 2a, a^2, c is
   (a) ac/2 (b) ac (c) 2/ac (d) none of these
6. If four numbers 1/2, 1/3, 1/5, 1/x are proportional then x is
   (a) 6/5 (b) 5/6 (c) 15/2 (d) none of these
7. The mean proportional between $12x^2$ and $27y^2$ is
   (a) 18xy (b) 81xy (c) 8xy (d) none of these
   (Hint: Let z be the mean proportional and z = $\sqrt{(12x^2 \text{ x } 27y^2)}$)
8. If A = B/2 = C/5, then A : B : C is
   (a) 3 : 5 : 2 (b) 2 : 5 : 3 (c) 1 : 2 : 5 (d) none of these
9. If a/3 = b/4 = c/7, then a + b + c/c is
   (a) 1 (b) 3 (c) 2 (d) none of these
10. If p/q = r/s = 2.5/1.5, the value of ps : qr is
    (a) 3/5 (b) 1:1 (c) 5/3 (d) none of these
11. If x : y = z : w = 2.5 : 1.5, the value of (x + z)/(y + w) is
    (a) 1 (b) 3/5 (c) 5/3 (d) none of these
12. If (5x – 3y)/(5y – 3x) = 3/4, the value of x : y is
    (a) 2 : 9 (b) 7 : 2 (c) 7 : 9 (d) none of these

13. If A : B = 3 : 2 and B : C = 3 : 5, then A : B : C is
(a) 9 : 6 : 10 (b) 6 : 9 : 10 (c) 10 : 9 : 6 (d) none of these

14. If x/2 = y/3 = z/7, then the value of (2x – 5y + 4z)/2y is
(a) 6/23 (b) 23/6 (c) 3/2 (d) 17/6

15. If x : y = 2 : 3, y : z = 4 : 3 then x : y : z is
(a) 2 : 3 : 4 (b) 4 : 3 : 2 (c) 3 : 2 : 4 (d) 8 : 12 : 9

16. Division of ₹ 750 into 3 parts in the ratio 4 : 5 : 6 is
(a) (200, 250, 300) (b) (250, 250, 250) (c) (350, 250, 150) (d) none of these

17. The sum of the ages of 3 persons is 150 years. 10 years ago their ages were in the ratio 7 : 8 : 9. Their present ages are
(a) (45, 50, 55) (b) (40, 60, 50) (c) (35, 45, 70) (d) none of these

18. The numbers 14, 16, 35, 42 are not in proportion. The fourth term for which they will be in proportion is
(a) 45 (b) 40 (c) 32 (d) none of these

19. If x/y = z/w, implies y/x = w/z, then the process is called
(a) Dividendo (b) Componendo (c) Alternendo (d) none of these

20. If p/q = r/s = p – r/q – s, the process is called
(a) Subtrahendo (b) Addendo (c) Invertendo (d) none of these

21. If a/b = c/d, implies (a + b)/(a – b) = (c + d)/(c – d), the process is called
(a) Componendo (b) Dividendo (c) Componendo and Dividendo (d) none of these

22. If u/v = w/p, then (u – v)/(u + v) = (w – p)/(w + p). The process is called
(a) Invertendo (b) Alternendo (c) Addendo (d) none of these

23. 12, 16, *, 20 are in proportion. Then * is
(a) 25 (b) 14 (c) 15 (d) none of these

24. 4, *, 9, 13½ are in proportion. Then * is
(a) 6 (b) 8 (c) 9 (d) none of these

25. The mean proportional between 1.4 gms and 5.6 gms is
(a) 28 gms (b) 2.8 gms (c) 3.2 gms (d) none of these

26. If $\frac{a}{4}=\frac{b}{5}=\frac{c}{9}$ then $\frac{a+b+c}{c}$ is
(a) 4 (b) 2 (c) 7 (d) none of these.

27. Two numbers are in the ratio 3 : 4; if 6 be added to each number of the ratio, then the new ratio will be 4 : 5, then the numbers are
(a) 14, 20 (b) 17, 19 (c) 18 and 24 (d) none of these

28. If $\frac{a}{4} = \frac{b}{5}$ then

(a) $\frac{a+4}{a-4} = \frac{b-5}{b+5}$ (b) $\frac{a+4}{a-4} = \frac{b+5}{b-5}$ (c) $\frac{a-4}{a+4} = \frac{b+5}{b-5}$ (d) none of these

29. If $a : b = 4 : 1$ then $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$ is

(a) 5/2 (b) 4 (c) 5 (d) none of these

30. If $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$ then

$(b - c)x + (c - a)y + (a - b)z$ is

(a) 1 (b) 0 (c) 5 (d) none of these

SUMMARY

- $p : q = r : s \Rightarrow q : p = s : r$ (Invertendo)
  $(p/q = r/s) \Rightarrow (q/p = s/r)$
- $a : b = c : d \Rightarrow a : c = b : d$ (Alternendo)
  $(a/b = c/d) \Rightarrow (a/c = b/d)$
- $a : b = c : d \Rightarrow a + b : b = c + d : d$ (Componendo)
  $(a/b = c/d) \Rightarrow (a + b)/b = (c + d)/d$
- $a : b = c : d \Rightarrow a - b : b = c - d : d$ (Dividendo)
  $(a/b = c/d) \Rightarrow (a - b)/b = (c - d)/d$
- $a : b = c : d \Rightarrow a + b : a - b = c + d : c - d$ (Componendo & Dividendo)
  $(a + b)/(a - b) = (c + d)/(c - d)$
- $a : b = c : d = a + c : b + d$ (Addendo)
  $(a/b = c/d = a + c/b + d)$
- $a : b = c : d = a - c : b - d$ (Subtrahendo)
  $(a/b = c/d = a - c/b - d)$
- If $a : b = c : d = e : f = \ldots\ldots\ldots\ldots$ then each of these ratios $= (a - c - e - \ldots\ldots) : (b - d - f - \ldots\ldots)$
  (Subtrahendo)
- The quantities a, b, c, d are called **terms** of the proportion; a, b, c and d are called its first, second, third and fourth terms respectively. First and fourth terms are called **extremes (or extreme terms)**. Second and third terms are called **means (or middle terms).**

- If a : b = c : d are in proportion then a/b = c/d i.e. ad = bc i.e. **product of extremes = product of means**. This is called *cross product rule.*
- Three quantities a, b, c of the same kind (in same units) are said to be in continuous proportion if a : b = b : c i.e. a/b = b/c i.e. $b^2 = ac$
- If a, b, c are in continuous proportion, then the middle term b is called the mean proportional between a and c, a is the first proportional and c is the third proportional.
- Thus, if b is mean proportional between a and c, then $b^2 = ac$ i.e. $b = \sqrt{ac}$.

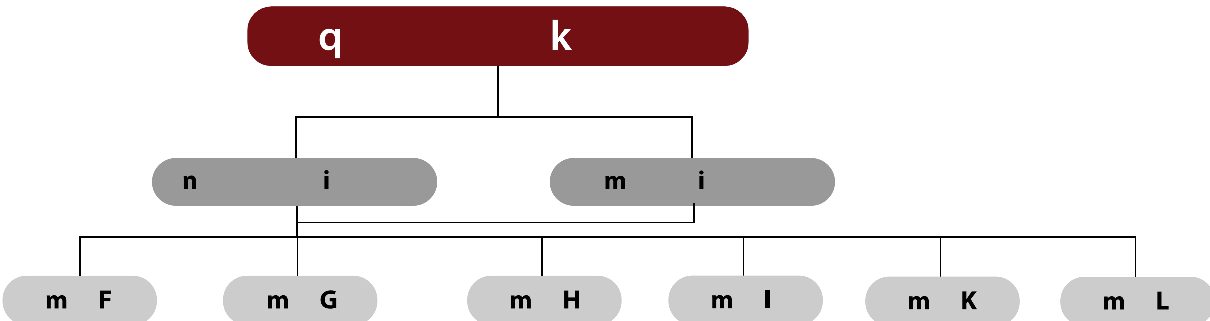
UNIT III: INDICES

LEARNING OBJECTIVES

After reading this unit a student will learn –

- A meaning of indices and their applications.
- Laws of indices which facilitates their easy applications.

UNIT OVERVIEW



1.3 INDICES:

We are aware of certain operations of addition and multiplication and now we take up certain higher order operations with powers and roots under the respective heads of indices.

We know that the result of a repeated addition can be held by multiplication e.g.

$$4 + 4 + 4 + 4 + 4 = 5(4) = 20$$

$$a + a + a + a + a = 5(a) = 5a$$

Now, $4 \times 4 \times 4 \times 4 \times 4 = 4^5;$

$$a \times a \times a \times a \times a = a^5.$$

It may be noticed that in the first case 4 is multiplied 5 times and in the second case 'a' is multiplied 5 times. In all such cases a factor which multiplies is called the "**base**" and the number of times it is multiplied is called the "**power**" or the "**index**". Therefore, "4" and "a" are the bases and "5" is the index for both. Any base raised to the power zero is defined to be 1; i.e. a # 0. We also define $\sqrt[r]{a} = a^{1/r}$.

If n is a positive integer, and 'a' is a real number, i.e. $n \in N$ and $a \in R$ (where N is the set of positive integers and R is the set of real numbers), 'a' is used to denote the continued product of n factors each equal to 'a' as shown below:

$a^n = a \times a \times a$ ………….. to n factors.

Here a^n is a power of "a" whose base is "a" and the index or power is "n".

For example, in $3 \times 3 \times 3 \times 3 = 3^4$, 3 is base and 4 is index or power.

Law 1

$a^m \times a^n = a^{m+n}$, when m and n are positive integers; by the above definition,
$a^m = a \times a$ ………….. to m factors and $a^n = a \times a$ ………….. to n factors.

$\therefore$ $a^m \times a^n = (a \times a$ ………….. to m factors$) \times (a \times a$ ………. to n factors$)$

$= a \times a$ ………….. to $(m + n)$ factors

$= a^{m+n}$

Now, we extend this logic to negative integers and fractions. First let us consider this for negative integer, that is m will be replaced by –n. By the definition of $a^m \times a^n = a^{m+n}$,

we get $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$

For example $3^4 \times 3^5 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3) = 3^{4+5} = 3^9$

Again, $3^{-5} = 1/3^5 = 1/(3 \times 3 \times 3 \times 3 \times 3) = 1/243$

Example 1: Simplify $2x^{1/2} 3x^{-1}$ if $x = 4$

Solution: We have $2x^{1/2} 3x^{-1}$

$= 6x^{1/2} x^{-1} = 6x^{1/2-1}$

$= 6x^{-1/2}$

$= \dfrac{6}{x^{1/2}} = \dfrac{6}{4^{1/2}} = \dfrac{6}{(2^2)^{1/2}} = \dfrac{6}{2} = 3$

Example 2: Simplify $6ab^2c^3 \times 4b^{-2}c^{-3}d$.

Solution: $6ab^2c^3 \times 4b^{-2}c^{-3}d$

$= 24 \times a \times b^2 \times b^{-2} \times c^3 \times c^{-3} \times d$

$= 24 \times a \times b^{2+(-2)} \times c^{3+(-3)} \times d$

$= 24 \times a \times b^{2-2} \times c^{3-3} \times d$

$= 24a\ b^0 \times c^0 \times d$

$= 24ad$

Law 2

$a^m / a^n = a^{m-n}$, when m and n are positive integers and m > n.

By definition, $a^m = a \times a$ ………….. to m factors

$$\text{Therefore, } a^m \div a^n = \frac{a^m}{a^n} = \frac{a \times a \text{.................to m factors}}{a \times a \text{................... to n factors}}$$

$= a \times a$ ……… to (m–n) factors

$= a^{m-n}$

Now we take a numerical value for a and check the validity of this Law

$$2^7 \div 2^4 = \frac{2^7}{2^4} = \frac{2 \times 2 \text{.............to 7 factors}}{2 \times 2 \text{................ to 4 factors}}$$

$= 2 \times 2 \times 2$ ……….. to (7–4) factors.

$= 2 \times 2 \times 2$ ……….. to 3 factors

$= 2^3 = 8$

$$\text{or} \quad 2^7 \div 2^4 = \frac{2^7}{2^4} = \frac{2\times2\times2\times2\times2\times2\times2}{2\times2\times2\times2}$$

$= 2 \times 2 \times 2 = 2^{1+1+1} = 2^3$

$= 8$

Example 3: Find the value of $\frac{4x^{-1}}{x^{-1/3}}$

Solution: $\frac{4x^{-1}}{x^{-1/3}}$

$= 4x^{-1-(-1/3)}$

$= 4x^{-1+1/3}$

$= 4x^{-2/3}$ or $\frac{4}{x^{2/3}}$

Example 4: Simplify $\dfrac{2a^{\frac{1}{2}} \times a^{\frac{2}{3}} \times 6a^{-\frac{7}{3}}}{9a^{\frac{-5}{3}} \times a^{\frac{3}{2}}}$ if $a = 4$

Solution: $\dfrac{2a^{\frac{1}{2}} \times a^{\frac{2}{3}} \times 6a^{-\frac{7}{3}}}{9a^{\frac{-5}{3}} \times a^{\frac{3}{2}}}$ if $a = 4$

$$= \frac{2.2.3.a^{\frac{1}{2}+\frac{2}{3}-\frac{7}{3}}}{3.3a^{-\frac{5}{3}+\frac{3}{2}}} = \frac{4}{3}\frac{a^{(3+4-14)/6}}{a^{(-10+9)/6}}$$

$$= \frac{4}{3}.\frac{a^{-7/6}}{a^{-1/6}} = \frac{4}{3}a^{\frac{-7}{6}+\frac{1}{6}}$$

$$= \frac{4}{3}a^{-1} = \frac{4}{3}.\frac{1}{a} = \frac{4}{3}.\frac{1}{4} = \frac{1}{3}$$

Law 3

$(a^m)^n = a^{mn}$. where m and n are positive integers

By definition $(a^m)^n = a^m \times a^m \times a^m$ …….. to n factors

$= (a \times a$ …….. to m factors) $a \times a \times$…….. to m factors……..to n times

$= a \times a$ …….. to mn factors

$= a^{mn}$

Following above, $(a^m)^n = (a^m)^{p/q}$

(We will keep m as it is and replace n by p/q, where p and q are positive integers)

$(a^m)^{p/q}$ is $\left\{(a^m)^{p/q}\right\}^q = (a^m)^{(p/q)\times q}$

$= (a^m)^{(p/q)\times q}$

$= a^{mp}$

If we take the qth root of the above we obtain

$$\left(a^{mp}\right)^{1/q} = \sqrt[q]{a^{mp}}$$

Now with the help of a numerical value for a let us verify this law.

$(2^4)^3 = 2^4 \times 2^4 \times 2^4$

$= 2^{4+4+4}$

$= 2^{12} = 4096$

Law 4

$(ab)^n = a^n \cdot b^n$ when n can take all of the values.

For example $6^3 = (2 \times 3)^3 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 3^3$

First, we look at n when it is a positive integer. Then by the definition, we have

$(ab)^n$ = ab × ab ……………. to n factors

= (a × a ……..……. to n factors) × (b × b ………… n factors)

$= a^n \times b^n$

When n is a positive fraction, we will replace n by p/q.

Then we will have $(ab)^n = (ab)^{p/q}$

The qth power of $(ab)^{p/q} = \{(ab)^{(p/q)}\}^q = (ab)^p$

Example 5: Simplify $(x^a.y^{-b})^3 \cdot (x^3 y^2)^{-a}$

Solution: $(x^a.y^{-b})^3 \cdot (x^3 y^2)^{-a}$

$= (x^a)^3 . (y^{-b})^3 \cdot (x^3)^{-a} . (y^2)^{-a}$

$= x^{3a-3a} . y^{-3b-2a.}$

$= x^0. y^{-3b-2a.}$

$= \dfrac{1}{y^{3b+2a}}$

Example 6: $\sqrt[6]{a^{4b} x^6}.(a^{2/3}x^{-1})^{-b}$

Solution: $\sqrt[6]{a^{4b} x^6}.(a^{2/3}x^{-1})^{-b}$

$= (a^{4b}x^6)^{\frac{1}{6}}.(a^{\frac{2}{3}})^{-b}.(x^{-1})^{-b}$

$= (a^{4b})^{\frac{1}{6}}.(x^6)^{\frac{1}{6}}.a^{-\frac{2}{3}b}.x^{(-1)\times(-b)}$

$= a^{\frac{2}{3}b}.x.a^{-\frac{2b}{3}}.x^b$

$= a^{\frac{2}{3}b-\frac{2}{3}b}.x^{1+b}$

$= a^0 . x^{1+b} = x^{1+b}$

Example 7: Find x, if $x\sqrt{x} = (x\sqrt{x})^x$

Solution: $x\,(x)^{1/2} = x^x.x^{x/2}$

or, $x^{1+1/2} = x^{x+x/2}$

or, $x^{3/2} = x^{3x/2}$

[If base is equal, then power is also equal]

i.e. $\frac{3}{2} = \frac{3x}{2}$ or, $x = \frac{3}{2} \times \frac{2}{3} = 1$

$\therefore$ x = 1

Example 8: Find the value of k from $(\sqrt{9})^{-7} \times (\sqrt{3})^{-5} = 3^k$

Solution: $(\sqrt{9})^{-7} \times (\sqrt{3})^{-5} = 3^k$

or, $(3^{2 \times 1/2})^{-7} \times (3^{½})^{-5} = 3^k$

or, $3^{-7-5/2} = 3^k$

or, $3^{-19/2} = 3^k$ or, k = –19/2

SUMMARY

- $a^m \times a^n = a^{m+n}$ (base must be same)

  Ex. $2^3 \times 2^2 = 2^{3+2} = 2^5$
- $a^m / a^n = a^{m-n}$

  Ex. $2^5 / 2^3 = 2^{5-3} = 2^2$
- $(a^m)^n = a^{mn}$

  Ex. $(2^5)^2 = 2^{5 \times 2} = 2^{10}$
- $a^o = 1$ (a # 1)

  Ex. $2^0 = 1, 3^0 = 1$
- $a^{-m} = 1/a^m$ and $1/a^{-m} = a^m$

  Ex. $2^{-3} = 1/2^3$ and $1/2^{-5} = 2^5$
- If $a^x = a^y$, then x=y (a # y, 0,1)
- If $x^a = y^a$, then x=y (x,y # -1,0,1)
- $\sqrt[m]{a} = a^{1/m}$, $\sqrt{x} = x^{½}$, $\sqrt{4} = 2$

  Ex. $\sqrt[3]{8} = 8^{1/3} = (2^3)^{1/3} = 2^{3 \times 1/3} = 2$

EXERCISE 1(C)

Choose the most appropriate option (a) (b) (c) or (d).

1. $4x^{-1/4}$ is expressed as

   (a) $-4x^{1/4}$ (b) x^{-1} (c) $4/x^{1/4}$ (d) none of these

2. The value of $8^{1/3}$ is

   (a) $\sqrt[3]{2}$ (b) 4 (c) 2 (d) none of these

3. The value of $2 \times (32)^{1/5}$ is

   (a) 2 (b) 10 (c) 4 (d) none of these

4. The value of $4/(32)^{1/5}$ is

   (a) 8 (b) 2 (c) 4 (d) none of these

5. The value of $(8/27)^{1/3}$ is

   (a) 2/3 (b) 3/2 (c) 2/9 (d) none of these

6. The value of $2(256)^{-1/8}$ is

   (a) 1 (b) 2 (c) 1/2 (d) none of these

7. $2^{½} . 4^{¾}$ is equal to

   (a) a fraction (b) a positive integer (c) a negative integer (d) none of these

8. $\left(\frac{81x^4}{y^{-8}}\right)^{\frac{1}{4}}$ has simplified value equal to

   (a) xy^2 (b) x^2y (c) $9xy^2$ (d) none of these

9. $x^{a-b} \times x^{b-c} \times x^{c-a}$ is equal to

   (a) x (b) 1 (c) 0 (d) none of these

10. The value of $\left(\frac{2p^2q^3}{3xy}\right)^0$ where p, q, x, y ≠ 0 is equal to

    (a) 0 (b) 2/3 (c) 1 (d) none of these

11. $\{(3^3)^2 \times (4^2)^3 \times (5^3)^2\} / \{(3^2)^3 \times (4^3)^2 \times (5^2)^3\}$ is

    (a) 3/4 (b) 4/5 (c) 4/7 (d) 1

12. Which is True ?

    (a) $2^0 > (1/2)^0$ (b) $2^0 < (1/2)^0$ (c) $2^0 = (1/2)^0$ (d) none of these

13. If $x^{1/p} = y^{1/q} = z^{1/r}$ and xyz = 1, then the value of p + q + r is

    (a) 1 (b) 0 (c) 1/2 (d) none of these

14. The value of $y^{a-b} \times y^{b-c} \times y^{c-a} \times y^{-a-b}$ is

    (a) y^{a+b} (b) y (c) 1 (d) $1/y^{a+b}$

15. The True option is
(a) $x^{2/3} = \sqrt[3]{x^2}$ (b) $x^{2/3} = \sqrt{x^3}$ (c) $x^{2/3} > \sqrt[3]{x^2}$ (d) $x^{2/3} < \sqrt[3]{x^2}$

16. The simplified value of $16x^{-3}y^2 \times 8^{-1}x^3y^{-2}$ is
(a) 2xy (b) xy/2 (c) 2 (d) none of these

17. The value of $(8/27)^{-1/3} \times (32/243)^{-1/5}$ is
(a) 9/4 (b) 4/9 (c) 2/3 (d) none of these

18. The value of $\left\{\frac{(x+y)^{2/3} \times (x-y)^{2/3}}{\sqrt{x+y} \times \sqrt{(x-y)^3}}\right\}^6$ is
(a) $(x + y)^2$ (b) (x – y) (c) x + y (d) none of these

19. Simplified value of $(125)^{2/3} \times \sqrt{25} \times \sqrt[3]{5^3} \times 5^{1/2}$ is
(a) 5 (b) 1/5 (c) 1 (d) none of these

20. $[\{(2)^{1/2} . (4)^{3/4} . (8)^{5/6} . (16)^{7/8} . (32)^{9/10}\}^4]^{3/25}$ is
(a) A fraction (b) an integer (c) 1 (d) none of these

21. $[1-\{1-(1-x^2)^{-1}\}^{-1}]^{-1/2}$ is equal to
(a) x (b) 1/x (c) 1 (d) none of these

22. $\left[\left(x^n\right)^{n-\frac{1}{n}}\right]^{\frac{1}{n+1}}$ is equal to
(a) x^n (b) x^{n+1} (c) x^{n-1} (d) none of these

23. If $a^3-b^3 = (a-b)(a^2 + ab + b^2)$, then the simplified form of
$$\left[\frac{x^l}{x^m}\right]^{l^2+lm+m^2} \times \left[\frac{x^m}{x^n}\right]^{m^2+mn+n^2} \times \left[\frac{x^n}{x^l}\right]^{l^2+ln+n^2}$$
(a) 0 (b) 1 (c) x (d) none of these

24. Using $(a-b)^3 = a^3-b^3-3ab(a-b)$ tick the correct of these when $x = p^{1/3} - p^{-1/3}$
(a) $x^3+3x = p + 1/p$ (b) $x^3 + 3x = p - 1/p$ (c) $x^3 + 3x = p + 1$ (d) none of these

25. On simplification, $1/(1+a^{m-n}+a^{m-p}) + 1/(1+a^{n-m}+a^{n-p}) + 1/(1+a^{p-m}+a^{p-n})$ is equal to
(a) 0 (b) a (c) 1 (d) 1/a

26. The value of $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a}$
(a) 1 (b) 0 (c) 2 (d) none of these

27. If $x = 3^{\frac{1}{3}} + 3^{-\frac{1}{3}}$, then $3x^3 - 9x$ is
(a) 15 (b) 10 (c) 12 (d) none of these

28. If $a^x = b, b^y = c, c^z = a$, then xyz is
(a) 1 (b) 2 (c) 3 (d) none of these

29. The value of $\left(\frac{x^a}{x^b}\right)^{(a^2+ab+b^2)} \times \left(\frac{x^b}{x^c}\right)^{(b^2+bc+c^2)} \times \left(\frac{x^c}{x^a}\right)^{(c^2+ca+a^2)}$
(a) 1 (b) 0 (c) –1 (d) none of these

30. If $2^x = 3^y = 6^{-z}$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is
(a) 1 (b) 0 (c) 2 (d) none of these

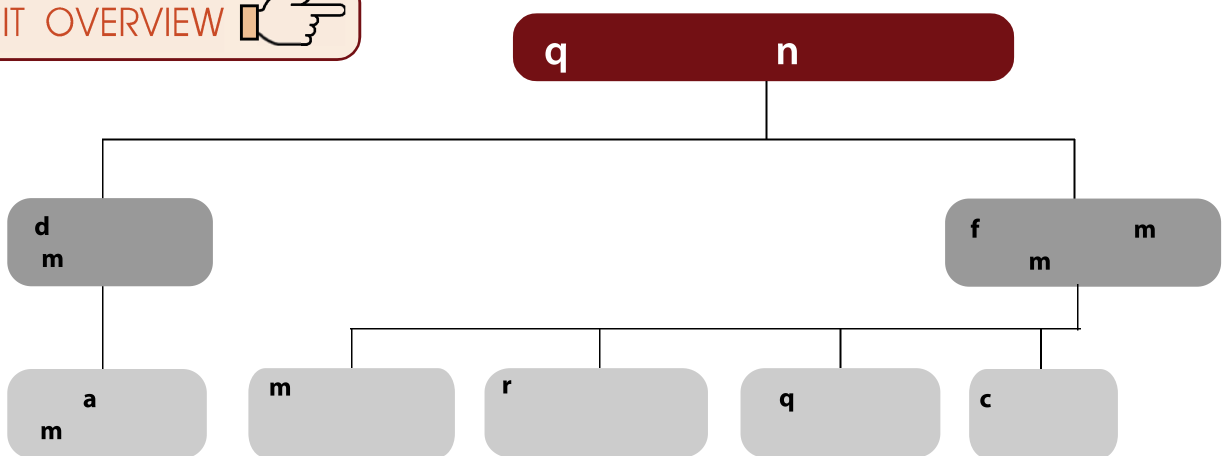
UNIT IV: LOGARITHM

LEARNING OBJECTIVES

After reading this unit a student will learn –

- After reading this unit, a student will get fundamental knowledge of logarithm and its application for solving business problems.

UNIT OVERVIEW



1.4 LOGARITHM:

The logarithm of a number to a given base is the index or the power to which the base must be raised to produce the number, i.e. to make it equal to the given number. If there are three quantities indicated by say a, x and n, they are related as follows:

If $a^x = n$, where $n > 0$, $a > 0$ and $a \neq 1$
then x is said to be the logarithm of the number n to the base 'a' symbolically it can be expressed as follows:

$\log_a n = x$, where $n > 0$, $a > 0$ and $a = 1$
i.e. the logarithm of n to the base 'a' is x. We give some illustrations below:

(i) $2^4 = 16 \Rightarrow \log_2 16 = 4$
i.e. the logarithm of 16 to the base 2 is equal to 4

(ii) $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$
i.e. the logarithm of 1000 to the base 10 is 3

(iii) $5^{-3} = \frac{1}{125} \Rightarrow \log_5\left(\frac{1}{125}\right) = -3$

i.e. the logarithm of $\frac{1}{125}$ to the base 5 is –3

(iv) $2^3 = 8 \Rightarrow \log_2 8 = 3$
i.e. the logarithm of 8 to the base 2 is 3

Remarks:

1. The two equations $a^x = n$ and $x = \log_a n$ are only transformations of each other and should be remembered to change one form of the relation into the other.
2. The logarithm of 1 to any base is zero. This is because any number raised to the power zero is one.

   Since $a^0 = 1$, $\log_a 1 = 0$
3. The logarithm of any quantity to the same base is unity. This is because any quantity raised to the power 1 is that quantity only.

   Since $a^1 = a$, $\log_a a = 1$

ILLUSTRATIONS:

1. If $\log_a \sqrt{2} = \frac{1}{6}$, find the value of a.

   We have $a^{1/6} = \sqrt{2} \Rightarrow a = (\sqrt{2})^6 = 2^3 = 8$
2. Find the logarithm of 5832 to the base 3√2.

   Let us take $\log_{3\sqrt{2}} 5832 = x$

   We may write, $(3\sqrt{2})^x = 5832 = 8 \times 729 = 2^3 \times 3^6 = (\sqrt{2})^6 (3)^6 = (3\sqrt{2})^6$

   Hence, x = 6

Logarithms of numbers to the base 10 are known as common logarithm.

1.4.1 Fundamental Laws of Logarithm

1. Logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers to the same base, i.e.

   $\log_a mn = \log_a m + \log_a n$

Proof:

Let $\log_a m = x$ so that $a^x = m$ – (I)

$\mathrm{Log}_a n = y$ so that $a^y = n$ – (II)

Multiplying (I) and (II), we get

$m \times n = a^x \times a^y = a^{x+y}$

$\log_a mn = x + y$ (by definition)

$\therefore \log_a mn = \log_a m + \log_a n$

2. The logarithm of the quotient of two numbers is equal to the difference of their logarithms to the same base, i.e.

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

Proof:

Let $\log_a m = x$ so that $a^x = m$ ————(I)

$\log_a n = y$ so that $a^y = n$ ——————(II)

Dividing (I) by (II) we get

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$$

Then by the definition of logarithm, we get

$$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$$

Similarly, $\log_a \frac{1}{n} = \log_a 1 - \log_a n = 0 - \log_a n = -\log_a n \; [\because \log_a 1 = 0]$

Illustration I: $\log ½ = \log 1 - \log 2 = -\log 2$

3. Logarithm of the number raised to the power is equal to the index of the power multiplied by the logarithm of the number to the same base i.e.

$\log_a m^n = n \log_a m$

Proof:

Let $\log_a m = x$ so that $a^x = m$

Raising the power n on both sides we get

$(a^x)^n = (m)^n$

$a^{xn} = m^n$ (by definition)

$\log_a m^n = nx$

i.e. $\log_a m^n = n \log_a m$

Illustration II: 1(a) Find the logarithm of 1728 to the base $2\sqrt{3}$.

Solution: We have $1728 = 2^6 \times 3^3 = 2^6 \times (\sqrt{3})^6 = (2\sqrt{3})^6$; and so, we may write

$$\log_{2\sqrt{3}} 1728 = 6$$

1(b) Solve $\frac{1}{2}\log_{10}25 - 2\log_{10}3 + \log_{10}18$

Solution: The given expression

$$= \log_{10} 25^{\frac{1}{2}} - \log_{10} 3^2 + \log_{10} 18$$

$$= \log_{10} 5 - \log_{10} 9 + \log_{10} 18$$

$$= \log_{10} \frac{5x18}{9} = \log_{10} 10 = 1$$

1.4.2 Change of Base

If the logarithm of a number to any base is given, then the logarithm of the same number to any other base can be determined from the following relation.

$$\log_a m = \log_b m \times \log_a b \Rightarrow \log_b m = \frac{\log_a m}{\log_a b}$$

Proof:

Let $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$

Then by definition,

$a^x = m$, $b^y = m$ and $a^z = b$

Also $a^x = b^y = (a^z)^y = a^{yz}$

Therefore, $x = yz$

$\Rightarrow \log_a m = \log_b m \times \log_a b$

$$\log_b m = \frac{\log_a m}{\log_a b}$$

Putting $m = a$, we have

$\log_a a = \log_b a \times \log_a b$

$\Rightarrow \log_b a \times \log_a b = 1$, since $\log_a a = 1$.

Example 1: Change the base of $\log_5 31$ into the common logarithmic base.

Solution: Since $\log_a x = \frac{\log_b x}{\log_b a}$

$$\therefore \log_5 31 = \frac{\log_{10} 31}{\log_{10} 5}$$

Example 2: Prove that $\dfrac{\log_3 8}{\log_9 16 \log_4 10} = 3\log_{10} 2$

Solution: Change all the logarithms on L.H.S. to the base 10 by using the formula.

$\log_b x = \dfrac{\log_a x}{\log_a b}$, we may write

$$\log_3 8 = \frac{\log_{10} 8}{\log_{10} 3} = \frac{\log_{10} 2^3}{\log_{10} 3} = \frac{3\log_{10} 2}{\log_{10} 3}$$

$$\log_9 16 = \frac{\log_{10} 16}{\log_{10} 9} = \frac{\log_{10} 2^4}{\log_{10} 3^2} = \frac{4\log_{10} 2}{2\log_{10} 3}$$

$$\log_4 10 = \frac{\log_{10} 10}{\log_{10} 4} = \frac{1}{\log_{10} 2^2} = \frac{1}{2\log_{10} 2} \quad [\log_{10} 10 = 1]$$

$$\therefore \text{L.H.S.} = \frac{3\log_{10} 2}{\log_{10} 3} \times \frac{2\log_{10} 3}{4\log_{10} 2} \times \frac{2\log_{10} 2}{1} \therefore [\log_{10} 10 = 1]$$

$= 3\log_{10} 2 =$ R.H.S.

Logarithm Tables:

The logarithm of a number consists of two parts, the whole part or the integral part is called the **characteristic** and the decimal part is called the **mantissa** where the former can be known by mere inspection, the latter has to be obtained from the logarithm tables.

Characteristic:

The characteristic of the logarithm of any number greater than 1 is positive and is one less than the number of digits to the left of the decimal point in the given number. The characteristic of the logarithm of any number less than one (1) is negative and numerically one more than the number of zeros to the right of the decimal point. If there is no zero then obviously it will be –1. The following table will illustrate it.

Number		Characteristic
3 7	1	[One less than the number of digits to the left of the decimal point]
4 6 2 3	3	
6.21	0	

Number		Characteristic
.8	–1	[One more than the number of zeros on the right immediately after the decimal point.]
.07	–2	

.00507	–3
.000670	–4

Zero on positive characteristic when the number under consideration is greater than unity:

Since	$10^0 = 1$,	log 1 = 0
	$10^1 = 10$,	log 10 = 1
	$10^2 = 100$,	log 100 = 2
	$10^3 = 1000$,	log 1000 = 3

All numbers lying between 1 and 10 i.e. numbers with 1 digit in the integral part have their logarithms lying between 0 and 1. Therefore, their integral parts are zero only.

All numbers lying between 10 and 100 have two digits in their integral parts. Their logarithms lie between 1 and 2. Therefore, numbers with two digits have integral parts with 1 as characteristic.

In general, the logarithm of a number containing n digits only in its integral parts is (n – 1) + a decimal. For example, the characteristics of log 75, log 79326, log 1.76 are 1, 4 and 0 respectively.

Negative characteristics

$$\textit{Since}\ 10^{-1} = \frac{1}{10} = 0.1 \rightarrow \log 0.1 = -1$$

$$10^{-2} = \frac{1}{100} = 0.01 \rightarrow \log 0.01 = -2$$

All numbers lying between 1 and 0.1 have logarithms lying between 0 and –1, i.e. greater than –1 and less than 0. Since the decimal part is always written positive, the characteristic is –1.

All numbers lying between 0.1 and 0.01 have their logarithms lying between –1 and –2 as characteristic of their logarithms.

In general, the logarithm of a number having n zeros just after the decimal point is

− (n + 1) + a decimal.

Hence, we deduce that the characteristic of the logarithm of a number less than unity is one more than the number of zeros just after the decimal point and is negative.

Mantissa

The mantissa is the fractional part of the logarithm of a given number.

Number	Mantissa	Logarithm
Log 4594	= (……… 6623)	= 3.6623
Log 459.4	= (……… 6623)	= 2.6623
Log 45.94	= (……… 6623)	= 1.6623
Log 4.594	= (……… 6623)	= 0.6623
Log .4594	= (……… 6623)	= $\bar{1}.6623$

Thus with the same figures there will be difference in the characteristic only. It should be remembered, that the mantissa is always a positive quantity. The other way to indicate this is

$\text{Log } .004594 = -3 + .6623 = \bar{3}.6623.$

Negative mantissa must be converted into a positive mantissa before reference to a logarithm table. For example

$-3.6872 = -4 + (1 - 3.6872) = \bar{4} + 0.3128 = \bar{4}.3128$

It may be noted that $\bar{4}.3128$ is different from – 4.3128 as – 4.3128 is a negative number whereas, in $\bar{4}.3128$, 4 is negative while .3128 is positive.

Illustration I: Add $\bar{4}.74628$ and 3.42367

– 4 + .74628

<u>3 + .42367</u>

–1 + 1.16995 = 0.16995

Antilogarithms

If x is the logarithm of a given number n with a given base then n is called the antilogarithm (antilog) of x to that base.

This can be expressed as follows:

If $\log_a n = x$ then $n = \text{antilog } x$

For example, if log 61720 = 4.7904 then 61720 = antilog 4.7904

Number	Logarithm
206	2.3139
20.6	1.3139
2.06	0.3139
.206	$\bar{1}.3139$
.0206	$\bar{2}.3139$

Example 1: Find the value of log 5 if log 2 is equal to .3010.

Solution: $\log 5 = \log \frac{10}{2} = \log 10 - \log 2$

= 1 – .3010

= .6990

Example 2: Find the number whose logarithm is 2.4678.

Solution: From the antilog table, for mantissa .467, the number = 2931

for mean difference 8, the number = 5

∴ for mantissa .4678, the number = 2936

The characteristic is 2, therefore, the number must have 3 digits in the integral part.

Hence, Antilog 2.4678 = 293.6

Example 3: Find the number whose logarithm is –2.4678.

Solution: $-2.4678 = -3 + 3 - 2.4678 = -3 + .5322 = \bar{3}.5322$

For mantissa .532, the number = 3404

For mean difference 2, the number = 2

$\therefore$ for mantissa .5322, the number = 3406

The characteristic is –3, therefore, the number is less than one and there must be two zeros just after the decimal point.

Thus, Antilog (–2.4678) = 0.003406

Relation between Indices and Logarithm

Let $x = \log_a m$ and $y = \log_a n$

$\therefore\ a^x = m$ and $a^y = n$

so $a^x . a^y = mn$

or $a^{x+y} = mn$

or $x^1 + y^1 = \log_a mn$

or $\log_a m + \log_a n = \log_a mn$ $\quad [\because \log_a a = 1]$

or $\log_a mn = \log_a m + \log_a n$

Also, $(m/n) = a^x/a^y$

or $(m/n) = a^{x-y}$

or $\log_a (m/n) = (x-y)$

or $\log_a (m/n) = \log_a m - \log_a n$ $\quad [\because \log_a a = 1]$

Again $m^n = m.m.m.$ ———————— to n times

so $\log_a m^n = \log_a (m.m.m$ ———————— to n times)

or $\log_a m^n = \log_a m + \log_a m + \log_a m +$ ———————— $+ \log_a m$

or $\log_a m^n = n \log_a m$

Now $a^0 = 1 \Rightarrow 0 = \log_a 1$

Let $\log_b a = x$ and $\log_a b = y$

$\therefore\ a = b^x$ and $b = a^y$

$\therefore$ so $a = (a^y)^x$

or $a^{xy} = a$

or $xy = 1$

or $\log_b a \times \log_a b = 1$

let $\log_b c = x$ & $\log_c b = y$

$\therefore$ $c = b^x$ & $b = c^y$

so $c = c^{xy}$ or $xy = 1$

$\log_b c \times \log_c b = 1$

Example 1: Find the logarithm of 64 to the base $2\sqrt{2}$

Solution: $\log_{2\sqrt{2}} 64 = \log_{2\sqrt{2}} 8^2 = 2\log_{2\sqrt{2}} 8 = 2\log_{2\sqrt{2}} (2\sqrt{2})^2 = 4\log_{2\sqrt{2}} 2\sqrt{2} = 4 \times 1 = 4$

Example 2: If $\log_a bc = x$, $\log_b ca = y$, $\log_c ab = z$, prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$$

Solution: $x+1 = \log_a bc + \log_a a = \log_a abc$

$y+1 = \log_b ca + \log_b b = \log_b abc$

$z+1 = \log_c ab + \log_c c = \log_c abc$

Therefore $$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = \frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$$

$= \log_{abc} a + \log_{abc} b + \log_{abc} c$

$= \log_{abc} abc = 1$ (proved)

Example 3: If $a = \log_{24} 12$, $b = \log_{36} 24$, and $c = \log_{48} 36$ then prove that

$1 + abc = 2bc$

Solution: $1 + abc = 1 + \log_{24} 12 \times \log_{36} 24 \times \log_{48} 36$

$= 1 + \log_{36} 12 \times \log_{48} 36$

$= 1 + \log_{48} 12$

$= \log_{48} 48 + \log_{48} 12$

$= \log_{48} 48 \times 12$

$= \log_{48} (2 \times 12)^2$

$= 2\log_{48} 24$

$= 2\log_{36} 24 \times \log_{48} 36$

$= 2bc$

SUMMARY

- $\log_a mn = \log_a m + \log_a n$

  Ex. log (2 × 3) = log 2 + log 3
- $\log_a (m/n) = \log_a m - \log_a n$

  Ex. log (3/2) = log3 – log2
- $\log_a m^n = n \log_a m$

  Ex. $\log 2^3 = 3 \log 2$
- $\log_a a = 1, a = 1$ (Since $a^1 = a$)

  Ex. $\log_{10} 10 = 1, \log_2 2 = 1, \log_3 3 = 1$ etc.
- $\log_a 1 = 0$ (Since $a^0 = 1$)

  Ex. $\log_2 1 = 0, \log_{10} 1 = 0$ etc.
- $\log_b a \times \log_a b = 1$

  Ex. $\log_3 2 \times \log_2 3 = 1$
- $\log_b a \times \log_c b = \log_c a$

  Ex. $\log_3 2 \times \log_5 3 = \log_5 2$
- $\log_b a = \log a / \log b$ (Base changing formula)

  Ex. $\log_3 2 = \log 2/\log 3$
- $\log_b a = 1/\log_a b$
- $a^{\log_a x} = x$ (Inverse logarithm Property)
- The two equations $a^x = n$ and $x = \log_a n$ are only transformations of each other and should be remembered to change one form of the relation into the other.

  Since $a_1 = a$, $\log_a^a = 1$

Notes:

(A) If base is understood, base is taken as 10

(B) Thus log 10 = 1, log 1 = 0

(C) Logarithm using base 10 is called Common logarithm and logarithm using base e is called Natural logarithm {e = 2.33 (approx.) called exponential number}.

EXERCISE 1(D)

Choose the most appropriate option. (a) (b) (c) or (d).

1. log 6 + log 5 is expressed as

   (a) log 11 (b) log 30 (c) log 5/6 (d) none of these

2. $\log_2 8$ is equal to
(a) 2 (b) 8 (c) 3 (d) none of these

3. log (32/4) is equal to
(a) log 32/log 4 (b) log 32 – log 4 (c) 2^3 (d) none of these

4. log (1 × 2 × 3) is equal to
(a) log 1 + log 2 + log 3 (b) log 3 (c) log 2 (d) none of these

5. The value of log 0.0001 to the base 0.1 is
(a) –4 (b) 4 (c) 1/4 (d) none of these

6. If 2 log x = 4 log 3, the *x* is equal to
(a) 3 (b) 9 (c) 2 (d) none of these

7. $\log_{\sqrt{2}} 64$ is equal to
(a) 12 (b) 6 (c) 1 (d) none of these

8. $\log_{2\sqrt{3}} 1728$ is equal to
(a) $2\sqrt{3}$ (b) 2 (c) 6 (d) none of these

9. log (1/81) to the base 9 is equal to
(a) 2 (b) ½ (c) –2 (d) none of these

10. log 0.0625 to the base 2 is equal to
(a) 4 (b) 5 (c) 1 (d) none of these

11. Given log2 = 0.3010 and log3 = 0.4771 the value of log 6 is
(a) 0.9030 (b) 0.9542 (c) 0.7781 (d) none of these

12. The value of $\log_2 \log_2 \log_2 16$
(a) 0 (b) 2 (c) 1 (d) none of these

13. The value of log $\frac{1}{3}$ to the base 9 is
(a) – ½ (b) ½ (c) 1 (d) none of these

14. If log x + log y = log (x+y), y can be expressed as
(a) x–1 (b) x (c) x/(x–1) (d) none of these

15. The value of $\log_2 [\log_2 \{\log_3 (\log_3 27^3)\}]$ is equal to
(a) 1 (b) 2 (c) 0 (d) none of these

16. If $\log_2 x + \log_4 x + \log_{16} x = 21/4$, these x is equal to
(a) 8 (b) 4 (c) 16 (d) none of these

17. Given that $\log_{10} 2 = x$ and $\log_{10} 3 = y$, the value of $\log_{10} 60$ is expressed as
(a) x – y + 1 (b) x + y + 1 (c) x – y – 1 (d) none of these

18. Given that $\log_{10} 2 = x$, $\log_{10} 3 = y$, then $\log_{10} 1.2$ is expressed in terms of *x* and y as
(a) x + 2y – 1 (b) x + y – 1 (c) 2x + y – 1 (d) none of these

19. Given that log x = m + n and log y = m – n, the value of log $(10x/y^2)$ is expressed in terms of m and n as
(a) 1 – m + 3n (b) m – 1 + 3n (c) m + 3n + 1 (d) none of these

20. The simplified value of $2 \log_{10}5 + \log_{10}8 - ½ \log_{10}4$ is
    (a) 1/2 (b) 4 (c) 2 (d) none of these
21. $\log [1 - \{1 - (1 - x^2)^{-1}\}^{-1}]^{-1/2}$ can be written as
    (a) $\log x^2$ (b) $\log x$ (c) $\log 1/x$ (d) none of these
22. The simplified value of $\log \left(\sqrt[4]{729 \sqrt[3]{9^{-1}.27^{-4/3}}} \right)$ is
    (a) log 3 (b) log 2 (c) log ½ (d) none of these
23. The value of $(\log_b a \times \log_c b \times \log_a c)^3$ is equal to
    (a) 3 (b) 0 (c) 1 (d) none of these
24. The logarithm of 64 to the base $2\sqrt{2}$ is
    (a) 2 (b) $\sqrt{2}$ (c) ½ (d) none of these
25. The value of $\log_8 25$ given $\log_{10} 2 = 0.3010$ is
    (a) 1 (b) 2 (c) 1.5482 (d) none of these

ANSWERS

Exercise 1(A)

1.	(a)	**2.**	(d)	**3.**	(c)	**4.**	(a)	**5.**	(c)	**6.**	(d)	**7.**	(a)	**8.**	(c)
9.	(a)	**10.**	(c)	**11.**	(d)	**12.**	(d)	**13.**	(a)	**14.**	(c)	**15.**	(d)	**16.**	(a)
17.	(c)	**18.**	(b)	**19.**	(b)	**20.**	(c)	**21.**	(a)	**22.**	(c)	**23.**	(a)	**24.**	(c)
25.	(c)														

Exercise 1(B)

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(d)	**5.**	(a)	**6.**	(c)	**7.**	(a)	**8.**	(c)
9.	(c)	**10.**	(b)	**11.**	(c)	**12.**	(d)	**13.**	(a)	**14.**	(d)	**15.**	(d)	**16.**	(a)
17.	(a)	**18.**	(b)	**19.**	(d)	**20.**	(a)	**21.**	(c)	**22.**	(d)	**23.**	(c)	**24.**	(a)
25.	(b)	**26.**	(b)	**27.**	(c)	**28.**	(b)	**29.**	(a)	**30.**	(b)				

Exercise 1(C)

1.	(c)	**2.**	(c)	**3.**	(c)	**4.**	(b)	**5.**	(a)	**6.**	(a)	**7.**	(b)	**8.**	(d)
9.	(b)	**10.**	(c)	**11.**	(d)	**12.**	(c)	**13.**	(b)	**14.**	(d)	**15.**	(a)	**16.**	(c)
17.	(a)	**18.**	(c)	**19.**	(d)	**20.**	(b)	**21.**	(a)	**22.**	(c)	**23.**	(b)	**24.**	(b)
25.	(c)	**26.**	(a)	**27.**	(b)	**28.**	(a)	**29.**	(a)	**30.**	(b)				

Exercise 1(D)

1.	(b)	**2.**	(c)	**3.**	(b)	**4.**	(a)	**5.**	(b)	**6.**	(b)	**7.**	(a)	**8.**	(c)
9.	(c)	**10.**	(d)	**11.**	(c)	**12.**	(c)	**13.**	(a)	**14.**	(c)	**15.**	(c)	**16.**	(a)
17.	(b)	**18.**	(c)	**19.**	(a)	**20.**	(c)	**21.**	(b)	**22.**	(a)	**23.**	(c)	**24.**	(d)
25.	(c)														

ADDITIONAL QUESTION BANK

1. The value of $\left(\frac{6^{-1}7^{2}}{6^{2}7^{-4}}\right)^{7/2} \times \left(\frac{6^{-2}7^{3}}{6^{3}7^{-5}}\right)^{-5/2}$ is

(a) 0 (b) 252 (c) 250 (d) 248

2. The value of $\frac{x^{2/7}}{z^{-1/2}} \times \frac{x^{2/5}}{z^{2/3}} \times \frac{x^{-9/7}}{z^{2/3}} \times \frac{z^{5/6}}{x^{-3/5}}$ is

(a) 1 (b) –1 (c) 0 (d) None

3. On simplification $\frac{2^{x+3} \times 3^{2x-y} \times 5^{x+y+3} \times 6^{y+1}}{6^{x+1} \times 10^{y+3} \times 15^{x}}$ reduces to

(a) –1 (b) 0 (c) 1 (d) 10

4. If $\frac{9^{y}.3^{2}.\left(3^{-y}\right)^{-1} - 27^{y}}{3^{3x}.2^{3}} = \frac{1}{27}$ then x – y is given by

(a) –1 (b) 1 (c) 0 (d) None

5. Show that $\left(x^{\frac{1}{a-b}}\right)^{\frac{1}{a-c}} \times \left(x^{\frac{1}{b-c}}\right)^{\frac{1}{b-a}} \times \left(x^{\frac{1}{c-a}}\right)^{\frac{1}{c-b}}$ is given by

(a) 1 (b) –1 (c) 3 (d) 0

6. Show that $\frac{16(32)^{x} - 2^{3x-2}.4^{x+1}}{15(2)^{x-1}(16)^{x}} - \frac{5(5)^{x-1}}{\sqrt{5^{2x}}}$ is given by

(a) 1 (b) –1 (c) 4 (d) 0

7. Show that $\left(\frac{x^{a}}{x^{b}}\right)^{a+b} \times \left(\frac{x^{b}}{x^{c}}\right)^{b+c} \times \left(\frac{x^{c}}{x^{a}}\right)^{c+a}$ is given by

(a) 0 (b) –1 (c) 3 (d) 1

8. Show that $\sqrt[(a+b)]{\frac{x^{a^2}}{x^{b^2}}} \times \sqrt[(b+c)]{\frac{x^{b^2}}{x^{c^2}}} \times \sqrt[(c+a)]{\frac{x^{c^2}}{x^{a^2}}}$ reduces to

(a) 1 (b) 0 (c) –1 (d) None

9. Show that $\left(x^{\frac{b+c}{c-a}}\right)^{\frac{1}{a-b}} \times \left(x^{\frac{c+a}{a-b}}\right)^{\frac{1}{b-c}} \times \left(x^{\frac{a+b}{b-c}}\right)^{\frac{1}{c-a}}$ reduces to

(a) 1 (b) 3 (c) –1 (d) None

10. Show that $\left(\frac{x^b}{x^c}\right)^a \times \left(\frac{x^c}{x^a}\right)^b \times \left(\frac{x^a}{x^b}\right)^c$ reduces to

(a) 1 (b) 3 (c) 0 (d) 2

11. Show that $\left(\frac{x^b}{x^c}\right)^{1/bc} \times \left(\frac{x^c}{x^a}\right)^{1/ca} \times \left(\frac{x^a}{x^b}\right)^{1/ab}$ reduces to

(a) –1 (b) 0 (c) 1 (d) None

12. Show that $\left(\frac{x^a}{x^b}\right)^{(a^2+ab+b^2)} \times \left(\frac{x^b}{x^c}\right)^{(b^2+bc+c^2)} \times \left(\frac{x^c}{x^a}\right)^{(c^2+ca+a^2)}$ is given by

(a) 1 (b) –1 (c) 0 (d) 3

13. Show that $2^{x+y} = 4 \times 8 \times 16$, then $(x + y)^2$ is equal to

(a) 16 (b) 81 (c) 32 (d) 64

14. Show that $\left(\frac{x^b}{x^c}\right)^{b+c-a} \times \left(\frac{x^c}{x^a}\right)^{c+a-b} \times \left(\frac{x^a}{x^b}\right)^{a+b-c}$ is given by

(a) 1 (b) 0 (c) –1 (d) None

15. Show that $\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$ is reduces to

(a) 1 (b) $x^{-2(a^2+b^2+c^2)}$ (c) $x^{2(a^3+b^3+c^3)}$ (d) $x^{-2(a^3+b^3+c^3)}$

16. $x^{a^2b^{-1}c^{-1}}.x^{b^2c^{-1}a^{-1}}.x^{c^2a^{-1}b^{-1}} - x^3$ would reduce to zero if $a+b+c$ is given by

(a) 1 (b) –1 (c) 0 (d) None

17. The value of z is given by the following if $z^{z\sqrt{z}} = \left(z\sqrt{z}\right)^z$

(a) 2 (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) $\frac{9}{4}$

18. $\frac{1}{x^b+x^{-c}+1} + \frac{1}{x^c+x^{-a}+1} + \frac{1}{x^a+x^{-b}+1}$ would reduce to one if $a+b+c$ is given by

(a) 1 (b) 0 (c) –1 (d) None

19. On simplification $\frac{1}{1+z^{a-b}+z^{a-c}} + \frac{1}{1+z^{b-c}+z^{b-a}} + \frac{1}{1+z^{c-a}+z^{c-b}}$ would reduces to

(a) $\frac{1}{z^{2(a+b+c)}}$ (b) $\frac{1}{z^{(a+b+c)}}$ (c) 1 (d) 0

20. If $(5.678)^x = (0.5678)^y = 10^z$ then

(a) $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 1$ (b) $\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 0$ (c) $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = -1$ (d) None

21. If $x = 4^{1/3} + 4^{-1/3}$ prove that $4x^3 - 12x$ is given by
(a) 12 (b) 13 (c) 15 (d) 17

22. If $x = 5^{1/3} + 5^{-1/3}$ prove that $5x^3 - 15x$ is given by
(a) 25 (b) 26 (c) 27 (d) 30

23. If $ax^{2/3} + bx^{1/3} + c = 0$ then the value of $a^3x^2 + b^3x + c^3$ is given by
(a) $3abcx$ (b) $-3abcx$ (c) $3abc$ (d) $-3abc$

24. If $a^p = b$, $b^q = c$, $c^r = a$ the value of pqr is given by
(a) 0 (b) 1 (c) –1 (d) None

25. If $a^p = b^q = c^r$ and $b^2 = ac$ the value of $q(p+r)/pr$ is given by
(a) 1 (b) –1 (c) 2 (d) None

26. On simplification $\left[\frac{x^{\frac{a}{a-b}}}{x^{\frac{a}{a+b}}} \div \frac{x^{\frac{b}{b-a}}}{x^{\frac{b}{b+a}}}\right]^{a+b}$ reduces to

(a) 1 (b) –1 (c) 0 (d) None

27. On simplification $\left[\frac{x^{ab}}{x^{a^2+b^2}}\right]^{a+b} \times \left[\frac{x^{b^2+c^2}}{x^{bc}}\right]^{b+c} \times \left[\frac{x^{ca}}{x^{c^2+a^2}}\right]^{c+a}$ reduces to

(a) x^{-2a^3} (b) x^{2a^3} (c) $x^{-2(a^3+b^3+c^3)}$ (d) $x^{2(a^3+b^3+c^3)}$

28. On simplification $\left[\frac{x^{ab}}{x^{a^2+b^2}}\right]^{a+b} \times \left[\frac{x^{bc}}{x^{b^2+c^2}}\right]^{b+c} \times \left[\frac{x^{ca}}{x^{c^2+a^2}}\right]^{c+a}$ reduces to

(a) x^{-2a^3} (b) x^{2a^3} (c) $x^{-2(a^3+b^3+c^3)}$ (d) $x^{2(a^3+b^3+c^3)}$

29. On simplification $\left(\frac{m^x}{m^y}\right)^{x+y} \times \left(\frac{m^y}{m^z}\right)^{y+z} \div 3\left(m^x m^z\right)^{x-z} m$ reduces to

(a) 3 (b) –3 (c) $-\frac{1}{3}$ (d) $\frac{1}{3}$

30. The value of $\frac{1}{1+a^{y-x}}+\frac{1}{1+a^{x-y}}$ is given by

(a) –1 (b) 0 (c) 1 (d) None

31. If $xyz=1$ then the value of $\frac{1}{1+x+y^{-1}}+\frac{1}{1+y+z^{-1}}+\frac{1}{1+z+x^{-1}}$ is

(a) 1 (b) 0 (c) 2 (d) None

32. If $2^a=3^b=(12)^c$, then $\frac{1}{c}-\frac{1}{b}-\frac{2}{a}$ reduces to

(a) 1 (b) 0 (c) 2 (d) None

33. If $2^a=3^b=6^{-c}$ then the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ reduce to

(a) 0 (b) 2 (c) 3 (d) 1

34. If $3^a=5^b=(75)^c$, then the value of $ab-c(2a+b)$ reduces to

(a) 1 (b) 0 (c) 3 (d) 5

35. If $2^a=3^b=(12)^c$, then the value of $ab-c(a+2b)$ reduces to

(a) 0 (b) 1 (c) 2 (d) 3

36. If $2^a=4^b=8^c$ and $abc=288$ then the value $\frac{1}{2a}+\frac{1}{4b}+\frac{1}{8c}$ is given by

(a) $\frac{1}{8}$ (b) $-\frac{1}{8}$ (c) $\frac{11}{96}$ (d) $-\frac{11}{96}$

37. If $a^p=b^q=c^r=d^s$ and ab = cd then the value of $\frac{1}{p}+\frac{1}{q}-\frac{1}{r}-\frac{1}{s}$ reduces to

(a) $1/a$ (b) $1/b$ (c) 0 (d) 1

38. If $a^b=b^a$, then the value of $\left(\frac{a}{b}\right)^{\frac{a}{b}}-a^{\frac{a}{b}-1}$ reduces to

(a) a (b) b (c) 0 (d) None

39. If $m=b^x, n=b^y$ and $(m^y n^x)=b^2$ the value of xy is given by

(a) –1 (b) 0 (c) 1 (d) None

40. If $a=xy^{m-1}$, $b=xy^{n-1}$, $c=xy^{p-1}$ then the value of $a^{n-p} \times b^{p-m} \times c^{m-n}$ reduces to
(a) 1 (b) –1 (c) 0 (d) None

41. If $a=x^{n+p}y^{m}$, $b=x^{p+m}y^{n}$, $c=x^{m+n}y^{p}$ then the value of $a^{n-p} \times b^{p-m} \times c^{m-n}$ reduces to
(a) 0 (b) 1 (c) –1 (d) None

42. If $a=\sqrt[3]{\sqrt{2}+1}-\sqrt[3]{\sqrt{2}-1}$ then the value of a^3+3a-2 is
(a) 3 (b) 0 (c) 2 (d) 1

43. If $a = x^{1/3}+x^{-1/3}$ then a^3-3a is
(a) $x + x^{-1}$ (b) $x - x^{-1}$ (c) $2x$ (d) 0

44. If $a = 3^{1/4} + 3^{-1/4}$ and $b = 3^{1/4} - 3^{-1/4}$ then the value of $3\left(a^2+b^2\right)^2$ is
(a) 67 (b) 65 (c) 64 (d) 62

45. If $x = \sqrt{3}+\dfrac{1}{\sqrt{3}}$ and $y = \sqrt{3}-\dfrac{1}{\sqrt{3}}$ then $x^2 - y^2$ is
(a) 5 (b) $\sqrt{3}$ (c) $\dfrac{1}{\sqrt{3}}$ (d) 4

46. If $a = \dfrac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$ then the value of $\dfrac{a+2\sqrt{2}}{a-2\sqrt{2}} + \dfrac{a+2\sqrt{3}}{a-2\sqrt{3}}$ is given by
(a) 1 (b) –1 (c) 2 (d) –2

47. If $P + \sqrt{3}Q + \sqrt{5}R + \sqrt{15}S = \dfrac{1}{1+\sqrt{3}+\sqrt{5}}$ then the value of P is
(a) 7/11 (b) 3/11 (c) -1/11 (d) -2/11

48. If $a = 3 + 2\sqrt{2}$ then the value of $a^{1/2} + a^{-1/2}$ is
(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) $2\sqrt{2}$ (d) $-2\sqrt{2}$

49. If $a = 3+2\sqrt{2}$ then the value of $a^{1/2} - a^{-1/2}$ is
(a) $2\sqrt{2}$ (b) 2 (c) $2\sqrt{2}$ (d) $-2\sqrt{2}$

50. If $a = \dfrac{1}{2}\left(5-\sqrt{21}\right)$ then the value of $a^3 + a^{-3} - 5a^2 - 5a^{-2} + a + a^{-1}$ is
(a) 0 (b) 1 (c) 5 (d) –1

51. If $a = \sqrt{\dfrac{7+4\sqrt{3}}{7-4\sqrt{3}}}$ then the value of $\left[a\left(a-14\right)\right]^2$ is
(a) 14 (b) 7 (c) 2 (d) 1

52. If $a = 3-\sqrt{5}$ then the value of $a^4 - a^3 - 20a^2 - 16a + 24$ is
(a) 10 (b) 14 (c) 0 (d) 15

53. If $a = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ then the value of $2a^4 - 21a^3 + 12a^2 - a + 1$ is

(a) 21 (b) 1 (c) 12 (d) None

54. The square root of $3+\sqrt{5}$ is

(a) $\sqrt{5/2} + \sqrt{1/2}$ (b) $-\left(\sqrt{5/2} + \sqrt{1/2}\right)$ (c) Both the above (d) None

55. If $x = \sqrt{2-\sqrt{2-\sqrt{2}}}$ ...∝ the value of x is given by

(a) –2 (b) 1 (c) 2 (d) 0

56. If $a = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, $b = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ then the value of $a + b$ is

(a) 10 (b) 100 (c) 98 (d) 99

57. If $a = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, $b = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ then the value of $a^2 + b^2$ is

(a) 10 (b) 100 (c) 98 (d) 99

58. If $a = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, $b = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ then the value of $\frac{1}{a^2} + \frac{1}{b^2}$ is

(a) 10 (b) 100 (c) 98 (d) 99

59. The square root of $x + \sqrt{x^2-y^2}$ is given by

(a) $\frac{1}{\sqrt{2}}\left[\sqrt{x+y} + \sqrt{x-y}\right]$ (b) $\frac{1}{2}\left[\sqrt{x+y} - \sqrt{x-y}\right]$ (c) $\left[\sqrt{x+y} + \sqrt{x-y}\right]$ (d) $\left[\sqrt{x+y} - \sqrt{x-y}\right]$

60. The square root of $11-\sqrt{120}$ is given by

(a) $\sqrt{6}+\sqrt{5}$ (b) $\sqrt{6}-\sqrt{5}$ (c) $2\sqrt{3}-3\sqrt{2}$ (d) $2\sqrt{3}+3\sqrt{2}$

61. log (1 + 2 + 3) is exactly equal to

(a) log 1 + log 2 + log 3 (b) log(1×2×3) (c) Both the above (d) None

62. The logarithm of 21952 to the base of $2\sqrt{7}$ and 19683 to the base of $3\sqrt{3}$ are

(a) Equal (b) Not equal (c) Have a difference of 2269 (d) None

63. The value of is $4\log\frac{8}{25} - 3\log\frac{16}{125} - \log 5$ is

(a) 0 (b) 1 (c) 2 (d) –1

64. $a^{\log b - \log c} \times b^{\log c - \log a} \times c^{\log a - \log b}$ has a value of

(a) 1 (b) 0 (c) –1 (d) None

65. $\frac{1}{\log_{ab}(abc)} + \frac{1}{\log_{bc}(abc)} + \frac{1}{\log_{ca}(abc)}$ is equal to

(a) 0 (b) 1 (c) 2 (d) –1

66. $\frac{1}{1+\log_a(bc)} + \frac{1}{1+\log_b(ca)} + \frac{1}{1+\log_c(ab)}$ is equal to

(a) 0 (b) 1 (c) 3 (d) –1

67. $\frac{1}{\log_{a/b}(x)} + \frac{1}{\log_{b/c}(x)} + \frac{1}{\log_{c/a}(x)}$ is equal to

(a) 0 (b) 1 (c) 3 (d) –1

68. $\log_b(a).\log_c(b).\log_a(c)$ is equal to

(a) 0 (b) 1 (c) –1 (d) None

69. $\log_b\left(a^{\frac{1}{2}}\right).\log_c(b^3).\log_a(c^{\frac{2}{3}})$ is equal to

(a) 0 (b) 1 (c) –1 (d) None

70. The value of is $a^{\log b/c}.b^{\log c/a}.c^{\log a/b}$

(a) 0 (b) 1 (c) –1 (d) None

71. The value of $(bc)^{\log b/c}.(ca)^{\log c/a}.(ab)^{\log a/b}$ is

(a) 0 (b) 1 (c) –1 (d) None

72. The value of $\log\frac{a^n}{b^n} + \log\frac{b^n}{c^n} + \log\frac{c^n}{a^n}$ is

(a) 0 (b) 1 (c) –1 (d) None

73. The value of $\log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab}$ is

(a) 0 (b) 1 (c) –1 (d) None

74. $\log(a^9) + \log a = 10$ if the value of *a* is given by

(a) 0 (b) 10 (c) –1 (d) None

75. If $\frac{\log a}{y\text{-}z} = \frac{\log b}{z\text{-}x} = \frac{\log c}{x\text{-}y}$ the value of *abc* is

(a) 0 (b) 1 (c) –1 (d) None

76. If $\frac{\log a}{y-z} = \frac{\log b}{z-x} = \frac{\log c}{x-y}$ the value of $a^{y+z}.b^{z+x}.c^{x+y}$ is given by

(a) 0 (b) 1 (c) –1 (d) None

77. If $\log a = \frac{1}{2}\log b = \frac{1}{5}\log c$ the value of $a^4b^3c^{-2}$ is

(a) 0 (b) 1 (c) –1 (d) None

78. If $\frac{1}{2}\log a = \frac{1}{3}\log b = \frac{1}{5}\log c$ the value of $a^4 - bc$ is

(a) 0 (b) 1 (c) –1 (d) None

79. If $\frac{1}{4}\log_2 a = \frac{1}{6}\log_2 b = -\frac{1}{24}\log_2 c$ the value of a^3b^2c is

(a) 0 (b) 1 (c) –1 (d) None

80. The value of $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)}$ is

(a) 0 (b) 1 (c) –1 (d) None

81. If $\frac{1}{\log_a t} + \frac{1}{\log_b t} + \frac{1}{\log_c t} = \frac{1}{\log_z t}$ then the value if z is given by

(a) abc (b) $a + b + c$ (c) $a(b + c)$ (d) $(a + b)c$

82. If $l = 1+\log_a bc$, $m = 1+\log_b ca$, , $n = 1+\log_c ab$ then the value of $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1$ is

(a) 0 (b) 1 (c) –1 (d) 3

83. If $a = b^2 = c^3 = d^4$ then the value of $\log_a(abcd)$ is

(a) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ (b) $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}$ (c) 1+2+3+4 (d) None

84. The sum of the series $\log_a b + \log_{a^2} b^2 + \log_{a^3} b^3 + \ldots\ldots\log_{a^n} b^n$ is given by

(a) $\log_a b^n$ (b) $\log_{a^n} b$ (c) $\log_{a^n} b^n$ (d) None

85. $a^{\left(\frac{1}{\log_b a}\right)}$ has a value of

(a) a (b) b (c) (a + b) (d) None

86. The value of the following expression $a^{\log_a b.\log_b c.\log_c d.\log_d t}$ is given by

(a) t (b) abcdt (c) (a + b +c + d + t) (d) None

87. For any three consecutive integers x y z the equation $\log(1+xz) - 2\log y = 0$ is

(a) True (b) False (c) Sometimes true

(d) cannot be determined in the cases of variables with cyclic order.

88. If $\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$ then the value of $\frac{a}{b}+\frac{b}{a}$ is

(a) 2 (b) 5 (c) 7 (d) 3

89. If $a^2 + b^2 = 7ab$ then the value of is $\log\frac{a+b}{3} - \frac{\log a}{2} - \frac{\log b}{2}$

(a) 0 (b) 1 (c) –1 (d) 7

90. If $a^3 + b^3 = 0$ then the value of $\log(a+b) - \frac{1}{2}(\log a + \log b + \log 3)$ is equal to

(a) 0 (b) 1 (c) –1 (d) 3

91. If $x = \log_a bc$; $y = \log_b ca$; $z = \log_c ab$ then the value of $xyz - x - y - z$ is

(a) 0 (b) 1 (c) –1 (d) 2

92. On solving the equation $\log t + \log(t-3) = 1$ we get the value of t as

(a) 5 (b) 2 (c) 3 (d) 0

93. On solving the equation $\log_3\left[\log_2\left(\log_3 t\right)\right]=1$ we get the value of t as

(a) 8 (b) 18 (c) 81 (d) 6561

94. On solving the equation $\log_{1/2}\left[\log_t\left(\log_4 32\right)\right]= 2$ we get the value of t as

(a) $\frac{5}{2}$ (b) $\frac{25}{4}$ (c) $\frac{625}{16}$ (d) None

95. If $(4.8)^x = (0.48)^y = 1{,}000$ then the value of $\frac{1}{x}-\frac{1}{y}$ is

(a) 3 (b) –3 (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

96. If $x^{2a-3}y^{2a} = x^{6-a}y^{5a}$ then the value of $a\log\left(\frac{x}{y}\right)$ is

(a) $3\log x$ (b) $\log x$ (c) $6\log x$ (d) $5\log x$

97. If $x = \frac{e^n - e^{-n}}{e^n + e^{-n}}$ then the value of n is

(a) $\frac{1}{2}\log_e\frac{1+x}{1-x}$ (b) $\log_e\frac{1+x}{1-x}$ (c) $\log_e\frac{1-x}{1+x}$ (d) $\frac{1}{2}\log_e\frac{1-x}{1+x}$

98. P and Q started a business investing Rs.85,000 and Rs. 15,000 respectively. In what ratio the profit earned after 2 years be divided between P and Q respectively?

(a) 3:4 (b) 3:5 (c) 15:23 (d) 17:23

99. Anand and Deepak started a business investing Rs. 22,500 and Rs. 35,000 respectively. Out of a total profit of Rs. 13,800, Deepak's share is

(a) Rs. 5,400 (b) Rs. 7,200 (c) Rs. 8,400 (d) Rs. 9,600

100. A, B, C enter into a partnership investing Rs. 35,000, Rs. 45,000 and Rs. 55,000 respectively. The respective shares of A,B,C in an annual profit of Rs. 40,500 are:

(a) Rs. 10,500 Rs. 13,500, Rs. 16,500

(b) Rs. 11,500 Rs. 13,000 Rs. 16,000

(c) Rs. 11,500 Rs. 14,000 Rs. 15,500

(d) Rs. 11,500 Rs. 12,500 Rs. 16,500

101. Reena and Shaloo are partner in a business. Reena invests Rs. 35,000 for 8 months and Shaloo invests Rs. 42,000 for 10 months. Out of a profit of Rs. 31,570, Reena's share is:

(a)Rs. 9471 (b) Rs. 12628 (c) Rs. 18040 (d) Rs. 18942

102. Kanchan started a business investing Rs. 9000. After five months, Sameer joined with a capital of Rs. 8000. If at the end of the year, they earn a profit of Rs. 6970, then what will be the share of Sameer in the profit?

(a) Rs. 1883.78 (b) Rs. 2380 (c) Rs. 3690 (d) Rs. 3864

103. Simran Started a software business by investing Rs.50, 000 . After six months , Nanda joined her with capital of Rs. 80,000. After three years , they earned a profit of Rs.24,500. What was Simran's share in the profit ?

(a) Rs.9423 (b) Rs.10500 (c) Rs.12,500 (d) Rs.14,000

104. A and B started a business in partnership by investing Rs.20, 000 and Rs.15,000 respectively. After six months , C joined them with Rs.20000. What will be B;s share in the total profit of Rs.20,000 earned at the end of 2 tears from the starting of the business ?

(a) Rs. 7500 (b) Rs.9000 (c) Rs.9500 (d) Rs. 10,000

105. Aman started a business investing Rs. 70000 . Rakhi joined him after six months with an amount of Rs. 1,05,000 and Sagar joined them with Rs.1.4 lakhs after another six months . The amount of profit earned should be distributed in what ratio among six months. The amount profit earned should be distributed in what ratio among Aman, Rakhi, and Sagar respectively, 3 years after Aman started the business ?

(a)7:6:10 (b) 12:15: 16 (c) 42:45:56 (d) None of these

106. A began a business with Rs.85,000 and is joined afterwards by B with Rs.42,500. For how much period does B join , if profits at the end of the year are divided in the ratio of 3:1 ?

(a) 4 months (b) 5 months (c) 6 months (d) 8 months

107. A, B and C enter into partnership by investing in the ratio of 3:2:4 . After one year , B invests another Rs.2,70, 000 and C , at the end of 2 years, also invests Rs.2,70, 000. At the end of three years , profits are shared in the ratio of 3:4: 5 . Find the initial investment of each .

(a) 2,70,000 : 1,80,000; 3, 60,000 (b) 2,70,000 : 1,50,000; 3, 60,000

(c) 2,50,000 : 1,80,000; 3, 60,000 (d) 2,70,000 : 1,80,000; 3, 00,000

108. A , B and C enter into partnership . A invests 3 times as much as B invests and B invests 2/3rd of what C invests . At the end of the year , the profit earned is Rs.6600. What is the share of B ?

(a) Rs.1200 (b) Rs.1500 (c) Rs.1800 (d) Rs.2000

109. A and B can do a work in 8 days , B and C can do the same work in 12 days . A, B and C together can finish it in 6 days . A and C together will do it in :

(a) 4 days (b) 6 days (c) 8 days (d) 12 days

ANSWERS

1.	(b)	18.	(b)	35.	(a)	52.	(c)	69.	(b)	86.	(a)	103	(b)
2.	(a)	19.	(c)	36.	(c)	53.	(b)	70.	(b)	87.	(a)	104	(a)
3.	(c)	20.	(b)	37.	(c)	54.	(a)	71.	(b)	88.	(c)	105	(b)
4.	(b)	21.	(d)	38.	(c)	55.	(b)	72.	(a)	89.	(a)	106	(d)
5.	(a)	22.	(b)	39.	(c)	56.	(a)	73.	(a)	90.	(a)	107	(a)
6.	(a)	23.	(a)	40.	(a)	57.	(c)	74.	(b)	91.	(d)	108	(a)
7.	(d)	24.	(b)	41.	(b)	58.	(c)	75.	(b)	92.	(a)	109	(c)
8.	(a)	25.	(c)	42.	(b)	59.	(a)	76.	(b)	93.	(d)		
9.	(a)	26.	(d)	43.	(a)	60.	(b)	77.	(b)	94.	(c)		
10.	(a)	27.	(a)	44.	(c)	61.	(c)	78.	(a)	95.	(c)		
11.	(c)	28.	(c)	45.	(d)	62.	(a)	79.	(b)	96.	(a)		
12.	(a)	29.	(d)	46.	(c)	63.	(a)	80.	(b)	97.	(a)		
13.	(b)	30.	(c)	47.	(a)	64.	(a)	81.	(a)	98	(d)		
14.	(a)	31.	(a)	48.	(c)	65.	(c)	82.	(a)	99	(c)		
15.	(c)	32.	(b)	49.	(b)	66.	(b)	83.	(a)	100	(a)		
16.	(c)	33.	(a)	50.	(a)	67.	(a)	84.	(a)	101	(b)		
17.	(d)	34.	(b)	51.	(d)	68.	(b)	85.	(b)	102	(b)		

NOTES

CHAPTER C

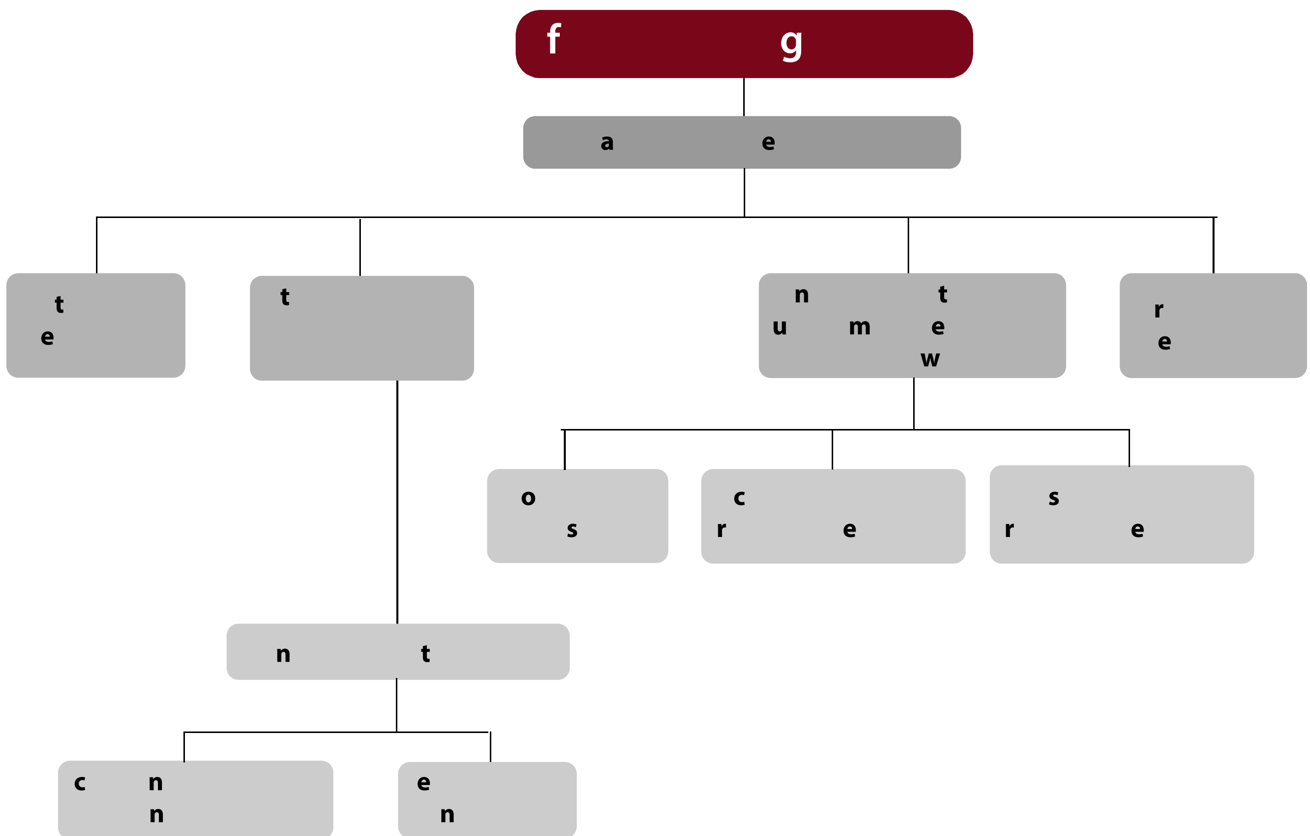
EQUATIONS

LEARNING OBJECTIVES

After studying this chapter, you will be able to:

- Understand the concept of equations and its various degrees – linear, simultaneous, quadratic and cubic equations;
- Know how to solve the different equations using different methods of solution; and

CHAPTER OVERVIEW



2.1 INTRODUCTION

Equation is defined to be a mathematical statement of equality. If the equality is true for certain value of the variable involved, the equation is often called a conditional equation and equality sign '=' is used; while if the equality is true for all values of the variable involved, the equation is called an identity.

For Example: $\frac{x+2}{3}+\frac{x+3}{2}=3$ holds true only for $x=1$.

So it is a conditional. On the other hand, $\frac{x+2}{3}+\frac{x+3}{2}=\frac{5x+13}{6}$

is an identity since it holds for all values of the variable x.

Determination of value of the variable which satisfy an equation is called solution of the equation or root of the equation. An equation in which highest power of the variable is 1 is called a Linear (or a simple) equation. This is also called the equation of degree 1. Two or more linear equations involving two or more variables are called *Simultaneous Linear Equations*. An equation of degree 2 (Highest Power of the variable is 2) is called *Quadratic equation* and the equation of degree 3 is called *Cubic Equation*.

For Example: $8x+17(x-3) = 4(4x-9) + 12$ is a Linear equation.

$3x^2 + 5x + 6 = 0$ is a Quadratic equation.

$4x^3 + 3x^2 + x - 7 = 1$ is a Cubic equation.

$x + 2y = 1$, $2x + 3y = 2$ are jointly called Simultaneous equations.

2.2 SIMPLE EQUATION

A simple equation in one unknown x is in the form $ax + b = 0$.

Where a, b are known constants and a # 0

Note: A simple equation has only one root.

Example: $\frac{4x}{3}-1=\frac{14}{15}x+\frac{19}{5}$.

Solution: By transposing the variables in one side and the constants in other side we have

$$\frac{4x}{3}-\frac{14x}{15}=\frac{19}{5}+1 \text{ or } \frac{(20-14)x}{15}=\frac{19+5}{5} \text{ or } \frac{6x}{15}=\frac{24}{5}.$$

$$x=\frac{24\text{x}15}{5\text{x}6}=12$$

EXERCISE (A)

Choose the most appropriate option (a) (b) (c) or (d).

1. The equation $-7x + 1 = 5 - 3x$ will be satisfied for x equal to:

   a) 2 b) –1 c) 1 d) none of these

2. The root of the equation $\frac{x+4}{4} + \frac{x-5}{3} = 11$ is

   a) 20 b) 10 c) 2 d) none of these

3. Pick up the correct value of x for $\frac{x}{30} = \frac{2}{45}$

   a) $x = 5$ b) $x = 7$ c) $x = 1\frac{1}{3}$ d) none of these

4. The solution of the equation $\frac{x+24}{5} = 4 + \frac{x}{4}$

   a) 6 b) 10 c) 16 d) none of these

5. 8 is the solution of the equation

   a) $\frac{x+4}{4} + \frac{x-5}{3} = 11$ b) $\frac{x+4}{2} + \frac{x+10}{9} = 8$

   c) $\frac{x+24}{5} = 4 + \frac{x}{4}$ d) $\frac{x-15}{10} + \frac{x+5}{5} = 4$

6. The value of y that satisfies the equation $\frac{y+11}{6} - \frac{y+1}{9} = \frac{y+7}{4}$ is

   a) –1 b) 7 c) 1 d) $-\frac{1}{7}$

7. The solution of the equation (p+2) (p–3) + (p+3) (p–4) = p(2p–5) is

   a) 6 b) 7 c) 5 d) none of these

8. The equation $\frac{12x+1}{4} = \frac{15x-1}{5} + \frac{2x-5}{3x-1}$ is true for

   a) $x=1$ b) $x=2$ c) $x=5$ d) $x=7$

9. Pick up the correct value x for which $\frac{x}{0.5} - \frac{1}{0.05} + \frac{x}{0.005} - \frac{1}{0.0005} = 0$

   a) $x=0$ b) $x = 1$ c) $x = 10$ d) none of these

ILLUSTRATIONS:

1. The denominator of a fraction exceeds the numerator by 5 and if 3 be added to both the fraction becomes $\frac{3}{4}$. Find the fraction.

Let x be the numerator and the fraction be $\frac{x}{x+5}$. By the question $\frac{x+3}{x+5+3} = \frac{3}{4}$ or

$4x + 12 = 3x + 24$ or $x = 12$

The required fraction is $\frac{12}{17}$.

2. If thrice of A's age 6 years ago be subtracted from twice his present age, the result would be equal to his present age. Find A's present age.

Let x years be A's present age. By the question

$2x–3(x–6) = x$

or $2x–3x + 18 = x$

or $–x + 18 = x$

or $2x = 18$

or $x=9$

$\therefore$ A's present age is 9 years.

3. A number consists of two digits the digit in the ten's place is twice the digit in the unit's place. If 18 be subtracted from the number the digits are reversed. Find the number.

Let x be the digit in the unit's place. So the digit in the ten's place is $2x$. Thus the number becomes $10(2x) + x$. By the question

$20x + x– 18 = 10x + 2x$

or $21x – 18 = 12x$

or $9x = 18$

or $x = 2$

So the required number is $10 (2 \times 2) + 2 = 42$.

4. For a certain commodity the demand equation giving demand 'd' in kg, for a price 'p' in rupees per kg. is d = 100 (10 – p). The supply equation giving the supply s in kg. for a price p in rupees per kg. is s = 75(p – 3). The market price is such at which demand equals supply. Find the market price and quantity that will be bought and sold.

Given d = 100(10 – p) and s = 75(p – 3).

Since the market price is such that demand (d) = supply (s) we have

100 (10 – p) = 75 (p – 3) or 1000 – 100p = 75p – 225

or – 175p = – 1225. $\therefore p = \frac{-1225}{-175} = 7$.

So market price of the commodity is ₹ 7 per kg.

$\therefore$ the required quantity bought = 100 (10 – 7) = 300 kg.

and the quantity sold = 75 (7 – 3) = 300 kg.

EXERCISE (B)

Choose the most appropriate option (a) (b) (c) or (d).

1. The sum of two numbers is 52 and their difference is 2. The numbers are

   a) 17 and 15 b) 12 and 10 c) 27 and 25 d) none of these

2. The diagonal of a rectangle is 5 cm and one of at sides is 4 cm. Its area is

   a) 20 sq.cm. b) 12 sq.cm. c) 10 sq.cm. d) none of these

3. Divide 56 into two parts such that three times the first part exceeds one third of the second by 48. The parts are.

   a) (20, 36) b) (25, 31) c) (24, 32) d) none of these

4. The sum of the digits of a two digit number is 10. If 18 be subtracted from it the digits in the resulting number will be equal. The number is

   a) 37 b) 73 c) 75 d) none of these numbers.

5. The fourth part of a number exceeds the sixth part by 4. The number is

   a) 84 b) 44 c) 48 d) none of these

6. Ten years ago the age of a father was four times of his son. Ten years hence the age of the father will be twice that of his son. The present ages of the father and the son are.

   a) (50, 20) b) (60, 20) c) (55, 25) d) none of these

7. The product of two numbers is 3200 and the quotient when the larger number is divided by the smaller is 2.The numbers are

   a) (16, 200) b) (160, 20) c) (60, 30) d) (80, 40)

8. The denominator of a fraction exceeds the numerator by 2. If 5 be added to the numerator the fraction increases by unity. The fraction is.

   a) $\frac{5}{7}$ b) $\frac{1}{3}$ c) $\frac{7}{9}$ d) $\frac{3}{5}$

9. Three persons Mr. Roy, Mr. Paul and Mr. Singh together have ₹ 51. Mr. Paul has ₹ 4 less than Mr. Roy and Mr. Singh has got ₹ 5 less than Mr. Roy. They have the money as.

   a) (₹ 20, ₹ 16, ₹ 15) b) (₹ 15, ₹ 20, ₹ 16)
   c) (₹ 25, ₹ 11, ₹ 15) d) none of these

10. A number consists of two digits. The digits in the ten's place is 3 times the digit in the unit's place. If 54 is subtracted from the number the digits are reversed. The number is

    a) 39 b) 92 c) 93 d) 94

11. One student is asked to divide a half of a number by 6 and other half by 4 and then to add the two quantities. Instead of doing so the student divides the given number by 5. If the answer is 4 short of the correct answer then the number was

    a) 320 b) 400 c) 480 d) none of these.

12. If a number of which the half is greater than $\frac{1}{5}$ th of the number by 15 then the number is

a) 50 b) 40 c) 80 d) none of these.

2.3 SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

The general form of a linear equations in two unknowns x and y is $ax + by + c = 0$ where a, b are non-zero coefficients and c is a constant. Two such equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ form a pair of simultaneous equations in x and y. A value for each unknown which satisfies simultaneously both the equations will give the roots of the equations.

2.4 METHOD OF SOLUTION

1. **Elimination Method:** In this method two given linear equations are reduced to a linear equation in one unknown by eliminating one of the unknowns and then solving for the other unknown.

Example 1: Solve: $2x + 5y = 9$ and $3x - y = 5$.

Solution: $2x + 5y = 9$ …….. (i)

$3x - y = 5$ ………(ii)

By making (i) x 1, $2x + 5y = 9$

and by making (ii) x 5, $15x - 5y = 25$

Adding $17x = 34$ or $x = 2$. Substituting this values of x in (i) i.e. $5y = 9 - 2x$ we find;

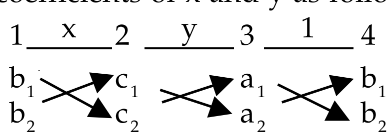
$5y = 9 - 4 = 5 \therefore y = 1 \therefore x = 2, y = 1$.

2. **Cross Multiplication Method:** Let two equations be:

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

We write the coefficients of x, y and constant terms and two more columns by repeating the coefficients of x and y as follows:



and the result is given by: $\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$

so the solution is : $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ $y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$.

Example 2: Solve $3x + 2y + 17 = 0, 5x - 6y - 9 = 0$

Solution: $3x + 2y + 17 = 0$ .... (i)

$5x - 6y - 9 = 0$ ....(ii)

Method of elimination: By (i) x 3 we get $9x + 6y + 51 = 0$ ...... (iii)

Adding (ii) & (iii) we get $14x + 42 = 0$

$$\text{or } x = -\frac{42}{14} = -3$$

Putting $x = -3$ in (i) we get $3(-3) + 2y + 17 = 0$

$$\text{or, } 2y + 8 = 0 \text{ or, } y = -\frac{8}{2} = -4$$

So $x = -3$ and $y = -4$

Method of cross-multiplication: $3x + 2y + 17 = 0$

$5x - 6y - 9 = 0$

$$\frac{x}{2(-9)-17(-6)} = \frac{y}{17\times(5)-3(-9)} = \frac{1}{3(-6)-5\times(2)}$$

$$\text{or, } \frac{x}{84} = \frac{y}{112} = \frac{1}{-28}$$

$$\text{or } \frac{x}{3} = \frac{y}{4} = \frac{1}{-1}$$

or $x = -3, y = -4$

2.5 METHOD OF SOLVING SIMULTANEOUS LINEAR EQUATION WITH THREE VARIABLES

Example 1: Solve for x, y and z:

$2x - y + z = 3, x + 3y - 2z = 11, 3x - 2y + 4z = 1$

Solution: (a) Method of elimination

$2x - y + z = 3$ ....(i)

$x + 3y - 2z = 11$ .... (ii)

$3x - 2y + 4z = 1$ .... (iii)

By (i) × 2 we get

$4x - 2y + 2z = 6$ .... (iv)

By (ii) + (iv), $5x + y = 17$ ....(v) [the variable z is thus eliminated]

By (ii) × 2, $2x + 6y - 4z = 22$ ....(vi)

By (iii) + (vi), $5x + 4y = 23$ ....(vii)

By (v) – (vii), –3y = – 6 or y = 2

Putting y = 2 in (v) $5x + 2 = 17$, or $5x = 15$ or, $x = 3$

Putting $x = 3$ and y = 2 in (i)

2 × 3 – 2 + z = 3

or 6 – 2 + z = 3

or 4 + z = 3

or z = –1

So $x = 3$, y = 2, z = –1 is the required solution.

(Any two of 3 equations can be chosen for elimination of one of the variables)

(b) Method of cross multiplication

We write the equations as follows:

$2x - y + (z - 3) = 0$

$x + 3y + (-2z - 11) = 0$

By cross multiplication

$$\frac{x}{-1(-2z-11)-3(z-3)} = \frac{y}{(z-3)-2(-2z-11)} = \frac{1}{(2\times 3)-(1(-1))}$$

$$\frac{x}{20\text{-}z} = \frac{y}{5z+19} = \frac{1}{7}$$

$$x = \frac{20 - z}{7}, \qquad y = \frac{5z+19}{7}$$

Substituting above values for x and y in equation (iii) i.e. $3x - 2y + yz = 1$, we have

$$3\left(\frac{20-z}{7}\right) - 2\left(\frac{5z+19}{7}\right) + 4z = 1$$

or 60 – 3z – 10z – 38 + 28z = 7

or 15z = 7 – 22 or 15z = –15 or z = –1

$$\text{Now } x = \frac{20\text{-}(\text{-}1)}{7} = \frac{21}{7} = 3, \qquad y = \frac{5(\text{-}1)+19}{7} = \frac{14}{7} = 2$$

Thus $x = 3$, y = 2, z = –1

Example 2: Solve for x, y and z :

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=5, \quad \frac{2}{x}-\frac{3}{y}-\frac{4}{z}=-11, \quad \frac{3}{x}+\frac{2}{y}-\frac{1}{z}=-6$$

Solution: We put u = $\frac{1}{x}$; v = $\frac{1}{y}$; w = $\frac{1}{z}$ and get

u + v + w = 5 .... (i)

2u – 3v – 4w = –11........ (ii)

3u + 2v – w = –6........ (iii)

By (i) + (iii) 4u + 3v = –1 .... (iv)

By (iii) x 4 12u + 8v – 4w = –24 .... (v)

By (ii) – (v) –10u – 11v = 13

or 10u + 11v = –13 .... (vi)

By (iv) × 11 44x + 33v = –11 ....(vii)

By (vi) × 3 30u + 33v = –39 ....(viii)

By (vii) – (viii) 14u = 28 or u = 2

Putting u = 2 in (iv) 4 × 2 + 3v = –1

or 8 + 3v = –1

or 3v = –9 or v = –3

Putting u = 2, v = –3 in (i) or 2–3 + w = 5

or –1 + w = 5 or w = 5 + 1 or w = 6

Thus $x = \frac{1}{u} = \frac{1}{2}$, $y = -\frac{1}{v} = \frac{1}{-3}$, $z = \frac{1}{w} = \frac{1}{6}$ is the solution.

Example 3: Solve for x, y and z:

$$\frac{xy}{x+y} = 70, \quad \frac{xz}{x+z} = 84, \quad \frac{yz}{y+z} = 140$$

Solution: We can write as

$$\frac{x+y}{xy} = \frac{1}{70} \text{ or } \frac{1}{x}+\frac{1}{y}=\frac{1}{70} \quad \text{....... (i)}$$

$$\frac{x+z}{xz} = \frac{1}{84} \text{ or } \frac{1}{z}+\frac{1}{x}=\frac{1}{84} \quad \text{........ (ii)}$$

$$\frac{y+z}{yz} = \frac{1}{140} \text{ or } \frac{1}{y}+\frac{1}{z}=\frac{1}{140} \quad \text{....... (iii)}$$

By (i) + (ii) + (iii), we get $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{70} + \frac{1}{84} + \frac{1}{140} = \frac{14}{420}$

or $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{420}=\frac{1}{60}$ ……(iv)

By (iv)–(iii) $\frac{1}{x}=\frac{1}{60}-\frac{1}{140}=\frac{4}{420}$ or $x = 105$

By (iv)–(ii) $\frac{1}{y}=\frac{1}{60}-\frac{1}{84}=\frac{2}{420}$ or $y = 210$

By (iv)–(i) $\frac{1}{z}=\frac{1}{60}-\frac{1}{70}$ or $z = 420$

Required solution is $x = 105$, $y = 210$, $z = 420$

EXERCISE (C)

Choose the most appropriate option (a), (b), (c) or (d).

1. The solution of the set of equations $3x + 4y = 7$, $4x - y = 3$ is

   a) (1, –1) b) (1, 1) c) (2, 1) d) (1, –2)

2. The values of x and y satisfying the equations $\frac{x}{2}+\frac{y}{3}=2$, $x + 2y = 8$ are given by the pair.

   a) (3, 2) b) (–2, –3) c) (2, 3) d) none of these

3. $\frac{x}{p}+\frac{y}{q}=2$, $x + y = p + q$ are satisfied by the values given by the pair.

   a) (x=p, y=q) b) (x=q, y=p) c) (x=1, y=1) d) none of these

4. The solution for the pair of equations

   $\frac{1}{16x}+\frac{1}{15y}=\frac{9}{20}$, $\frac{1}{20x}-\frac{1}{27y}=\frac{4}{45}$ is given by

   (a) $\left(\frac{1}{4},\frac{1}{3}\right)$ (b) $\left(\frac{1}{3},\frac{1}{4}\right)$ (c) (3 4) (d) (4 3)

5. Solve for x and y: $\frac{4}{x}-\frac{5}{y}=\frac{x+y}{xy}+\frac{3}{10}$ and $3xy = 10(y-x)$.

   a) (5, 2) b) (–2, –5) c) (2, –5) d) (2, 5)

6. The pair satisfying the equations $x + 5y = 36$, $\frac{x+y}{x-y}=\frac{5}{3}$ is given by

   a) (16, 4) b) (4, 16) c) (4, 8) d) none of these.

7. Solve for x and y : $x-3y = 0$, $x+2y = 20$.

   a) $x = 4, y = 12$ b) $x = 12, y = 4$ c) $x = 5, y = 4$ d) none of these

8. The simultaneous equations $7x–3y = 31$, $9x–5y = 41$ have solutions given by

a) (–4, –1) b) (–1, 4) c) (4, –1) d) (3, 7)

9. $1.5x + 2.4y = 1.8$, $2.5(x+1) = 7y$ have solutions as

a) (0.5, 0.4) b) (0.4, 0.5) c) $\left(\frac{1}{2}, \frac{2}{5}\right)$ d) (2, 5)

10. The values of x and y satisfying the equations

$\frac{3}{x+y}+\frac{2}{x-y}=3$, $\frac{2}{x+y}+\frac{3}{x-y}=3\frac{2}{3}$ are given by

a) (1, 2) b) (–1, –2) c) $\left(1, \frac{1}{2}\right)$ d) (2, 1)

EXERCISE (D)

Choose the most appropriate option (a), (b), (c) or (d) as the solution to the given set of equations :

1. $1.5x + 3.6y = 2.1$, $2.5(x+1) = 6y$

a) (0.2, 0.5) b) (0.5, 0.2) c) (2, 5) d) (–2, –5)

2. $\frac{x}{5} + \frac{y}{6} + 1 = \frac{x}{6} + \frac{y}{5} = 28$

a) (6, 9) b) (9, 6) c) (60, 90) d) (90, 60)

3. $\frac{x}{4} = \frac{y}{3} = \frac{z}{2}$; $7x + 8y + 5z = 62$

a) (4, 3, 2) b) (2, 3, 4) c) (3, 4, 2) d) (4, 2, 3)

4. $\frac{xy}{x+y}=20$, $\frac{yz}{y+z}=40$, $\frac{zx}{z+x}=24$

a) (120, 60, 30) b) (60, 30, 120) c) (30, 120, 60) d) (30, 60, 120)

5. $2x + 3y + 4z = 0$, $x + 2y – 5z = 0$, $10x + 16y – 6z = 0$

a) (0,0,0) b) (1, –1, 1) c) (3, 2, –1) d) (1, 0, 2)

6. $\frac{1}{3}(x+y) + 2z = 21$, $3x - \frac{1}{2}(y+z) = 65$, $x + \frac{1}{2}(x+y–z) = 38$

a) (4, 9, 5) b) (2, 9, 5) c) (24, 9, 5) d) (5, 24, 9)

7. $\frac{4}{x}-\frac{5}{y} = \frac{x+y}{xy}+\frac{3}{10}$ $3xy = 10(y–x)$

a) (2, 5) b) (5, 2) c) (2, 7) d) (3, 4)

8. $\frac{x}{0.01}+\frac{y+0.03}{0.05}=\frac{y}{0.02}+\frac{x+0.03}{0.04}=2$

a) (1, 2) b) (0.1, 0.2) c) (0.01, 0.02) d) (0.02, 0.01)

9. $\frac{xy}{y-x}=110, \frac{yz}{z-y}=132, \frac{zx}{z+x}=\frac{60}{11}$

a) (12, 11, 10) b) (10, 11, 12) c) (11, 10, 12) d) (12, 10, 11)

10. $3x-4y+70z = 0$, $2x+3y-10z = 0$, $x+2y+3z = 13$

a) (1, 3, 7) b) (1, 7, 3) c) (2, 4, 3) d) (–10, 10, 1)

2.6 PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS

ILLUSTRATIONS:

1. If the numerator of a fraction is increased by 2 and the denominator by 1 it becomes 1. Again if the numerator is decreased by 4 and the denominator by 2 it becomes 1/2 . Find the fraction.

SOLUTION: Let x/y be the required fraction.

By the question $\frac{x+2}{y+1}=1, \frac{x-4}{y-2}=\frac{1}{2}$

Thus $x + 2 = y + 1$ or $x - y = -1$ .... (i)

and $2x - 8 = y - 2$ or $2x - y = 6$ .... (ii)

By (i) – (ii) $-x = -7$ or $x = 7$

from (i) $7 - y = -1$ or $y = 8$

So the required fraction is 7/8.

2. The age of a man is three times the sum of the ages of his two sons and 5 years hence his age will be double the sum of their ages. Find the present age of the man?

SOLUTION: Let x years be the present age of the man and sum of the present ages of the two sons be y years.

By the condition $x = 3y$ .... (i)

and $x + 5 = 2 (y + 5 + 5)$ ....(ii)

From (i) & (ii) $3y + 5 = 2 (y + 10)$

or $3y + 5 = 2y + 20$

or $3y - 2y = 20 - 5$

or $y = 15$

$\therefore x = 3 \times y = 3 \times 15 = 45$

Hence the present age of the man is 45 years

3. A number consist of three digit of which the middle one is zero and the sum of the other digits is 9. The number formed by interchanging the first and third digits is more than the original number by 297 find the number.

SOLUTION: Let the number be $100x + y$.we have $x + y = 9$……(i)

Also $100y + x = 100x + y + 297$ ……………………………….. (ii)

From (ii) $99(x - y) = -297$

or $x - y = -3$ ……………………………………………………. (iii)

Adding (i) and (iii) $2x = 6$ or $x = 3$ $\therefore$ from (i) $y = 6$

$\therefore$ Hence the number is 306.

EXERCISE (E)

Choose the most appropriate option (a), (b), (c) or (d).

1. Monthly incomes of two persons are in the ratio 4 : 5 and their monthly expenses are in the ratio 7 : 9. If each saves ₹ 50 per month find their monthly incomes.

   a) (500, 400) b) (400, 500) c) (300, 600) d) (350, 550)

2. Find the fraction which is equal to 1/2 when both its numerator and denominator are increased by 2. It is equal to 3/4 when both are increased by 12.

   a) 3/8 b) 5/8 c) 2/8 d) 2/3

3. The age of a person is twice the sum of the ages of his two sons and five years ago his age was thrice the sum of their ages. Find his present age.

   a) 60 years b) 52 years c) 51 years d) 50 years

4. A number between 10 and 100 is five times the sum of its digits. If 9 be added to it the digits are reversed find the number.

   a) 54 b) 53 c) 45 d) 55

5. The wages of 8 men and 6 boys amount to ₹ 33. If 4 men earn ₹ 4.50 more than 5 boys determine the wages of each man and boy.

   a) (₹ 1.50, ₹ 3) b) (₹ 3, ₹ 1.50)

   c) (₹ 2.50, ₹ 2) d) (₹ 2, ₹ 2.50)

6. A number consisting of two digits is four times the sum of its digits and if 27 be added to it the digits are reversed. The number is :

   a) 63 b) 35 c) 36 d) 60

7. Of two numbers, 1/5th of the greater is equal to 1/3rd of the smaller and their sum is 16. The numbers are:

   a) (6, 10) b) (9, 7) c) (12, 4) d) (11, 5)

8. y is older than x by 7 years 15 years back x's age was 3/4 of y's age. Their present ages are:

   a) (x=36, y=43) b) (x=50, y=43)

   c) (x=43, y=50) d) (x=40, y=47)

9. The sum of the digits in a three digit number is 12. If the digits are reversed the number is increased by 495 but reversing only of the ten's and unit digits increases the number by 36. The number is

   a) 327 b) 372 c) 237 d) 273

10. Two numbers are such that twice the greater number exceeds twice the smaller one by 18 and 1/3rd of the smaller and 1/5th of the greater number are together 21. The numbers are:

   a) (36, 45) b) (45, 36) c) (50, 41) d) (55, 46)

11. The demand and supply equations for a certain commodity are 4q + 7p = 17 and $p = \frac{q}{3} + \frac{7}{4}$. respectively where p is the market price and q is the quantity then the equilibrium price and quantity are:

   (a) $2, \frac{3}{4}$ (b) $3, \frac{1}{2}$ (c) $5, \frac{3}{5}$ (d) None of these.

2.7 QUADRATIC EQUATION

An equation of the form $ax^2 + bx + c = 0$ where x is a variable and a, b, c are constants with $a \neq 0$ is called a quadratic equation or equation of the second degree.

When b=0 the equation is called a pure quadratic equation; when $b \neq 0$ the equation is called an affected quadratic.

Examples: i) $2x^2 + 3x + 5 = 0$

ii) $x^2 - x = 0$

iii) $5x^2 - 6x - 3 = 0$

The value of the variable say x is called the root of the equation. A quadratic equation has got two roots.

How to find out the roots of a quadratic equation:

$ax^2 + bx + c = 0 \quad (a \neq 0)$

or $x^2 + \frac{b}{a} x + \frac{c}{a} = 0$

or $x^2 + 2\frac{b}{2a} x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

or $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

$$\text{or } x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and Product of the Roots:

Let one root be α and the other root be β

$$\text{Now } \alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$$\text{Thus sum of roots} = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coeffient of } x^2}$$

$$\text{Next } \alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}$$

$$\text{So the product of the roots} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

2.8 HOW TO CONSTRUCT A QUADRATIC EQUATION

For the equation $ax^2 + bx + c = 0$ we have

$$\text{or } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\text{or } x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

or x^2 – (Sum of the roots) x + Product of the roots = 0

2.9 NATURE OF THE ROOTS

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i) If $b^2 – 4ac = 0$ the roots are real and equal;

ii) If $b^2 – 4ac > 0$ then the roots are real and unequal (or distinct);

iii) If $b^2 – 4ac < 0$ then the roots are imaginary;

iv) If $b^2–4ac$ is a perfect square ($\neq 0$) the roots are real, rational and unequal (distinct);

v) If $b^2–4ac > 0$ but not a perfect square the roots are real, irrational and unequal.

Since $b^2 – 4ac$ discriminates the roots $b^2 – 4ac$ is called the discriminant in the equation $ax^2 + bx + c = 0$ as it actually discriminates between the roots.

Note: (a) Irrational roots occur in conjugate pairs that is if $(m + \sqrt{n})$ is a root then $(m - \sqrt{n})$ is the other root of the same equation.

(b) If one root is reciprocal to the other root then their product is 1 and so $\frac{c}{a} = 1$ i.e. c = a

(c) If one root is equal to other root but opposite in sign then.

their sum = 0 and so $\frac{b}{a} = 0$. i.e. b = 0.

Example 1: Solve $x^2 – 5x + 6 = 0$

Solution: 1st method : $x^2 – 5x + 6 = 0$

or $x^2 –2x –3x +6 = 0$

or $x(x–2) – 3(x–2) = 0$

or $(x–2)(x–3) = 0$

or $x = 2$ or 3

2nd method (By formula) $x^2 – 5x + 6 = 0$

Here a = 1, b = –5 , c = 6 (comparing the equation with $ax^2 + bx + c = 0$)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{25-24}}{2}$$

$$= \frac{5 \pm 1}{2} = \frac{6}{2} \text{ and } \frac{4}{2}; \quad \therefore x = 3 \text{ and } 2$$

Example 2: Examine the nature of the roots of the following equations.

i) $x^2 – 8x + 16 = 0$

ii) $3x^2 – 8x + 4 = 0$

iii) $5x^2 – 4x + 2 = 0$

iv) $2x^2 – 6x – 3 = 0$

Solution: (i) a = 1, b = –8, c = 16

$b^2 – 4ac = (–8)^2 – 4 \times 1 \times 16 = 64 – 64 = 0$

The roots are real and equal.

(ii) $3x^2 – 8x + 4 = 0$

a = 3, b = –8, c = 4

$b^2 - 4ac = (-8)^2 - 4 \times 3 \times 4 = 64 - 48 = 16 > 0$ and a perfect square

The roots are real, rational and unequal

(iii) $5x^2 - 4x + 2 = 0$

$b^2 - 4ac = (-4)^2 - 4 \times 5 \times 2 = 16 - 40 = -24 < 0$

The roots are imaginary and unequal

(iv) $2x^2 - 6x - 3 = 0$

$b^2 - 4ac = (-6)^2 - 4 \times 2\,(-3)$

$= 36 + 24 = 60 > 0$

The roots are real and unequal. Since $b^2 - 4ac$ is not a perfect square the roots are real irrational and unequal.

ILLUSTRATIONS:

1. If α and ß be the roots of $x^2 + 7x + 12 = 0$ find the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.

SOLUTION: Now sum of the roots of the required equation

$= (\alpha+\beta)^2 + (\alpha-\beta)^2 = (-7)^2 + (\alpha+\beta)^2 - 4\alpha\beta$

$= 49 + (-7)^2 - 4x12$

$= 49 + 49 - 48 = 50$

Product of the roots of the required equation $= (\alpha+\beta)^2(\alpha-\beta)^2$

$= 49\,(49-48) = 49$

Hence the required equation is

x^2 – (sum of the roots) x + product of the roots = 0

or $x^2 - 50x + 49 = 0$

2. If α, β be the roots of $2x^2 - 4x - 1 = 0$ find the value of $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

SOLUTION: $\alpha + \beta = \frac{-(-4)}{2} = 2,\ \alpha\beta = \frac{-1}{2}$

$$\therefore \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$\frac{2^3 - 3\left(-\frac{1}{2}\right).2}{\left(-\frac{1}{2}\right)} = -22$$

3. Solve x : $4^x - 3.2^{x+2} + 2^5 = 0$

SOLUTION: $4^x - 3.2^{x+2} + 2^5 = 0$

or $(2^x)^2 - 3.2^x.\ 2^2 + 32 = 0$

or $(2^x)^2 - 12.\ 2^x + 32 = 0$

or $y^2 - 12y + 32 = 0$ (taking $y = 2^x$)

or $y^2 - 8y - 4y + 32 = 0$

or $y(y - 8) - 4(y - 8) = 0$ $\quad \therefore (y - 8)(y - 4) = 0$

either $y - 8 = 0$ or $y - 4 = 0$ $\quad \therefore y = 8$ or $y = 4$.

$\Rightarrow 2^x = 8 = 2^3$ or $2^x = 4 = 2^2$ Therefore $x = 3$ or $x = 2$.

4. Solve $\left(x - \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = 7\frac{1}{4}$.

SOLUTION: $\left(x - \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = 7\frac{1}{4}$.

$$\left(x - \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = \frac{29}{4}.$$

or $$\left(x + \frac{1}{x}\right)^2 - 4 + 2\left(x + \frac{1}{x}\right) = \frac{29}{4}$$

[as $(a - b)^2 = (a + b)^2 - 4ab$]

or $p^2 + 2p - \frac{45}{4} = 0$ $\quad$ Taking $p = x + \frac{1}{x}$

or $4p^2 + 8p - 45 = 0$

or $4p^2 + 18p - 10p - 45 = 0$

or $2p(2p + 9) - 5(2p + 9) = 0$

or $(2p - 5)(2p + 9) = 0$.

$\therefore$ Either $2p + 9 = 0$ or $2p - 5 = 0$ $\quad \Rightarrow p = -\frac{9}{2}$ or $p = \frac{5}{2}$

$\therefore$ Either $x+\frac{1}{x} = -\frac{9}{2}$ or $x+\frac{1}{x} = \frac{5}{2}$

i.e. Either $2x^2 + 9x + 2 = 0$ or $2x^2 - 5x + 2 = 0$

i.e. Either $x = \frac{-9\pm\sqrt{81-16}}{4}$ or, $x = \frac{5\pm\sqrt{25-16}}{4}$

i.e. Either $x = \frac{-9\pm\sqrt{65}}{4}$ or $x = 2$ or $\frac{1}{2}$.

5. Solve $2^{x-2} + 2^{3-x} = 3$

SOLUTION: $2^{x-2} + 2^{3-x} = 3$

or $2^x. 2^{-2} + 2^3. 2^{-x} = 3$

or $\frac{2^x}{2^2}+\frac{2^3}{2^x}=3$

or $\frac{t}{4}+\frac{8}{t}=3$ when $t = 2^x$

or $t^2 + 32 = 12t$

or $t^2 - 12t + 32 = 0$

or $t^2 - 8t - 4t + 32 = 0$

or $t(t-8) - 4(t-8) = 0$

or $(t-4)(t-8) = 0$

$\therefore$ t = 4, 8

For t = 4, $2^x = 4 = 2^2$ i.e. $x = 2$

For t = 8, $2^x = 8 = 2^3$ i.e. $x = 3$

6. If one root of the equation is $2-\sqrt{3}$ form the equation given that the roots are irrational

SOLUTION: Other root is $2+\sqrt{3}$ $\therefore$ sum of two roots $= 2-\sqrt{3} + 2 + \sqrt{3} = 4$

Product of roots $= (2-\sqrt{3})(2+\sqrt{3}) = 4 - 3 = 1$

$\therefore$ Required equation is : $x^2 -$ (sum of roots)x + (product of roots) = 0

or $x^2 - 4x + 1 = 0$.

7. If α β are the two roots of the equation $x^2 - px + q = 0$ form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

SOLUTION: As α, β are the roots of the equation $x^2 - px + q = 0$

$\alpha + \beta = -(-p) = p$ and $\alpha\beta = q$.

Now $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2-2q}{q}$; and $\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1$

∴ Required equation is $x^2 - \left(\frac{p^2-2q}{q}\right)x + 1 = 0$

or $q x^2 - (p^2 - 2q)x + q = 0$

8. If the roots of the equation $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$ are equal show that $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$.

SOLUTION: Since the roots of the given equation are equal the discriminant must be zero ie. $q^2(r - p)^2 - 4.\ p(q - r)\ r(p - q) = 0$

or $q^2 r^2 + q^2 p^2 - 2q^2 rp - 4pr(pq - pr - q^2 + qr) = 0$

or $p^2q^2 + q^2r^2 + 4p^2r^2 + 2q^2pr - 4p^2qr - 4pqr^2 = 0$

or $(pq + qr - 2rp)^2 = 0$

∴ $pq + qr = 2pr$

or $\frac{pq+qr}{2pr} = 1$ or, $\frac{q}{2}\cdot\frac{(p+r)}{pr} = 1$ or, $\frac{1}{r} + \frac{1}{p} = \frac{2}{q}$

EXERCISE (F)

Choose the most appropriate option (a) (b) (c) or (d).

1. If the roots of the equation $2x^2 + 8x - m^3 = 0$ are equal then value of m is

(a) – 3 (b) – 1 (c) 1 (d) – 2

2. If $2^{2x+3} - 3^2.\ 2^x + 1 = 0$ then values of x are

(a) 0, 1 (b) 1, 2 (c) 0, 3 (d) 0, – 3

3. The values of $4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ldots\infty}}}$

(a) $1 \pm \sqrt{2}$ (b) $2 + \sqrt{5}$ (c) $2 \pm \sqrt{5}$ (d) none of these

4. If α, β be the roots of the equation $2x^2 - 4x - 3 = 0$ the value of $\alpha^2 + \beta^2$ is

a) 5 b) 7 c) 3 d) – 4

5. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals then $\frac{b^2}{ac} + \frac{bc}{a^2}$ is equal to

a) 2 b) –2 c) 1 d) –1

6. The equation $x^2 - (p+4)x + 2p + 5 = 0$ has equal roots the values of p will be.

a) ± 1 b) 2 c) ± 2 d) –2

7. The roots of the equation $x^2 + (2p-1)x + p^2 = 0$ are real if.

a) $p \geq 1$ b) $p \leq 4$ c) $p \geq 1/4$ d) $p \leq 1/4$

8. If $x = m$ is one of the solutions of the equation $2x^2 + 5x - m = 0$ the possible values of m are

a) (0, 2) b) (0, –2) c) (0, 1) d) (1, –1)

9. If p and q are the roots of $x^2 + 2x + 1 = 0$ then the values of $p^3 + q^3$ becomes

a) 2 b) –2 c) 4 d) – 4

10. If L + M + N = 0 and L, M, N are rationals the roots of the equation $(M+N-L)x^2+(N+L-M)x+(L+M-N) = 0$ are

a) real and irrational b) real and rational

c) imaginary and equal d) real and equal

11. If α and β are the roots of $x^2 = x + 1$ then value of $\frac{\alpha^2}{\beta} - \frac{\beta^2}{\alpha}$ is

a) $2\sqrt{5}$ b) $\sqrt{5}$ c) $3\sqrt{5}$ d) $-2\sqrt{5}$

12. If $p \neq q$ and $p^2 = 5p - 3$ and $q^2 = 5q - 3$ the equation having roots as $\frac{p}{q}$ and $\frac{q}{p}$ is

a) $x^2 - 19x + 3 = 0$ b) $3x^2 - 19x - 3 = 0$

c) $3x^2 - 19x + 3 = 0$ d) $3x^2 + 19x + 3 = 0$

13. If one root of $5x^2 + 13x + p = 0$ be reciprocal of the other then the value of p is

a) –5 b) 5 c) 1/5 d) –1/5

EXERCISE (G)

Choose the most appropriate option (a) (b) (c) or (d).

1. A solution of the quadratic equation $(a+b-2c)x^2 + (2a-b-c)x + (c+a-2b) = 0$ is

a) $x = 1$ b) $x = -1$ c) $x = 2$ d) $x = -2$

2. If the root of the equation $x^2-8x+m = 0$ exceeds the other by 4 then the value of m is

a) m = 10 b) m = 11 c) m = 9 d) m = 12

3. The values of x in the equation

$7(x+2p)^2 + 5p^2 = 35xp + 117p^2$ are

a) (4p, –3p) b) (4p, 3p) c) (–4p, 3p) d) (–4p, –3p)

4. The solutions of the equation $\frac{6x}{x+1}+\frac{6(x+1)}{x}=13$ are

a) (2, 3) b) (3, –2) c) (–2, –3) d) (2, –3)

5. The satisfying values of x for the equation

$\frac{1}{x+p+q}=\frac{1}{x}+\frac{1}{p}+\frac{1}{q}$ are

a) (p, q) b) (–p, –q) c) (p, –p) d) (–p, q)

6. The values of x for the equation $x^2 + 9x + 18 = 6 - 4x$ are

a) (1, 12) b) (–1, –12) c) (1, –12) d) (–1, 12)

7. The values of x satisfying the equation

$\sqrt{(2x^2+5x-2)}-\sqrt{(2x^2+5x-9)} = 1$ are

a) (2, –9/2) b) (4, –9) c) (2, 9/2) d) (–2, 9/2)

8. The solution of the equation $3x^2-17x + 24 = 0$ are

a) (2, 3) b) $(2, 3\frac{2}{3})$ c) $(3, 2\frac{2}{3})$ d) $(3, \frac{2}{3})$

9. The equation $\frac{3(3x^2+15)}{6} + 2x^2 + 9 = \frac{2x^2+96}{7} + 6$

has got the solution as

a) (1, 1) b) (1/2, –1) c) (1, –1) d) (2, –1)

10. The equation $\left(\frac{l-m}{2}\right)x^2-\left(\frac{l+m}{2}\right)x + m = 0$ has got two values of x to satisfy the equation given as

a) $\left(1,\frac{2m}{l-m}\right)$ b) $\left(1,\frac{m}{l-m}\right)$ c) $\left(1,\frac{2l}{l-m}\right)$ d) $\left(1,\frac{1}{l-m}\right)$

2.10 PROBLEMS ON QUADRATIC EQUATION

1. Difference between a number and its positive square root is 12; find the numbers?

Solution: Let the number be x.

Then $x - \sqrt{x} = 12$ …………… (i)

$(\sqrt{x})^2 - \sqrt{x} - 12 = 0$. Taking y = $\sqrt{x}$, $y^2 - y - 12 = 0$

or (y – 4) (y + 3) = 0 ∴ Either y = 4 or y = – 3 i.e. Either $\sqrt{x} = 4$ or $\sqrt{x} = -3$

If $\sqrt{x} = -3$ $x = 9$ if does not satisfy equation (i) so $\sqrt{x}$ =4 or x=16.

2. A piece of iron rod costs ₹ 60. If the rod was 2 metre shorter and each metre costs ₹ 1.00 more, the cost would remain unchanged. What is the length of the rod?

Solution: Let the length of the rod be x metres. The rate per meter is ₹ $\frac{60}{x}$.

New Length = (x – 2); as the cost remain the same the new rate per meter is $\frac{60}{x-2}$

As given $\frac{60}{x-2} = \frac{60}{x} + 1$

or $\frac{60}{x-2} - \frac{60}{x} = 1$

or $\frac{120}{x(x-2)} = 1$

or $x^2 - 2x = 120$

or $x^2 - 2x - 120 = 0$ or $(x - 12)(x + 10) = 0$.

Either $x = 12$ or $x = -10$ (not possible)

∴ Hence the required length = 12m.

3. Divide 25 into two parts so that sum of their reciprocals is 1/6.

Solution: let the parts be x and $25 - x$

By the question $\frac{1}{x} + \frac{1}{25-x} = \frac{1}{6}$

or $\frac{25-x+x}{x(25-x)} = \frac{1}{6}$

or $150 = 25x - x^2$

or $x^2–25x+150 = 0$

or $x^2–15x–10x+150 = 0$

or $x(x–15) – 10(x–15) = 0$

or $(x–15)(x–10) = 0$

or $x = 10, 15$

So the parts of 25 are 10 and 15.

EXERCISE (H)

Choose the most appropriate option (a) (b) (c) or (d).

1. The sum of two numbers is 8 and the sum of their squares is 34. Taking one number as x form an equation in x and hence find the numbers. The numbers are

   a) (7, 10) b) (4, 4) c) (3, 5) d) (2, 6)

2. The difference of two positive integers is 3 and the sum of their squares is 89. Taking the smaller integer as x form a quadratic equation and solve it to find the integers. The integers are.

   a) (7, 4) b) (5, 8) c) (3, 6) d) (2, 5)

3. Five times of a positive whole number is 3 less than twice the square of the number. The number is

   a) 3 b) 4 c) –3 d) 2

4. The area of a rectangular field is 2000 sq.m and its perimeter is 180m. Form a quadratic equation by taking the length of the field as x and solve it to find the length and breadth of the field. The length and breadth are

   a) (205m, 80m) b) (50m, 40m) c) (60m, 50m) d) none

5. Two squares have sides p cm and (p + 5) cms. The sum of their squares is 625 sq. cm. The sides of the squares are

   a) (10 cm, 30 cm) b) (12 cm, 25 cm)
   c) 15 cm, 20 cm) d) none of these

6. Divide 50 into two parts such that the sum of their reciprocals is 1/12. The numbers are

   a) (24, 26) b) (28, 22) c) (27, 23) d) (20, 30)

7. There are two consecutive numbers such that the difference of their reciprocals is 1/240. The numbers are

   a) (15, 16) b) (17, 18) c) (13, 14) d) (12, 13)

8. The hypotenuse of a right–angled triangle is 20cm. The difference between its other two sides be 4cm. The sides are

   a) (11cm, 15cm) b) (12cm, 16cm) c) (20cm, 24cm) d) none of these

9. The sum of two numbers is 45 and the mean proportional between them is 18. The numbers are

a) (15, 30) b) (32, 13) c) (36, 9) d) (25, 20)

10. The sides of an equilateral triangle are shortened by 12 units 13 units and 14 units respectively and a right angle triangle is formed. The side of the equilateral triangle is

a) 17 units b) 16 units c) 15 units d) 18 units

11. A distributor of apple Juice has 5000 bottle in the store that it wishes to distribute in a month. From experience it is known that demand D (in number of bottles) is given by $D = -2000p^2 + 2000p + 17000$. The price per bottle that will result zero inventory is

a) ₹ 3 b) ₹ 5 c) ₹ 2 d) none of these.

12. The sum of two irrational numbers multiplied by the larger one is 70 and their difference is multiplied by the smaller one is 12; the two numbers are

a) $3\sqrt{2},\ 2\sqrt{3}$ (b) $5\sqrt{2},\ 3\sqrt{5}$ (c) $2\sqrt{2},\ 5\sqrt{2}$ d) none of these.

2.11 SOLUTION OF CUBIC EQUATION

On trial basis putting if some value of x stratifies the equation then we get a factor. This is a trial and error method. With this factor to factorise the LHS and then other get values of x.

ILLUSTRATIONS:

1. Solve $x^3 - 7x + 6 = 0$

Putting $x = 1$ L.H.S is Zero. So $(x-1)$ is a factor of $x^3 - 7x + 6$

We write $x^3 - 7x + 6 = 0$ in such a way that $(x-1)$ becomes its factor. This can be achieved by writing the equation in the following form.

or $x^3-x^2+x^2-x-6x+6 = 0$

or $x^2(x-1) + x(x-1) - 6(x-1) = 0$

or $(x-1)(x^2+x-6) = 0$

or $(x-1)(x^2+3x-2x-6) = 0$

or $(x-1)\{ x(x+3) - 2(x+3) \} = 0$

or $(x-1)(x-2)(x+3) = 0$

$\therefore$ or $x = 1, 2, -3$

2. Solve for real ***x*:** $x^3 + x + 2 = 0$

SOLUTION: By trial we find that $x = -1$ makes the LHS zero. So $(x + 1)$ is a factor of $x^3 + x + 2$

We write $x^3 + x + 2 = 0$ as $x^3 + x^2 - x^2 - x + 2x + 2 = 0$

or $x^2(x + 1) - x(x + 1) + 2(x + 1) = 0$

or $(x + 1)(x^2 - x + 2) = 0$.

Either $x + 1 = 0$; $x = -1$

or $x^2 - x + 2 = 0$ i.e. $x = -1$

i.e. $x = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm \sqrt{-7}}{2}$

As $x = \frac{1 \pm \sqrt{-7}}{2}$ is not real, $x = -1$ is the required solution.

EXERCISE (I)

Choose the most appropriate option (a), (b), (c) or (d)

1. The solution of the cubic equation $x^3–6x^2+11x–6 = 0$ is given by the triplet :

   a) (–1, 1 –2) b) (1, 2, 3) c) (–2, 2, 3) d) (0, 4, –5)

2. The cubic equation $x^3 + 2x^2 - x - 2 = 0$ has 3 roots namely.

   a) (1, –1, 2) b) (–1, 1, –2) c) (–1, 2, –2) d) (1, 2, 2)

3. x, $x - 4$, $x + 5$ are the factors of the left–hand side of the equation.

   a) $x^3 + 2x^2 - x - 2 = 0$ b) $x^3 + x^2 - 20x = 0$

   c) $x^3 - 3x^2 - 4x + 12 = 0$ d) $x^3 - 6x^2 + 11x - 6 = 0$

4. The equation $3x^3 + 5x^2 = 3x + 5$ has got 3 roots and hence the factors of the left–hand side of the equation $3x^3 + 5x^2 - 3x - 5 = 0$ are

   a) $x - 1, x - 2, x - 5/3$ b) $x - 1, x + 1, 3x + 5$ c) $x + 1, x - 1, 3x - 5$ d) $x - 1, x + 1, x - 2$

5. The roots of the equation $x^3 + 7x^2 - 21x - 27 = 0$ are

   a) (– 3, – 9, – 1) b) (3, – 9, – 1) c) (3, 9, 1) d) (– 3, 1, 9)

6. The roots of $x^3 + x^2 - x - 1 = 0$ are

   a) (– 1, – 1, 1) b) (1, 1, – 1) c) (– 1, – 1, – 1) d) (1, 1, 1)

7. The satisfying value of $x^3 + x^2 - 20x = 0$ are

   a) (1, 4, – 5) b) (2, 4, – 5) c) (0, – 4, 5) d) (0, 4, – 5)

8. The roots of the cubic equation $x^3 - 6x^2 + 9x - 4 = 0$ are

   a) (4, 1, –1) b) (–4, –1, –1) c) (–4, –1, 1) d) (1, 1, 4)

9. If $4x^3 + 8x^2 – x – 2 = 0$ then value of $(2x+3)$ is given by

   a) 4, –1, 2 b) –4, 2, 1 c) 2, –4, –1 d) none of these.

10. The rational root of the equation $2x^3 - x^2 - 4x + 2 = 0$ is

    a) $\frac{1}{2}$ b) $-\frac{1}{2}$ c) 2 d) – 2.

SUMMARY

- A simple equation in one unknown x is in the form $ax + b = 0$.

  Where a, b are known constants and $a \neq 0$

- The general form of a linear equations in two unknowns x and y is $ax + by + c = 0$ where a, b are non-zero coefficients and c is a constant. Two such equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ form a pair of simultaneous equations in x and y. A value for each unknown which satisfies simultaneously both the equations will give the roots of the equations.
- **Elimination Method:** In this method two given linear equations are reduced to a linear equation in one unknown by eliminating one of the unknowns and then solving for the other unknown.
- **Cross Multiplication Method:** Let two equations be:

  $a_1x + b_1y + c_1 = 0$

  $a_2x + b_2y + c_2 = 0$

  $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \qquad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

- An equation of the form $ax^2 + bx + c = 0$ where x is a variable and a, b, c are constants with a ¹ 0 is called a quadratic equation or equation of the second degree.

  When b=0 the equation is called a pure quadratic equation; when b= 0 the equation is called an affected quadratic.

- The roots of a quadratic equation:

  $$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The Sum and Product of the Roots of quadratic equation

  $$\text{sum of roots} = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coeffent of } x^2}$$

  $$\text{product of the roots} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

- To construct a quadratic equation for the equation $ax^2 + bx + c = 0$ we have

  x^2 – (Sum of the roots) x + Product of the roots = 0

- Nature of the roots

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i) If $b^2–4ac = 0$ the roots are real and equal;

ii) If $b^2–4ac >0$ then the roots are real and unequal (or distinct);

iii) If $b^2–4ac <0$ then the roots are imaginary;

iv) If $b^2–4ac$ is a perfect square ($\neq$ 0) the roots are real, rational and unequal (distinct);

v) If $b^2–4ac >0$ but not a perfect square the rots are real, irrational and unequal.

Since $b^2 – 4ac$ discriminates the roots $b^2 – 4ac$ is called the discriminant in the equation $ax^2 + bx + c = 0$ as it actually discriminates between the roots.

ANSWERS

Exercise (A)

1. (b) **2.** (a) **3.** (c) **4.** (c) **5.** (b) **6.** (d) **7.** (a) **8.** (d)
9. (c)

Exercise (B)

1. (c) **2.** (b) **3.** (a) **4.** (b) **5.** (c) **6.** (a) **7.** (d) **8.** (d)
9. (a) **10.** (c) **11.** (c) **12.** (a)

Exercise (C)

1. (b) **2.** (c) **3.** (a) **4.** (a) **5.** (d) **6.** (a) **7.** (b) **8.** (c)
9. (b) **10.** (d)

Exercise (D)

1. (a) **2.** (c) **3.** (a) **4.** (d) **5.** (a) **6.** (c) **7.** (a) **8.** (c)
9. (b) **10.** (d)

Exercise (E)

1. (b) **2.** (a) **3.** (d) **4.** (c) **5.** (b) **6.** (c) **7.** (a) **8.** (a)
9. (c) **10.** (b) **11.** (a)

Exercise (F)

1. (d) **2.** (d) **3.** (b) **4.** (b) **5.** (a) **6.** (c) **7.** (d) **8.** (b)
9. (b) **10.** (b) **11.** (d) **12.** (c) **13.** (b)

Exercise (G)

1. (b) **2.** (d) **3.** (a) **4.** (d) **5.** (b) **6.** (b) **7.** (a) **8.** (c)
9. (c) **10.** (a)

Exercise (H)

1.	(c)	**2.**	(b)	**3.**	(a)	**4.**	(b)	**5.**	(c)	**6.**	(d)	**7.**	(a)	**8.**	(b)
9.	(c)	**10.**	(a)	**11.**	(a)	**12.**	(c)								

Exercise (I)

1.	(b)	**2.**	(b)	**3.**	(b)	**4.**	(b)	**5.**	(b)	**6.**	(a)	**7.**	(d)	**8.**	(d)
9.	(a)	**10.**	(a)												

ADDITIONAL QUESTION BANK

1. Solving equation $x^2 - (a+b)\,x + ab = 0$ are, value(s) of x

   (a) a, b (b) a (c) b (d) None

2. Solving equation $x^2 - 24x + 135 = 0$ are, value(s) of x

   (a) 9, 6 (b) 9, 15 (c) 15, 6 (d) None

3. If $\frac{x}{b} + \frac{b}{x} = \frac{a}{b} + \frac{b}{a}$ the roots of the equation are

   (a) $a, b^2/a$ (b) $a^2, b/a^2$ (c) $a^2, b^2/a$ (d) a, b^2

4. Solving equation $\frac{6x+2}{4} + \frac{2x^2-1}{2x^2+2} = \frac{10x-1}{4x}$ we get roots as

   (a) ±1 (b) +1 (c) -1 (d) 0

5. Solving equation $3x^2 - 14x + 16 = 0$ we get roots as

   (a) ±1 (b) 2 and $\frac{8}{3}$ (c) 0 (d) None

6. Solving equation $3x^2 - 14x + 8 = 0$ we get roots as

   (a) ±4 (b) ±2 (c) $4, \frac{2}{3}$ (d) None

7. Solving equation $(b-c)\,x^2 + (c-a)\,x + (a-b) = 0$ following roots are obtained

   (a) $\frac{b-a}{b-c}, 1$ (b) $(a-b)(a-c), 1$ (c) $\frac{b-c}{a-b}, 1$ (d) None

8. Solving equation $7\sqrt{\frac{x}{1-x}} + 8\sqrt{\frac{1-x}{x}} = 15$ following roots are obtained

   (a) $\frac{64}{113}, \frac{1}{2}$ (b) $\frac{1}{50}, \frac{1}{65}$ (c) $\frac{49}{50}, \frac{1}{65}$ (d) $\frac{1}{50}, \frac{64}{65}$

9. Solving equation $6\left[\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}\right]=13$ following roots are obtained

(a) $\frac{4}{13},\frac{9}{13}$ (b) $\frac{-4}{13},\frac{-9}{13}$ (c) $\frac{4}{13},\frac{5}{13}$ (d) $\frac{6}{13},\frac{7}{13}$

10. Solving equation $z^2-6z+9=4\sqrt{z^2-6z+6}$ following roots are obtained

(a) $3+2\sqrt{3}, 3-2\sqrt{3}$ (b) 5, 1 (c) all the above (d) None

11. Solving equation $\frac{x+\sqrt{12p-x}}{x-\sqrt{12p-x}}=\frac{\sqrt{p}+1}{\sqrt{p}-1}$ following roots are obtained

(a) $3p$ (b) both $3p$ and $-4p$ (c) only $-4p$ (d) $-3p$, $4p$

12. Solving equation $(1+x)^{2/3}+(1-x)^{2/3}=4(1-x^2)^{1/3}$ are, values of x

(a) $\frac{5}{\sqrt{3}}$ (b) $-\frac{5}{\sqrt{3}}$ (c) $\pm\frac{5}{3\sqrt{3}}$ (d) $\pm\frac{15}{\sqrt{3}}$

13. Solving equation $(2x+1)(2x+3)(x-1)(x-2)=150$ the roots available are

(a) $\frac{1\pm\sqrt{129}}{4}$ (b) $\frac{7}{2}, -3$ (c) $-\frac{7}{2}, 3$ (d) None

14. Solving equation $(2x+3)(2x+5)(x-1)(x-2)=30$ the roots available are

(a) $0,\frac{1}{2},-\frac{11}{4},\frac{9}{4}$ (b) $0,-\frac{1}{2},\frac{-1\pm\sqrt{105}}{4}$ (c) $0,-\frac{1}{2},-\frac{11}{4},-\frac{9}{4}$ (d) None

15. Solving equation $z+\sqrt{z}=\frac{6}{25}$ the value of z works out to

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{25}$ (d) $\frac{2}{25}$

16. Solving equation $z^{10}-33z^5+32=0$ the following values of z are obtained

(a) 1, 2 (b) 2, 3 (c) 2, 4 (d) 1, 2, 3

17. When $\sqrt{2z+1}+\sqrt{3z+4}=7$ the value of z is given by

(a) 1 (b) 2 (c) 3 (d) 4

18. Solving equation $\sqrt{x^2-9x+18}+\sqrt{x^2+2x-15}=\sqrt{x^2-4x+3}$ following roots are obtained

(a) 3, $\frac{2\pm\sqrt{91}}{3}$ (b) $\frac{2\pm\sqrt{94}}{3}$ (c) 4, $-\frac{8}{3}$ (d) 3, $4-\frac{8}{3}$

19. Solving equation $\sqrt{y^2+4y-21}+\sqrt{y^2-y-6}=\sqrt{6y^2-5y-39}$ following roots are obtained

(a) 2, 3, 5/3 (b) 2, 3, -5/3 (c) -2, -3, 5/3 (d) -2, -3, -5/3

20. Solving equation $6x^4+11x^3-9x^2-11x+6=0$ following roots are obtained

(a) $\frac{1}{2},-2,\frac{-1\pm\sqrt{37}}{6}$ (b) $-\frac{1}{2},2,\frac{-1\pm\sqrt{37}}{6}$ (c) $\frac{1}{2},-2,\frac{5}{6},\frac{-7}{6}$ (d) None

21. If $\frac{x-bc}{d+c}+\frac{x-ca}{c+a}+\frac{x-ab}{a+b}=a+b+c$ the value of x is

(a) $a^2+b^2+c^2$ (b) $a(a+b+c)$ (c) $(a+b)(b+c)$ (d) $ab+bc+ca$

22. If $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{x-1}{x+3}-\frac{x+3}{x-3}$ then the values of x are

(a) $0,\pm\sqrt{6}$ (b) $0,\pm\sqrt{3}$ (c) $0,\pm2\sqrt{3}$ (d) None

23. If $\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}$ then the values of x are

(a) $0,(a+b),(a-b)$ (b) $0,(a+b),\frac{a^2+b^2}{a+b}$ (c) $0,(a-b),\frac{a^2+b^2}{a+b}$ (d) $\frac{a^2+b^2}{a+b}$

24. If $\frac{x-a^2-b^2}{c^2}+\frac{c^2}{x-a^2-b^2}=2$ the value of is

(a) $a^2+b^2+c^2$ (b) $-a^2-b^2-c^2$ (c) $\frac{1}{a^2+b^2+c^2}$ (d) $-\frac{1}{a^2+b^2+c^2}$

25. Solving equation $\left(x-\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)+12=0$ we get roots as follows

(a) 0 (b) 1 (c) -1 (d) None

26. Solving equation $\left(x-\frac{1}{x}\right)^2-10\left(x-\frac{1}{x}\right)+24=0$ we get roots as follows

(a) 0 (b) 1 (c) -1 (d) $(2\pm\sqrt{5}),(3\pm\sqrt{10})$

27. Solving equation $2\left(x-\frac{1}{x}\right)^2-5\left(x+\frac{1}{x}+2\right)+18=0$ we get roots as under

(a) 0 (b) 1 (c) -1 (d) 2,1/2

28. If α, β are the roots of equation x^2-5x+6=0 and α > β thenthe equation with roots (α + β) and (α - β) is

(a) x^2-6x+5=0 (b) 2x^2-6x+5=0 (c) 2x^2-5x+6=0 (d) x^2-5x+6=0

29. If α, β are the roots of equation x^2-5x+6=0 and α > β then the equation with roots (α^2 + β) and (α + β^2) is

(a) x^2-9x+99=0 (b) x^2-18x+90=0 (c) x^2-18x+77=0 (d) None

30. If α, β are the roots of equation x^2-5x+6=0 and α > β then the equation with roots (αβ+α+β) and (αβ-α-β) is

(a) x^2-12x+11=0 (b) 2x^2-6x+12=0 (c) x^2-12x+12=0 (d) None

31. The condition that one of ax^2+bx+c=0 the roots of is twice the other is

(a) b^2=4ca (b) 2b^2=9(c+a) (c) 2b^2=9ca (d) 2b^2=9(c-a)

32. The condition that one of ax^2+bx+c=0 the roots of is thrice the other is

(a) 3b^2=16ca (b) b^2=9ca (c) 3b^2=-16ca (d) b^2=-9ca

33. If the roots of ax^2+bx+c=0 are in the ratio $\frac{p}{q}$ then the value of $\frac{b^2}{(ca)}$ is

(a) $\frac{(p+q)^2}{(pq)}$ (b) $\frac{(p+q)}{(pq)}$ (c) $\frac{(p-q)^2}{(pq)}$ (d) $\frac{(p-q)}{(pq)}$

34. Solving 6x+5y-16=0 and 3x-y-1=0 we get values of x and y as

(a) 1, 1 (b) 1, 2 (c) -1, 2 (d) 0, 2

35. Solving x^2+y^2-25=0 and x-y-1=0 we get the roots as under

(a) ±3 ±4 (b) ±2 ±3 (c) 0, 3, 4 (d) 0, -3, -4

36. Solving $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}-\frac{5}{2}=0$ and x+y-5=0 we get the roots as under

(a) 1, 4 (b) 1, 2 (c) 1, 3 (d) 1, 5

37. Solving $\frac{1}{x^2}+\frac{1}{y^2}-13=0$ and $\frac{1}{x}+\frac{1}{y}-5=0$ we get the roots as under

(a) $\frac{1}{8},\frac{1}{5}$ (b) $\frac{1}{2},\frac{1}{3}$ (c) $\frac{1}{13},\frac{1}{5}$ (d) $\frac{1}{4},\frac{1}{5}$

38. Solving $x^2+xy-21=0$ and $xy-2y^2+20=0$ we get the roots as under

(a) ±1, ±2 (b) ±2, ±3 (c) ±3, ±4 (d) None

39. Solving $x^2+xy+y^2=37$ and $3xy+2y^2=68$ we get the following roots

(a) ±3, ±4 (b) ±4, ±5 (c) ±2, ±3 (d) None

40. Solving $4^x.2^y=128$ and $3^{3x+2y}=9^{xy}$ we get the following roots

(a) $\frac{7}{4},\frac{7}{2}$ (b) 2, 3 (c) 1, 2 (d) 1, 3

41. Solving $9^x=3^y$ and $5^{x+y+1}=25^{xy}$ we get the following roots

(a) 1, 2 (b) 0, 1 (c) 0, 3 (d) 1, 3

42. Solving $9x+3y-4z=3$, $x+y-z=0$ and $2x-5y-4z=-20$ following roots are obtained

(a) 2, 3, 4 (b) 1, 3, 4 (c) 1, 2, 3 (d) None

43. Solving $x+2y+2z=0$, $3x-4y+z=0$ and $x^2+3y^2+z^2=11$ following roots are obtained

(a) 2, 1, -2 and -2, -1, 2 (b) 2, 1, 2 and -2, -1, -2

(c) only 2, 1, -2 (d) only -2, -1, 2

44. Solving $x^3-6x^2+11x-6=0$ we get the following roots

(a) -1, -2, 3 (b) 1, 2, -3 (c) 1, 2, 3 (d) -1, -2, -3

45. Solving $x^3+9x^2-x-9=0$ we get the following roots

(a) ±1, -9 (b) ±1, ±9 (c) ±1, 9 (d) None

46. It is being given that one of the roots is half the sum of the other two solving $x^3-12x^2+47x-60=0$ we get the following roots:

(a) 1, 2, 3 (b) 3, 4, 5 (c) 2, 3, 4 (d) -3, -4, -5

47. Solve $x^3+3x^2-x-3=0$ given that the roots are in arithmetical progression

(a) -1, 1, 3 (b) 1, 2, 3 (c) -3, -1, 1 (d) -3, -2, -1

48. Solve $x^3-7x^2+14x-8=0$ given that the roots are in geometrical progression

(a) ½, 1, 2 (b) 1, 2, 4 (c) ½ , -1, 2 (d) -1, 2, -4

49. Solve $x^3-6x^2+5x+12=0$ given that the product of the two roots is 12

(a) 1, 3, 4 (b) -1, 3, 4 (c) 1, 6, 2 (d) 1, -6, -2

50. Solve $x^3-5x^2-2x+24=0$ given that two of its roots being in the ratio of 3:4

(a) -2, 4, 3 (b) -1, 4, 3 (c) 2, 4, 3 (d) -2, -4, -3

ANSWERS

1.	(a)	**18.**	(a)	**35.**	(a)
2.	(b)	**19.**	(b)	**36.**	(a)
3.	(a)	**20.**	(a)	**37.**	(b)
4.	(b)	**21.**	(d)	**38.**	(c)
5.	(b)	**22.**	(d)	**39.**	(a)
6.	(c)	**23.**	(b)	**40.**	(a), (b)
7.	(a)	**24.**	(a)	**41.**	(a)
8.	(a)	**25.**	(b)	**42.**	(c)
9.	(a)	**26.**	(d)	**43.**	(a)
10.	(c)	**27.**	(d)	**44.**	(c)
11.	(a)	**28.**	(a)	**45.**	(a)
12.	(c)	**29.**	(c)	**46.**	(b)
13.	(a)	**30.**	(a)	**47.**	(c)
14.	(b)	**31.**	(c)	**48.**	(b)
15.	(c)	**32.**	(a)	**49.**	(b)
16.	(a)	**33.**	(a)	**50.**	(a)
17.	(d)	**34.**	(b)		

NOTES

CHAPTER D

LINEAR INEQUALITIES

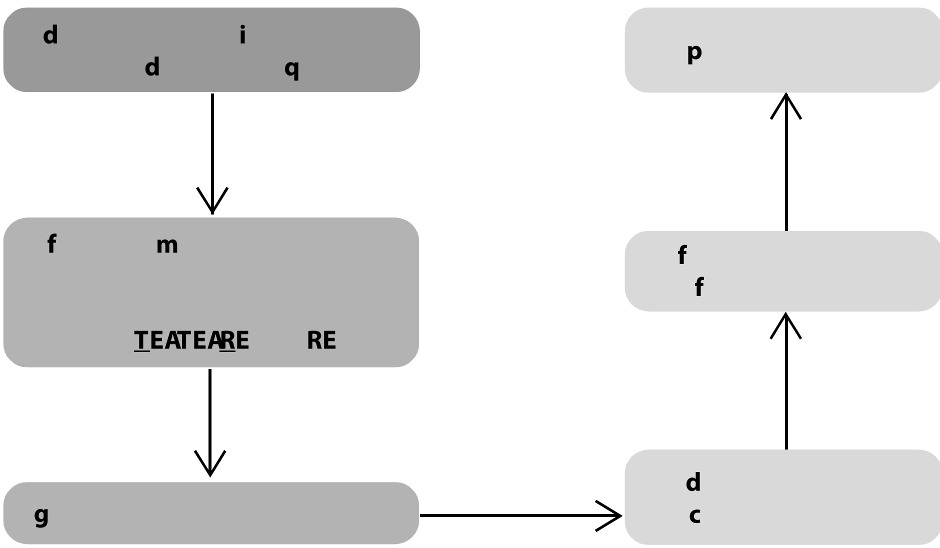
LEARNING OBJECTIVES

One of the widely used decision making problems, nowadays, is to decide on the optimal mix of scarce resources in meeting the desired goal. In simplest form, it uses several linear inequations in two variables derived from the description of the problem.

The objective in this section is to make a foundation of the working methodology for the above by way of introduction of the idea of :

- development of inequations from the descriptive problem;
- graphing of linear inequations; and
- determination of common region satisfying the inequations.

CHAPTER OVERVIEW



3.1 INEQUALITIES

Inequalities are statements where two quantities are unequal but a relationship exists between them. These type of inequalities occur in business whenever there is a limit on supply, demand, sales etc. For example, if a producer requires a certain type of raw material for his factory and there is an upper limit in the availability of that raw material, then any decision which he takes about production should involve this constraint also. We will see in this chapter more about such situations.

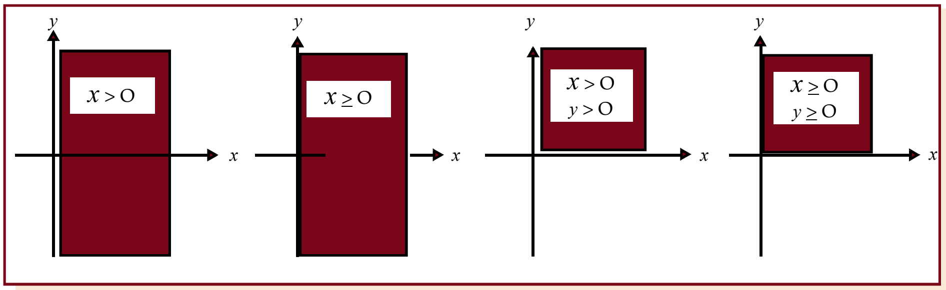
3.2 LINEAR INEQUALITIES IN ONE VARIABLE AND THE SOLUTION SPACE

Any linear function that involves an inequality sign is a linear inequality. It may be of one variable, or, of more than one variable. Simple example of linear inequalities are those of one variable only; viz., $x > 0$, $x \leq 0$ etc.

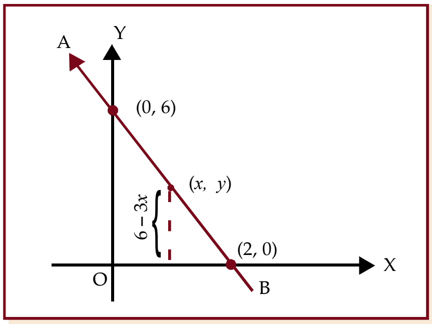
The values of the variables that satisfy an inequality are called the *solution space*, and is abbreviated as S.S. The solution spaces for (i) $x > 0$, (ii) $x \leq 0$ are shaded in the above diagrams, by using deep lines.

Linear inequalities in two variables: Now we turn to linear inequalities in two variables x and y and shade a few S.S.

Let us now consider a linear inequality in two variables given by $3x + y < 6$



The inequality mentioned above is true for certain pairs of numbers (x, y) that satisfy $3x + y < 6$. By trial, we may arbitrarily find such a pair to be (1,1) because $3 \times 1 + 1 = 4$, and $4 < 6$.



Linear inequalities in two variables may be solved easily by extending our knowledge of straight lines.

For this purpose, we replace the inequality by an equality and seek the pairs of number that satisfy $3x + y = 6$. We may write $3x + y = 6$ as $y = 6 - 3x$, and draw the graph of this linear function.

Let $x = 0$ so that $y = 6$. Let $y = 0$, so that $x = 2$.

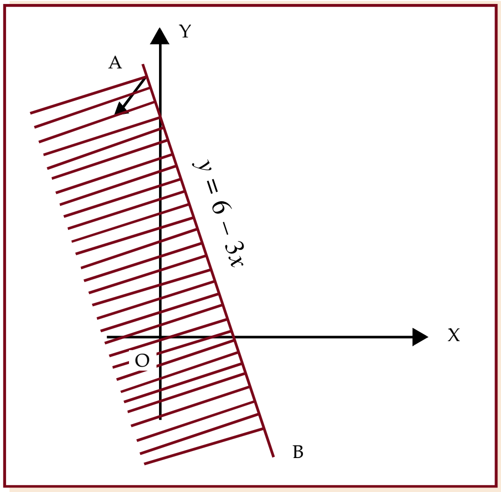
Any pair of numbers (x, y) that satisfies the equation $y = 6 - 3x$ falls on the line AB.

Note: The pair of inequalities $x \geq 0, y \geq 0$ play an important role in linear programming problems.

Therefore, if y is to be less than $6 - 3x$ for the same value of x, it must assume a value that is less than the ordinate of length $6 - 3x$.

All such points (x, y) for which the ordinate is less than $6 - 3x$ lie below the line AB.

The region where these points fall is indicated by an arrow and is shaded too in the adjoining diagram. Now we consider two inequalities $3x + y \leq 6$ and $x - y \leq -2$ being satisfied simultaneously by x and y. The pairs of numbers (x, y) that satisfy both the inequalities may be found by drawing the graphs of the two lines $y = 6 - 3x$ and $y = 2 + x$, and determining the region where both the inequalities hold. It is convenient to express each equality with y on the left-side and the remaining terms in the right side. The first inequality $3x + y \leq 6$ is equivalent to $y \leq 6 - 3x$ and it requires the value of y for each x to be less than or equal to that of and on $6 - 3x$. The inequality is therefore satisfied by all points lying below the line $y = 6 - 3x$. The region where these points fall has been shaded in the adjoining diagram.

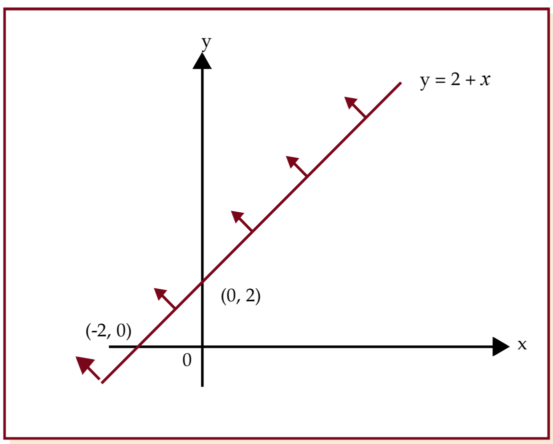


We consider the second inequality $x - y \leq -2$, and note that this is equivalent to $y \geq 2 + x$. It requires the value of y for each x to be larger than or equal to that of $2 + x$. The inequality is, therefore, satisfied by all points lying on and above the line $y = 2 + x$.

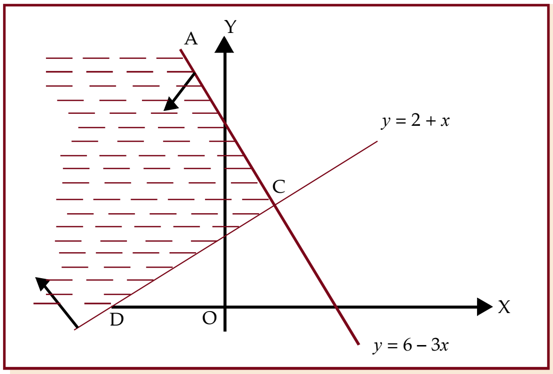
The region of interest is indicated by an arrow on the line $y = 2 + x$ in the diagram below.

For $x = 0$, $y = 2 + 0 = 2$;

For $y = 0$, $0 = 2 + x$ i.e, $x = -2$.



By superimposing the above two graphs we determine the common region ACD in which the pairs (x, y) satisfy both inequalities.



Example 1: We now consider the problem of drawing graphs of the following inequalities

$$x \geq 0,\ y \geq 0,\ x \leq 6,\ y \leq 7,\ \ x + y \leq 12$$

and shading the common region.

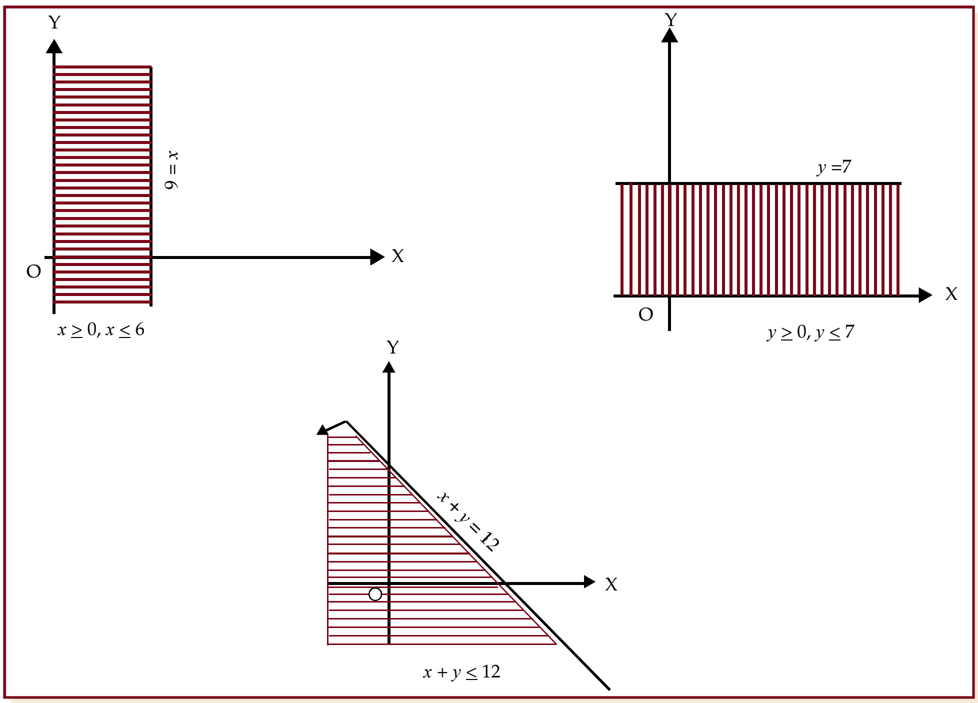
Note: [1] The inequalities $3x + y \leq 6$ and $x - y \leq -2$ differ from the preceding ones in that these also include equality signs. It means that the points lying on the corresponding lines are also included in the region.

[2] The procedure may be extended to any number of inequalities.

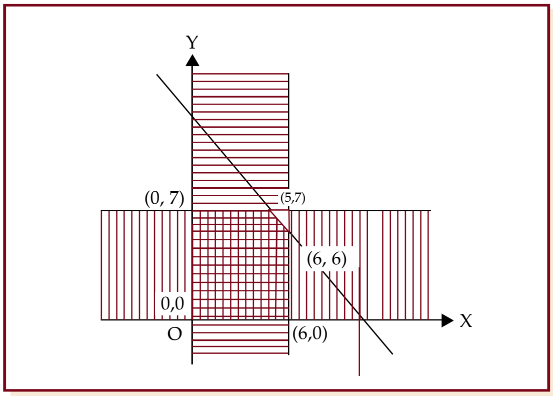
We note that the given inequalities may be grouped as follows :

$x \geq 0, \quad y \geq 0$

$x \leq 6, \quad y \leq 7, \quad x + y \leq 12$



By superimposing the above three graphs, we determine the common region in the *xy* plane where all the five inequalities are simultaneously satisfied.



This common region is known as feasible region or the solution set (or the polygonal convex sets).

A region is said to be *bounded* if it can be totally included within a (very large) circle. The shaded region enclosed by deep lines in the previous diagram is bounded, since it can be included within a circle.

The objective function attains a maximum or a minimum value at one of the corner points of the feasible solution known as extreme points of the solution set. Once these extreme points (the points of intersection of lines bounding the region) are known, a compact matrix representation of these points is possible. We shall denote the matrix of the extreme points by E.

The coefficients of the objective function may also be represented by a column vector. We shall represent this column vector by C.

The elements in the product matrix EC shows different values, which the objective function attains at the various extreme points. The largest and the smallest elements in matrix EC are respectively the maximum and the minimum values of the objective function. The row in matrix EC in which this happens is noted and the elements in that row indicate the appropriate pairing and is known as the *optimal solution.*

In the context of the problem under consideration.

$$\begin{matrix} & X \quad Y \\ E = & \begin{bmatrix} 0 & 0 \\ 0 & 7 \\ 5 & 7 \\ 6 & 0 \\ 6 & 6 \end{bmatrix} \end{matrix}, C = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} x \\ y \end{matrix}$$

$$EC = \begin{bmatrix} 0 & 0 \\ 0 & 7 \\ 5 & 7 \\ 6 & 0 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0\times1+0\times2 \\ 0\times1+7\times2 \\ 5\times1+7\times2 \\ 6\times1+0\times2 \\ 6\times1+6\times2 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 19 \\ 6 \\ 18 \end{bmatrix}$$

The given objective function viz. Z = x + 2y is maximum at the points (5, 7) present in the third row of the matrix E. Thus the optimal solution is $x = 5$, $y = 7$, and the maximum value of the objective function is 19.

We now list the steps to be followed under graphical solution to a linear programming problem.

Step 1 Determine the region that satisfies the set of given inequalities.

Step 2 Ensure that the region is bounded*. If the region is not bounded, either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.

Step 3 Construct the matrix E of the extreme points, and the column vector C of the objective function.

Step 4 Find the matrix product EC. For maximization, determine the row in EC where the largest element appears; while for minimization, determine the row in EC where the smallest element appears.

Step 5 The objective function is optimized corresponding to the same row elements of the extreme point matrix E.

Note: If the slope of the objective function be same as that of one side of feasible region, there are multiple solutions to the problem. However, the optimized value of the objective function remains the same.

Example 2:

A manufacturer produces two products A and B, and has his machines in operation for 24 hours a day. Production of A requires 2 hours of processing in machine M_1 and 6 hours in machine M_2. Production of B requires 6 hours of processing in machine M_1 and 2 hours in machine M_2. The manufacturer earns a profit of ₹ 5 on each unit of A and ₹ 2 on each unit of B. How many units of each product should be produced in a day in order to achieve maximum profit?

Solution:

Let x_1 be the number of units of type A product to be produced, and x_2 is that of type B product to be produced. The formulation of the L.P.P. in this case is as below:

Maximize $Z = 5x_1 + 2x_2$

* It is inconceivable for a practical problem to have an unbounded solution.

subject to the constraints.

$2x_1 + 6x_2 < 24$

$6x_1 + 2x_2 < 24$

$x_1 \geq 0, x_2 \geq 0$

For the line $2x_1 + 6x_2 = 24$

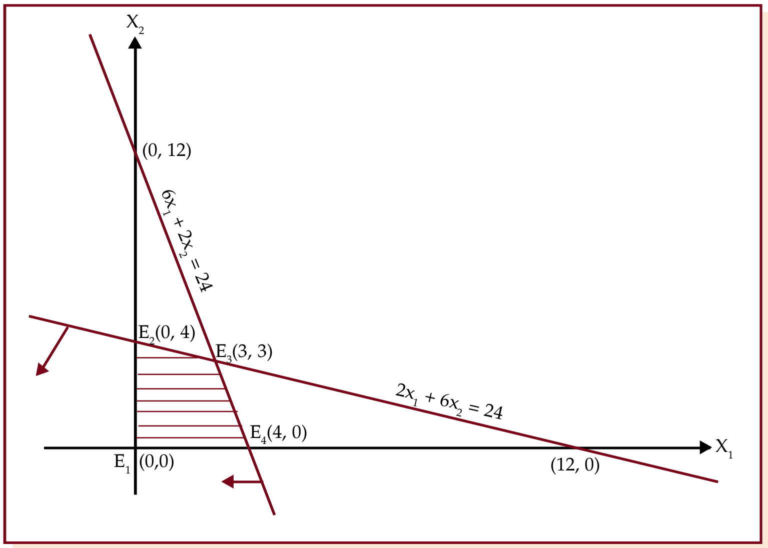
Let $x_1 = 0$, so that $x_2 = 4$

Let $x_2 = 0$, so that $x_1 = 12$

For the line $6x_1 + 2x_2 = 24$

Let $x_1 = 0$, so that $x_2 = 12$

Let $x_2 = 0$, so that $x_1 = 4$



The shaded portion in the diagram is the feasible region and the matrix of the extreme points E_1, E_2, E_3 and E_4 is

$$E = \begin{array}{c} X_1 \quad Y_2 \\ \begin{bmatrix} 0 & 0 \\ 0 & 4 \\ 3 & 3 \\ 4 & 0 \end{bmatrix} \begin{array}{l} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \end{array}$$

The column vector for the objective function is $C = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

The column vector the values of the objective function is given by

$$EC = \begin{bmatrix} 0 & 0 \\ 0 & 4 \\ 3 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0\times5+0\times2 \\ 0\times5+4\times2 \\ 3\times5+3\times2 \\ 4\times5+0\times2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 21 \\ 20 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}$$

Since 21 is the largest element in matrix EC, therefore the maximum value is reached at the extreme point E_3 whose coordinates are (3,3).

Thus, to achieve maximum profit the manufacturer should produce 3 units each of both the products A and B.

Summary of Graphical Method

It involves:

(i) Formulating the linear programming problem, i.e. expressing the objective function and constraints in the standardised format.

(ii) Plotting the capacity constraints on the graph paper. For this purpose normally two terminal points are required. This is done by presuming simultaneously that one of the constraints is zero. When constraints concerns only one factor, then line will have only one origin point and it will run parallel to the other axis.

(iii) Identifying feasible region and coordinates of corner points. Mostly it is done by breading the graph, but a point can be identified by solving simultaneous equation relating to two lines which intersect to form a point on graph.

(iv) Testing the corner point which gives maximum profit. For this purpose the coordinates relating to the corner point should put in objectives function and the optimal point should be ascertained.

(v) For decision – making purpose, sometimes, it is required to know whether optimal point leaves some resources unutilized. For this purpose value of coordinates at the optimal point should be put with constraint to find out which constraints are not fully utilized.

Example 3: A company produces two products A and B, each of which requires processing in two machines. The first machine can be used at most for 60 hours, the second machine can be used at most for 40 hours. The product A requires 2 hours on machine one and one hour on machine two. The product B requires one hour on machine one and two hours on machine two. Express above situation using linear inequalities.

Solution: Let the company produce, x number of product A and y number of product B. As each of product A requires 2 hours in machine one and one hour in machine two, x number of product A requires $2x$ hours in machine one and x hours in machine two. Similarly, y number of product

B requires y hours in machine one and $2y$ hours in machine two. But machine one can be used for 60 hours and machine two for 40 hours. Hence $2x + y$ cannot exceed 60 and $x + 2y$ cannot exceed 40. In other words,

$2x + y \leq 60$ and $x + 2y \leq 40$.

Thus, the conditions can be expressed using linear inequalities.

Example 4: A fertilizer company produces two types of fertilizers called grade I and grade II. Each of these types is processed through two critical chemical plant units. Plant A has maximum of 120 hours available in a week and plant B has maximum of 180 hours available in a week. Manufacturing one bag of grade I fertilizer requires 6 hours in plant A and 4 hours in plant B. Manufacturing one bag of grade II fertilizer requires 3 hours in plant A and 10 hours in plant B. Express this using linear inequalities.

Solution: Let us denote by x_1, the number of bags of fertilizers of grade I and by x_2, the number of bags of fertilizers of grade II produced in a week. We are given that grade I fertilizer requires 6 hours in plant A and grade II fertilizer requires 3 hours in plant A and plant A has maximum of 120 hours available in a week. Thus $6x_1 + 3x_2 \leq 120$.

Similarly grade I fertilizer requires 4 hours in plant B and grade II fertilizer requires 10 hours in Plant B and Plant B has maximum of 180 hours available in a week. Hence, we get the inequality $4x_1 + 10x_2 \leq 180$.

Example 5: Graph the inequalities $5x_1 + 4x_2 \geq 9$, $x_1 + x_2 \geq 3$, $x_1 \geq 0$ and $x_2 \geq 0$ and mark the common region.

Solution: We draw the straight lines $5x_1 + 4x_2 = 9$ and $x_1 + x_2 = 3$.

Table for $5x_1 + 4x_2 = 9$

x_1	0	9/5
x_2	9/4	0

Table for $x_1 + x_2 = 3$

x_1	0	3
x_2	3	0

Now, if we take the point (4, 4), we find

$5x_1 + 4x_2 \geq 9$

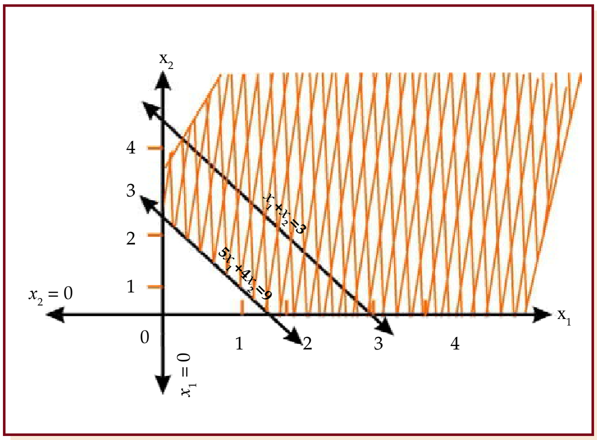
i.e., $5.4 + 4.4 \geq 9$

or, $36 \geq 9$ (True)

$x_1 + x_2 \geq 3$

i.e., $4 + 4 \geq 3$

$8 \geq 3$ (True)



Hence (4, 4) is in the region which satisfies the inequalities.

We mark the region being satisfied by the inequalities and note that the cross-hatched region is satisfied by all the inequalities.

Example 6: Draw the graph of the solution set of the following inequality and equality:

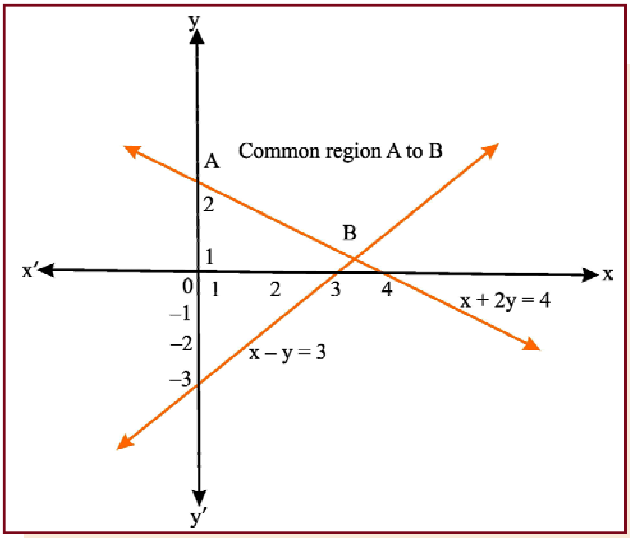
$$x + 2y = 4.$$

$$x - y \leq 3.$$

Mark the common region.

Solution: We draw the graph of both $x + 2y = 4$ and $x - y \leq 3$ in the same plane.

The solution set of system is that portion of the graph of $x + 2y = 4$ that lies within the half-plane representing the inequality $x - y \leq 3$.



For $x + 2y = 4$,

x	4	0
y	0	2

For $x - y = 3$,

x	3	0
y	0	–3

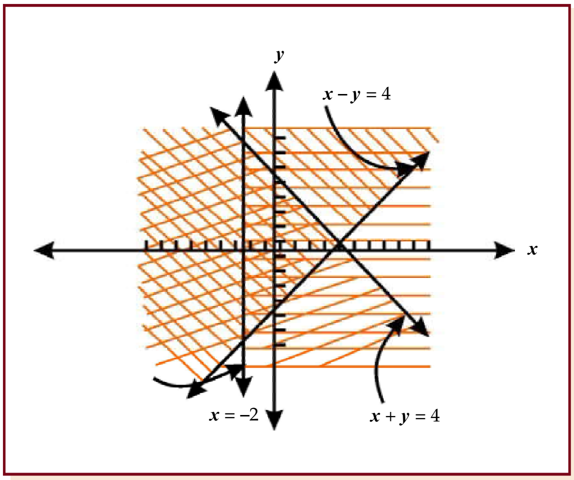
Example 7: Draw the graphs of the following inequalities:

$$x + y \leq 4,$$

$$x - y \leq 4,$$

$$x \geq -2.$$

and mark the common region.



For $x - y = 4$,

x	4	0
y	0	–4

For $x + y = 4$,

x	0	4
y	4	0

The common region is the one represented by overlapping of the shadings.

Example 8: Draw the graphs of the following linear inequalities:

$5x + 4y \leq 100$, $5x + y \geq 40$,

$3x + 5y \leq 75$, $x \geq 0, y \geq 0$.

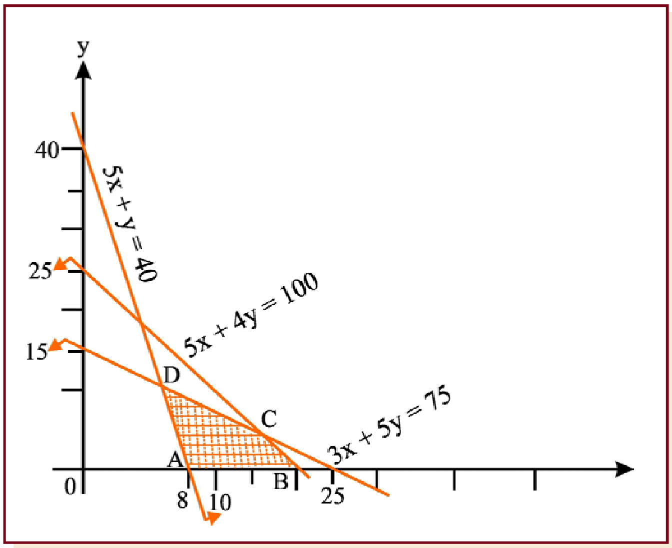
and mark the common region.

Solution:

$5x + 4y = 100$ or, $\frac{x}{20} + \frac{y}{25} = 1$

$3x + 5y = 75$ or, $\frac{x}{25} + \frac{y}{15} = 1$

$5x + y = 40$ or, $\frac{x}{8} + \frac{y}{40} = 1$



Plotting the straight lines on the graph paper we have the above diagram:

The common region of the given inequalities is shown by the shaded portion ABCD.

Example 9: Draw the graphs of the following linear inequalities:

$5x + 8y \leq 2000$, $x \leq 175$, $x \geq 0$.

$7x + 4y \leq 1400$, $y \leq 225$, $y \geq 0$.

and mark the common region:

Solution: Let us plot the line AB ($5x + 8y = 2,000$) by joining the points A(400, 0) and B(0, 250).

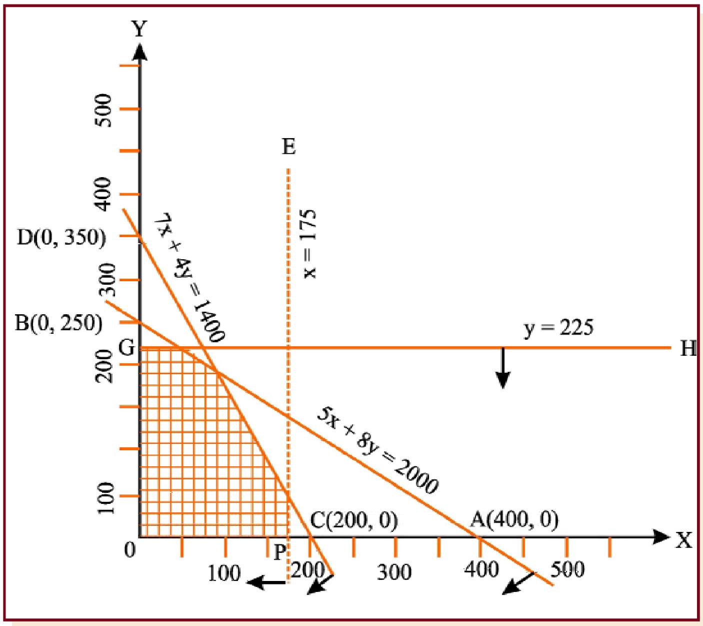
x	400	0
y	0	250

Similarly, we plot the line CD ($7x + 4y = 1400$) by joining the points C(200, 0) and D(0, 350).

x	200	0
y	0	350

Also, we draw the lines EF($x = 175$) and GH ($y = 225$).

The required graph is shown alongside in which the common region is shaded.



Example 10: Draw the graphs of the following linear inequalities:

$x + y \geq 1,\quad 7x + 9y \leq 63,$

$y \leq 5,\quad x \leq 6,\quad x \geq 0,\ y \geq 0.$

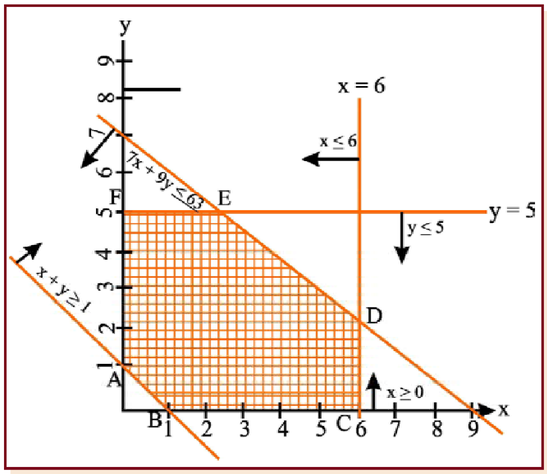
and mark the common region.

Solution: $x + y = 1$; $\begin{array}{c|c|c} x & 1 & 0 \\ \hline y & 0 & 1 \end{array}$; $7x + 9y = 63$, $\begin{array}{c|c|c} x & 9 & 0 \\ \hline y & 0 & 7 \end{array}$.

We plot the line AB ($x + y = 1$), CD ($y = 5$), EF ($x = 6$), DE ($7x + 9y = 63$).

Given inequalities are shown by arrows.

Common region *ABCDEF* is the shaded region.



Example 11: Two machines (I and II) produce two grades of plywood, grade A and grade B. In one hour of operation machine I produces two units of grade A and one unit of grade B, while machine II, in one hour of operation produces three units of grade A and four units of grade B. The machines are required to meet a production schedule of at least fourteen units of grade A and twelve units of grade B. Express this using linear inequalities and draw the graph.

Solution: Let the number of hours required on machine I be x and that on machine II be y. Since in one hour, machine I can produce 2 units of grade A and one unit of grade B, in x hours it will produce $2x$ and x units of grade A and B respectively. Similarly, machine II, in one hour, can produce 3 units of grade A and 4 units of grade B. Hence, in y hours, it will produce $3y$ and $4y$ units Grade A & B respectively.

The given data can be expressed in the form of linear inequalities as follows:

$2x + 3y \geq 14$ (Requirement of grade A)

$x + 4y \geq 12$ (Requirement of grade B)

Moreover x and y cannot be negative, thus $x \geq 0$ and $y \geq 0$

Let us now draw the graphs of above inequalities. Since both x and y are positive, it is enough to draw the graph only on the positive side.

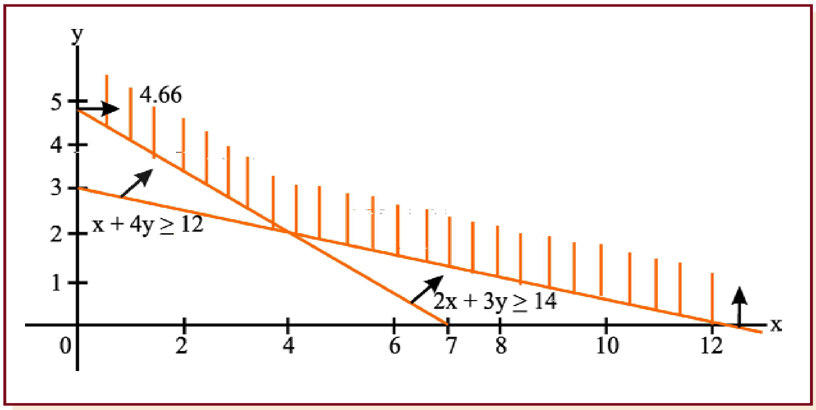
The inequalities are drawn in the following graph:

For $2x + 3y = 14$,

x	7	0
y	0	4.66

For $x + 4y = 12$,

x	0	12
y	3	0



In the above graph we find that the shaded portion is moving towards infinity on the positive side. Thus the result of these inequalities is unbounded.

EXERCISE: 3 (A)

Choose the correct answer/answers

1 (i) An employer recruits experienced (x) and fresh workmen (y) for his firm under the condition that he cannot employ more than 9 people. x and y can be related by the inequality

(a) $x + y \neq 9$ (b) $x + y \leq 9 \quad x \geq 0, y \geq 0$ (c) $x + y \geq 9 \quad x \geq 0, y \geq 0$ (d) none of these

(ii) On the average experienced person does 5 units of work while a fresh one 3 units of work daily but the employer has to maintain an output of at least 30 units of work per day. This situation can be expressed as

(a) $5x + 3y \leq 30$ (b) $5x + 3y > 30$ (c) $5x+3y \geq 30 \; x \geq 0, y \geq 0$ (d) none of these

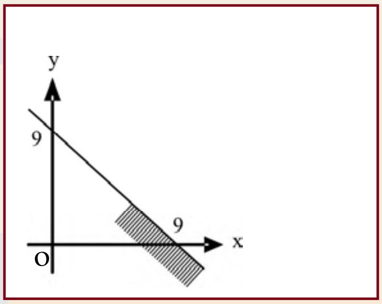
(iii) The rules and regulations demand that the employer should employ not more than 5 experienced hands to 1 fresh one and this fact can be expressed as

(a) $y \geq x/5$ (b) $5y \leq x$ (c) $5y \geq x$ (d) none of these

(iv) The union however forbids him to employ less than 2 experienced person to each fresh person. This situation can be expressed as

(a) $x \le y/2$ (b) $y \le x/2$ (c) $y \ge x/2$ (d) $x > 2y$

(v) The graph to express the inequality $x + y \le 9$ is

(a)


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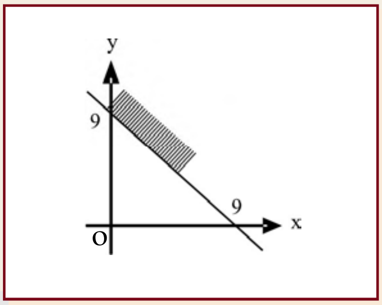
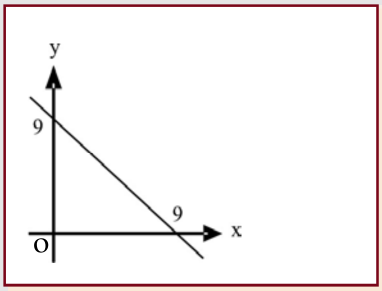
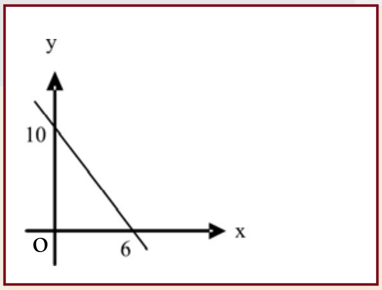
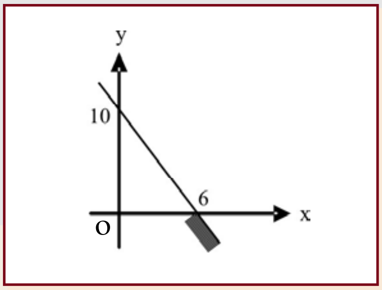


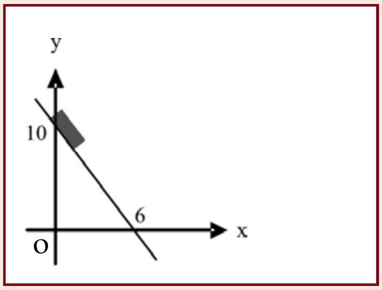
(c)


(d) none of these

(vi) The graph to express the inequality $5x + 3y \ge 30$ is

(a)


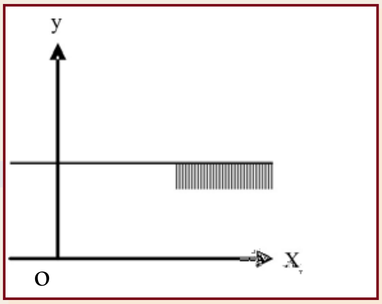
(b)


(c)


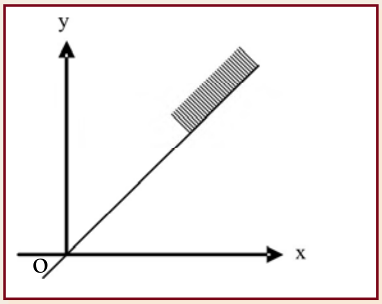
(d) none of these

(vii) The graph to express the inequality $y \leq \left(\frac{1}{2}\right) x$ is indicated by

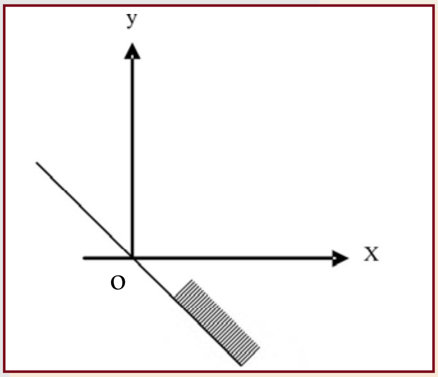
(a)



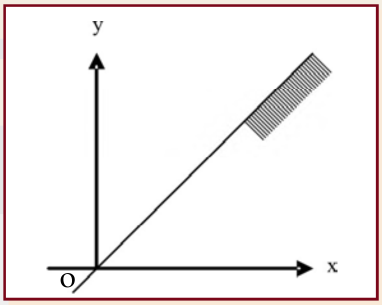
(b)



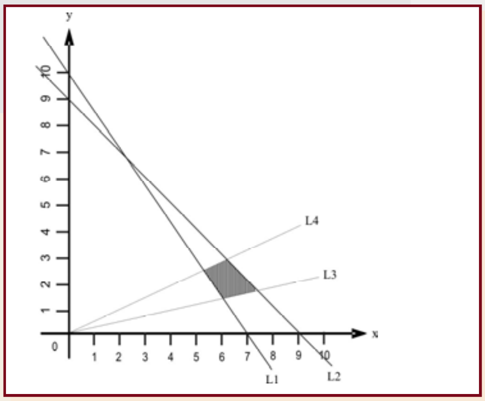
(c)



(d)



(viii)



L1 : $5x + 3y = 30$ L2 : $x+y = 9$ L3 : $y = x/3$ L4 : $y = x/2$

The common region (shaded part) shown in the diagram refers to

(a) $5x + 3y \leq 30$, $x + y \leq 9$, $y \leq 1/5\, x$, $y \leq x/2$

(b) $5x + 3y \geq 30$, $x + y \leq 9$, $y \geq x/3$, $y \leq x/2$, $x \geq 0, y \geq 0$

(c) $5x + 3y \geq 30$, $x + y \geq 9$, $y \leq x/3$, $y \geq x/2$, $x \geq 0, y \geq 0$

(d) $5x + 3y > 30$, $x + y < 9$, $y \geq 9$, $y \leq x/2$, $x \geq 0, y \geq 0$

(e) None of these

2. A dietitian wishes to mix together two kinds of food so that the vitamin content of the mixture is at least 9 units of vitamin A, 7 units of vitamin B, 10 units of vitamin C and 12 units of vitamin D. The vitamin content per Kg. of each food is shown below:

	A	B	C	D
Food I :	2	1	1	2
Food II:	1	1	2	3

Assuming x units of food I is to be mixed with y units of food II the situation can be expressed as

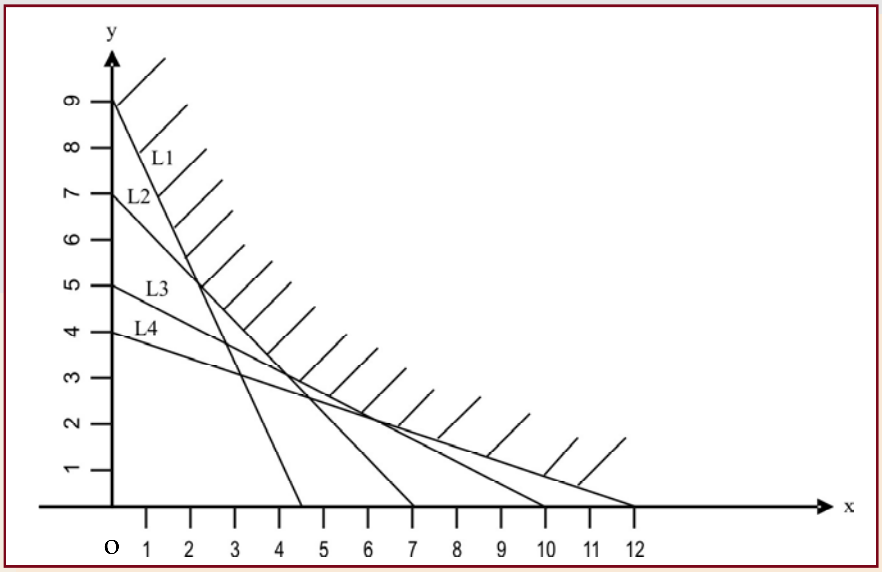
(a) $2x + y \leq 9$, $x + y \leq 7$, $x + 2y \leq 10$, $2x + 3y \leq 12$, $x > 0, y > 0$

(b) $2x + y \geq 30$, $x + y \leq 7$, $x + 2y \geq 10$, $x + 3y \geq 12$

(c) $2x + y \geq 9$, $x + y \geq 7$, $x + y \leq 10$, $x + 3y \geq 12$

(d) $2x + y \geq 9$, $x + y \geq 7$, $x + 2y \geq 10$, $2x + 3y \geq 12$, $x \geq 0, y \geq 0$,

3. Graphs of the inequations are drawn below :



L1 : $2x + y = 9$ L2 : $x + y = 7$ L3 : $x+2y= 10$ L4 : $x + 3y = 12$

The common region (shaded part) indicated on the diagram is expressed by the set of inequalities

(a) $2x + y \leq 9$, $x + y \geq 7$, $x + 2y \geq 10$, $x + 3y \geq 12$

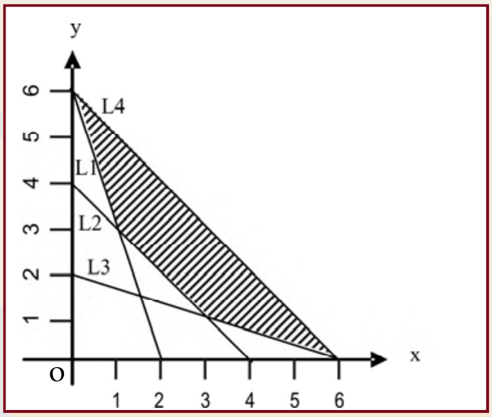
(b) $2x + y \geq 9$, $x + y \leq 7$, $x + 2y \geq 10$, $x + 3y \geq 12$

(c) $2x + y \geq 9$, $x + y \geq 7$, $x + 2y \geq 10$, $x + 3y \geq 12$, $x \geq 0, y \geq 0$

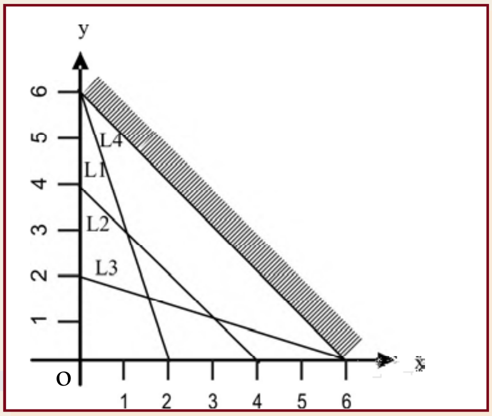
(d) none of these

4. The common region satisfied by the inequalities L1: $3x + y \geq 6$, L2: $x + y \geq 4$, L3: $x + 3y \geq 6$, and L4: $x + y \leq 6$ is indicated by

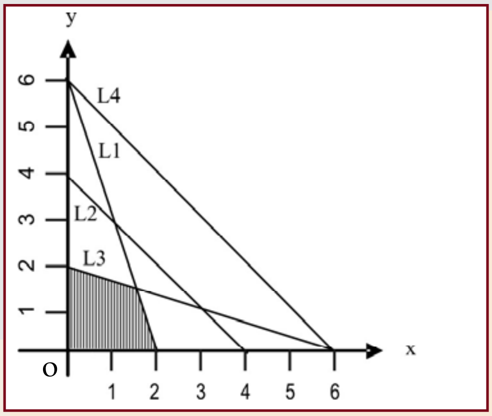
(a)



(b)

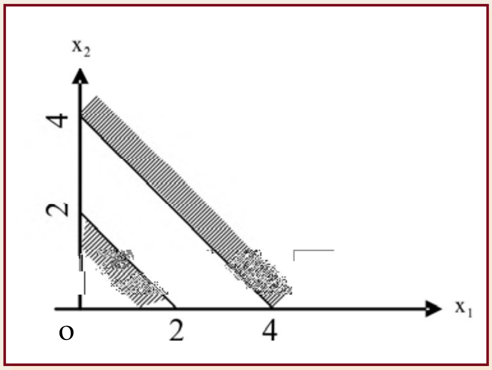


(c)



(d) none of these

5. The region indicated by the shading in the graph is expressed by inequalities



(a) $x_1 + x_2 \le 2$
$2x_1 + 2x_2 \ge 8$
$x_1 \ge 0$, $x_2 \ge 0$,

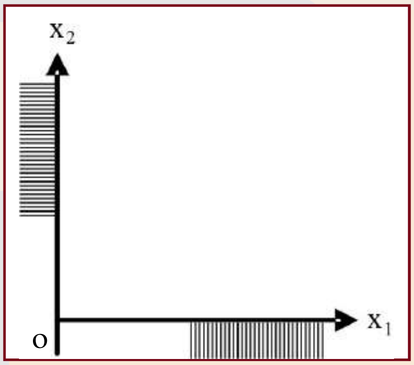
(b) $x_1 + x_2 \le 2$
$x_2 x_1 + x_2 \le 4$

(c) $x_1 + x_2 \ge 2$
$2x_1 + 2x_2 \ge 8$

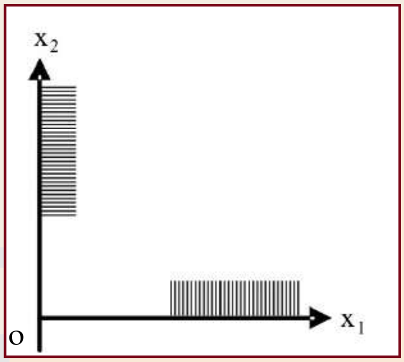
(d) $x_1 + x_2 \le 2$
$2x_1 + 2x_2 > 8$

6. (i) The inequalities $x_1 \ge 0$, $x_2 \ge 0$, are represented by one of the graphs shown below:

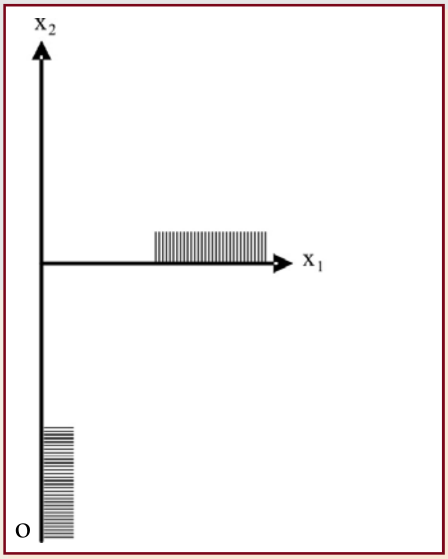
(a)



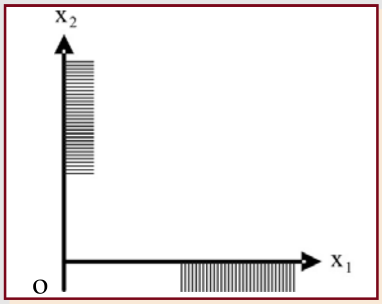
(b)



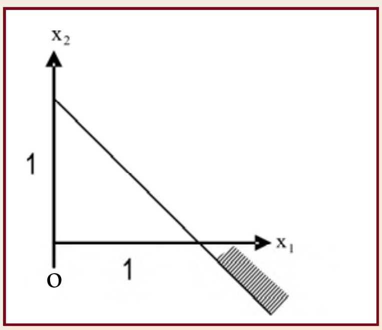
(c)



(d)



(ii)



The region is expressed as

(a) $x_1 - x_2 \ge 1$

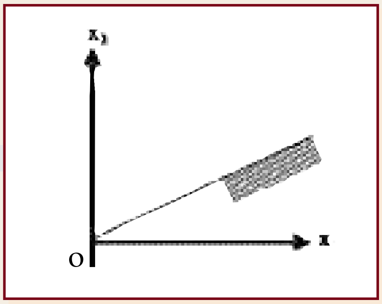
(b) $x_1 + x_2 \le 1$

(c) $x_1 + x_2 \ge 1$

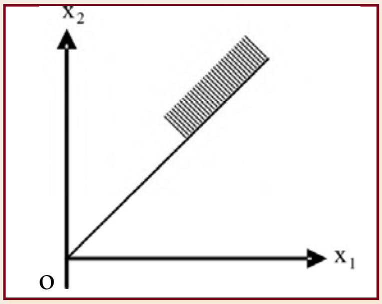
(d) none of these

(iii) The inequality $-x_1 + 2x_2 \le 0$ is indicated on the graph as

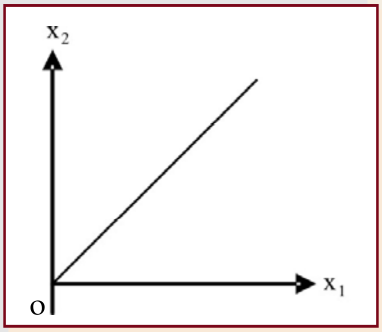
(a)



(b)

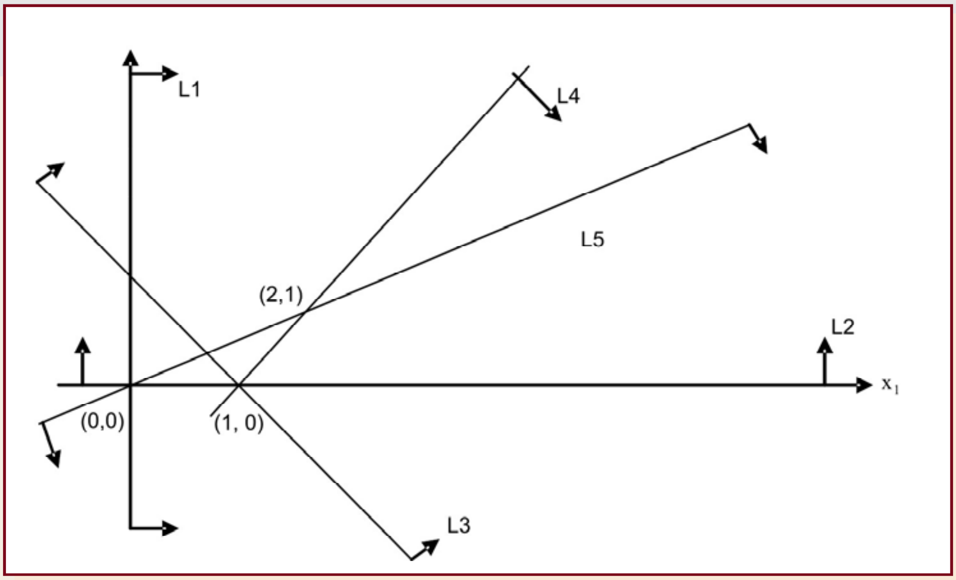


(c)



(d) none of these

7.



The common region indicated on the graph is expressed by the set of five inequalities

(a)	(b)	(c)	(d)
L1 : $x_1 \geq 0$	L1 : $x_1 \geq 0$	L1 : $x_1 \leq 0$	None of these
L2 : $x_2 \geq 0$	L2 : $x_2 \geq 0$	L2 : $x_2 \leq 0$	
L3 : $x_1 + x_2 \leq 1$	L3 : $x_1+x_2 \geq 1$	L3 : $x_1+ x_2 \geq 1$	
L4 : $x_1 - x_2 \geq 1$	L4 : $x_1-x_2 \geq 1$	L4 : $x_1-x_2 \geq 1$	
L5 : $-x_1 + 2x_2 \leq 0$	L5 :$- x_1+2x_2 \leq 0$	L5 :$- x_1+2x_2 \leq 0$	

8. A firm makes two types of products : Type A and Type B. The profit on product A is ₹ 20 each and that on product B is ₹ 30 each. Both types are processed on three machines M1, M2 and M3. The time required in hours by each product and total time available in hours per week on each machine are as follows:

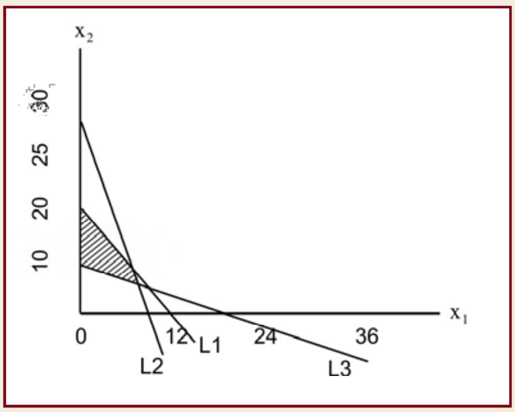
Machine	Product A	Product B	Available Time
M1	3	3	36
M2	5	2	50
M3	2	6	60

The constraints can be formulated taking x_1 = number of units A and x_2 = number of unit of B as

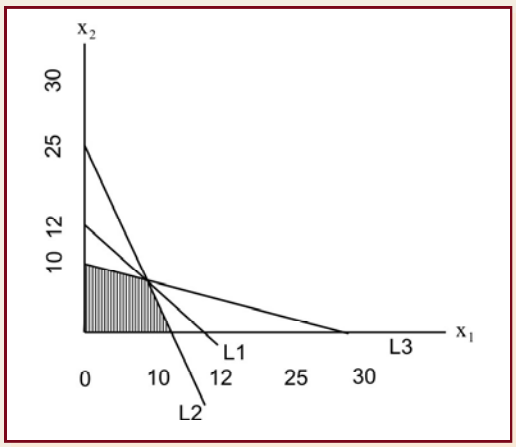
(a)	(b)	(c)	(d)
$x_1 + x_2 \leq 12$	$3x_1 + 3x_2 \geq 36$	$3x_1 + 3x_2 \leq 36$	none of these
$5x_1 + 2x_2 \leq 50$	$5x_1 + 2x_2 \leq 50$	$5x_1 + 2x_2 \leq 50$	
$2x_1 + 6x_2 \leq 60$	$2x_1 + 6x_2 \geq 60$	$2x_1 + 6x_2 \leq 60$	
	$x_1 \geq 0, x_2 \geq 0$	$x_1 \geq 0, x_2 \geq 0$	

9. The set of inequalities L1: $x_1 + x_2 \leq 12$, L2: $5x_1 + 2x_2 \leq 50$, L3: $x_1 + 3x_2 \leq 30$, $x_1 \geq 0$, and $x_2 \geq 0$ is represented by

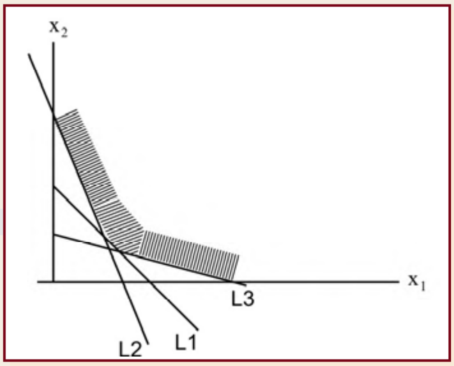
(a)



(b)



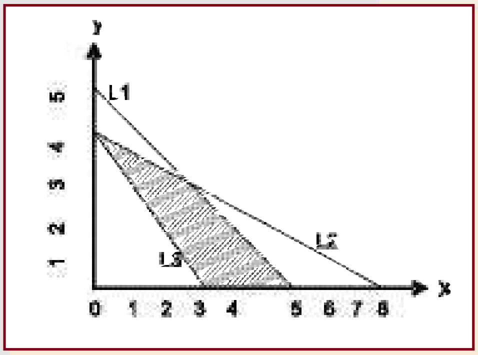
(c)



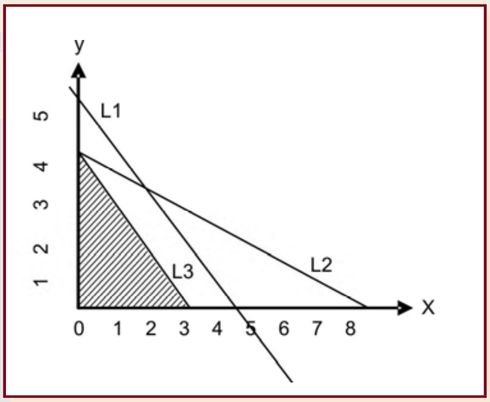
(d) none of these

10. The common region satisfying the set of inequalities x ≥ 0, y ≥ 0, L1: x+y ≤ 5, L2: x +2y ≤ 8 and L3: 4x +3y ≥ 12 is indicated by

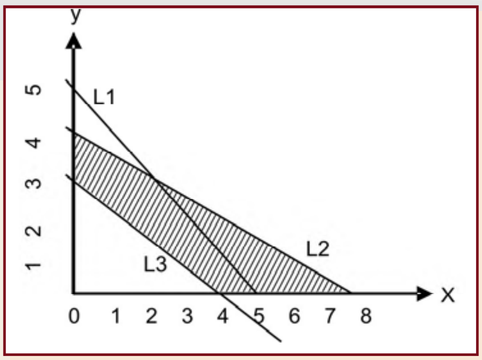
(a)



(b)



(c)



(d) none of these

SUMMARY

- Any linear function that involves an inequality sign is a linear inequality. It may be of one variable, or, of more than one variable. Simple example of linear inequalities are those of one variable only; viz., $x > 0$, $x < 0$.
- The values of the variables that satisfy an inequality are called the *solution space*, and is abbreviated as S.S. The solution spaces for (i) $x > 0$, (ii) $x \leq 0$ are shaded in the diagrams, by using deep lines.
- Linear inequalities in two variables may be solved easily by extending our knowledge of straight lines.

ANSWERS

1.	**(i)** (b)	**(ii)**	(c)	**(iii)** (a or c)	**(iv)**	(b)	**(v)**	(a)	**(vi)**	(c)	**(vii)**	(d) **(viii)** (b)			
2.	(d)	**3.**	(c)	**4.**	(a)	**5.**	(a)	**6.**	**(i)**	(b)	**(ii)**	(c)	**(iii)**	(a)	
7.	(b)	**8.**	(c)	**9.**	(b)	**10.**	(a)								

ADDITIONAL QUESTION BANK

1. On solving the inequalities $2x + 5y \le 20$, $3x + 2y \le 12$, $x \ge 0, y \ge 0$, we get the following situation

   (a) (0, 0), (0, 4), (4, 0) and $\left(20/11, 36/11\right)$ (b) (0, 0), (10, 0), (0, 6) and $\left(20/11, 36/11\right)$

   (c) (0, 0), (0, 4), (4, 0) and (2, 3) (d) (0, 0), (10, 0), (0, 6) and (2, 3)

2. On solving the inequalities $6x + y \ge 18$, $x + 4y \ge 12$, $2x + y \ge 10$, we get the following situation

   (a) (0, 18), (12, 0), (4, 2) and (2, 6)

   (b) (3, 0), (0, 3), (4, 2) and (7, 6)

   (c) (5, 0), (0, 10), (4, 2) and (7, 6)

   (d) (0, 18), (12, 0), (4, 2), (0, 0) and (7, 6)

ANSWERS

1. (a) **2.** (a)

NOTES

CHAPTER E

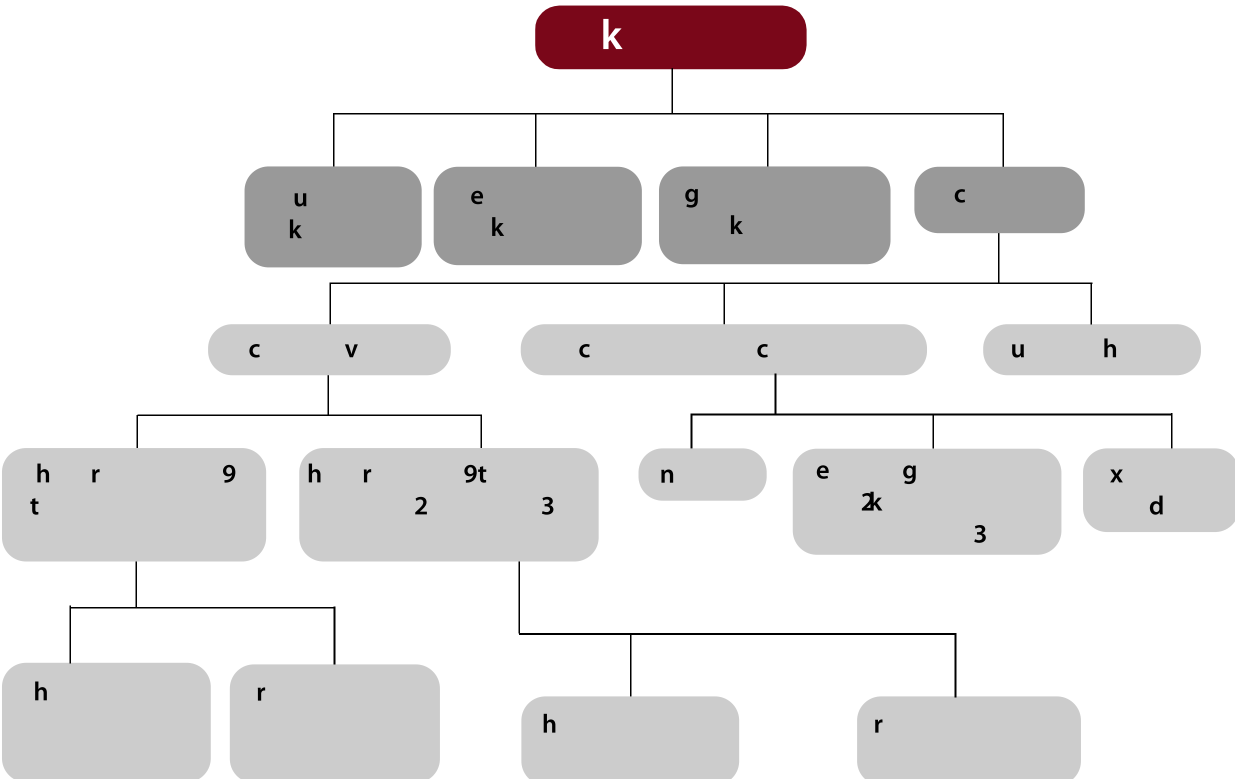
MATHEMATICS OF FINANCE

LEARNING OBJECTIVES

After studying this chapter students will be able to understand:

- The concept of interest, related terms and computation thereof;
- Difference between simple and compound interest;
- The concept of annuity;
- The concept of present value and future value;
- Use of present value concept in Leasing, Capital expenditure and Valuation of Bond.

CHAPTER OVERVIEW



4.1 INTRODUCTION

People earn money for spending it on housing, food, clothing, education, entertainment etc. Sometimes extra expenditures have also to be met with. For example there might be a marriage in the family; one may want to buy house, one may want to set up his or her business, one may want to buy a car and so on. Some people can manage to put aside some money for such expected and unexpected expenditures. But most people have to borrow money for such contingencies. From where they can borrow money?

Money can be borrowed from friends or money lenders or Banks. If you can arrange a loan from your friend it might be interest free but if you borrow money from lenders or Banks you will have to pay some charge periodically for using money of money lenders or Banks. This charge is called interest.

Let us take another view. People earn money for satisfying their various needs as discussed above. After satisfying those needs some people may have some savings. People may invest their savings in debentures or lend to other person or simply deposit it into bank. In this way they can earn interest on their investment.

Most of you are very much aware of the term interest. Interest can be defined as the price paid by a borrower for the use of a lender's money.

We will know more about interest and other related terms later.

4.2 WHY IS INTEREST PAID?

Now question arises why lenders charge interest for the use of their money. There are a variety of reasons. We will now discuss those reasons.

1. **Time value of money:** Time value of money means that the value of a unity of money is different in different time periods. The sum of money received in future is less valuable than it is today. In other words the present worth of money received after some time will be less than money received today. Since money received today has more value rational investors would prefer current receipts to future receipts. If they postpone their receipts they will certainly charge some money i.e. interest.

2. **Opportunity Cost:** The lender has a choice between using his money in different investments. If he chooses one he forgoes the return from all others. In other words lending incurs an opportunity cost due to the possible alternative uses of the lent money.

3. **Inflation:** Most economies generally exhibit inflation. Inflation is a fall in the purchasing power of money. Due to inflation a given amount of money buys fewer goods in the future than it will now. The borrower needs to compensate the lender for this.

4. **Liquidity Preference:** People prefer to have their resources available in a form that can immediately be converted into cash rather than a form that takes time or money to realize.

5. **Risk Factor:** There is always a risk that the borrower will go bankrupt or otherwise default on the loan. Risk is a determinable factor in fixing rate of interest.

A lender generally charges more interest rate (risk premium) for taking more risk.

4.3 DEFINITION OF INTEREST AND SOME OTHER RELATED TERMS

Now we can define interest and some other related terms.

4.3.1 Interest

Interest is the price paid by a borrower for the use of a lender's money. If you borrow (or lend) some money from (or to) a person for a particular period you would pay (or receive) more money than your initial borrowing (or lending). This excess money paid (or received) is called interest. Suppose you borrow (or lend) ₹ 50,000 for a year and you pay (or receive) ₹ 55,000 after one year the difference between initial borrowing (or lending) ₹ 50,000 and end payment (or receipts) ₹ 55,000 i.e. ₹ 5,000 is the amount of interest you paid (or earned).

4.3.2 Principal

Principal is initial value of lending (or borrowing). If you invest your money the value of initial investment is also called principal. Suppose you borrow (or lend) ₹ 50,000 from a person for one year. ₹ 50,000 in this example is the 'Principal.' Take another example suppose you deposit ₹ 20,000 in your bank account for one year. In this example ₹ 20,000 is the principal.

4.3.3 Rate of Interest

The rate at which the interest is charged for a defined length of time for use of principal generally on a yearly basis is known to be the rate of interest. Rate of interest is usually as expressed as percentages. Suppose you invest ₹ 20,000 in your bank account for one year with the interest rate of 5% per annum. It means you would earn ₹ 5 as interest every ₹ 100 of principal amount in a year.

Per annum means for a year.

4.3.4 Accumulated amount (or Balance)

Accumulated amount is the final value of an investment. It is the sum total of principal and interest earned. Suppose you deposit ₹ 50,000 in your bank for one year with an interest rate of 5% p.a. you would earn interest of ₹ 2,500 after one year. (method of computing interest will be illustrated later). After one year you will get ₹ 52,500 (principal+ interest), ₹ 52,500 is accumulated amount here.

Amount is also known as the balance.

4.4 SIMPLE INTEREST AND COMPOUND INTEREST

Now we can discuss the method of computing interest. Interest accrues as either simple interest or compound interest. We will discuss simple interest and compound interest in the following paragraphs:

4.4.1 Simple Interest

Now we would know what is simple interest and the methodology of computing simple interest and accumulated amount for an investment (principal) with a simple rate over a period of time. As you already know the money that you borrow is known as principal and the additional money that you pay for using somebody else's money is known as interest. The interest paid for keeping ₹ 100 for one year is known as the rate percent per annum. Thus if money is borrowed at the rate of 8% per annum then the interest paid for keeping ₹ 100 for one year is ₹ 8. The sum of principal and interest is known as the Amount.

Clearly the interest you pay is proportionate to the money that you borrow and also to the period of time for which you keep the money; the more the money and the time, the more the interest. Interest is also proportionate to the rate of interest agreed upon by the lending and the borrowing parties. Thus interest varies directly with principal, time and rate.

Simple interest is the interest computed on the principal for the entire period of borrowing. It is calculated on the outstanding principal balance and not on interest previously earned. It means no interest is paid on interest earned during the term of loan.

Simple interest can be computed by applying following formulas:

$$I = Pit$$

$$A = P + I$$

$$= P + Pit$$

$$= P(1 + it)$$

$$I = A - P$$

Here,

A = Accumulated amount (final value of an investment)

P = Principal (initial value of an investment)

i = Annual interest rate in decimal.

I = Amount of Interest

t = Time in years

Let us consider the following examples in order to see how exactly are these quantities related.

Example 1: How much interest will be earned on ₹ 2000 at 6% simple interest for 2 years?

Solution: Required interest amount is given by

$$I = P \times i \times t$$

$$= 2{,}000 \times \frac{6}{100} \times 2$$

$$= ₹\ 240$$

Example 2: Sania deposited ₹ 50,000 in a bank for two years with the interest rate of 5.5% p.a. How much interest would she earn?

Solution: Required interest amount is given by

$I = P \times i \times t$

$= ₹\ 50{,}000 \times \dfrac{5.5}{100} \times 2$

$= ₹\ 5{,}500$

Example 3: In example 2 what will be the final value of investment?

Solution: Final value of investment is given by

$A = P(1 + it)$

$= ₹\ 50{,}000 \left(1+\dfrac{5.5}{100}\times 2\right)$

$= ₹\ 50{,}000 \left(1+\dfrac{11}{100}\right)$

$= ₹\ \dfrac{50{,}000\times 111}{100}$

$= ₹\ 55{,}500$

or

$A = P + I$

$= ₹\ (50{,}000 + 5{,}500)$

$= ₹\ 55{,}500$

Example 4: Sachin deposited ₹ 1,00,000 in his bank for 2 years at simple interest rate of 6%. How much interest would he earn? How much would be the final value of deposit?

Solution: (a) Required interest amount is given by

$I = P \times it$

$= ₹\ 1{,}00{,}000 \times \dfrac{6}{100} \times 2$

$= ₹\ 12{,}000$

(b) Final value of deposit is given by

$A = P + I$

$= ₹\ (1{,}00{,}000 + 12{,}000)$

$= ₹\ 1{,}12{,}000$

Example 5: Find the rate of interest if the amount owed after 6 months is ₹ 1050, borrowed amount being ₹ 1000.

Solution: We know $A = P + Pit$

i.e. $1050 = 1000 + 1000 \times i \times (6/12)$

- $50 = 500\ i$
- $i = 1/10 = 10\%$

Example 6: Rahul invested ₹ 70,000 in a bank at the rate of 6.5% p.a. simple interest rate. He received ₹ 85,925 after the end of term. Find out the period for which sum was invested by Rahul.

Solution: We know A = P (1+it)

$$\text{i.e. } 85{,}925 = 70{,}000\left(1+\frac{6.5}{100}\times t\right)$$

- $85{,}925/70{,}000 = \dfrac{100+6.5\,t}{100}$
- $\dfrac{85{,}925\times 100}{70{,}000} - 100 = 6.5t$
- $22.75 = 6.5t$
- $t = 3.5$

∴ time = 3.5 years

Example 7: Kapil deposited some amount in a bank for 7 ½ years at the rate of 6% p.a. simple interest. Kapil received ₹ 1,01,500 at the end of the term. Compute initial deposit of Kapil.

Solution: We know A = P(1+ it)

$$\text{i.e. } 1{,}01{,}500 = P\left(1+\frac{6}{100}\times\frac{15}{2}\right)$$

- $1{,}01{,}500 = P\left(1+\dfrac{45}{100}\right)$
- $1{,}01{,}500 = P\left(\dfrac{145}{100}\right)$
- $P = \dfrac{1{,}01{,}500\times 100}{145}$

$= ₹\ 70{,}000$

∴ Initial deposit of Kapil = ₹ 70,000

Example 8: A sum of ₹ 46,875 was lent out at simple interest and at the end of 1 year 8 months the total amount was ₹ 50,000. Find the rate of interest percent per annum.

Solution: We know A = P (1 + it)

i.e. $50,000 = 46,875\left(1+i\times1\frac{8}{12}\right)$

➢ $50,000/46,875 = 1 + \frac{5}{3}\ i$

➢ $(1.067 - 1) \times 3/5 = i$

➢ $i = 0.04$

➢ rate = 4%

Example 9: What sum of money will produce ₹ 28,600 as an interest in 3 years and 3 months at 2.5% p.a. simple interest?

Solution: We know I = P × it

i.e. $28,600 = P \times \frac{2.5}{100} \times 3\frac{3}{12}$

➢ $28,600 = \frac{2.5}{100}\ P \times \frac{13}{4}$

➢ $28,600 = \frac{32.5}{400}P$

➢ $P = \frac{28,600\times400}{32.5}$

$= ₹\ 3,52,000$

∴ ₹ 3,52,000 will produce ₹ 28,600 interest in 3 years and 3 months at 2.5% p.a. simple interest

Example 10: In what time will ₹ 85,000 amount to ₹ 1,57,675 at 4.5 % p.a. ?

Solution: We know

$A = P\ (1 + it)$

➢ $1,57,675 = 85,000\left(1+\frac{4.5}{100}\times t\right)$

➢ $\frac{1,57,675}{85,000} = \frac{100\ +\ 4.5\,t}{100}$

➢ $4.5t = \left[\frac{1,57,675}{85,000}\times100\right] - 100$

➢ $t = \frac{85.5}{4.5} = 19$

∴ In 19 years ₹ 85,000 will amount to ₹ 1,57,675 at 4.5% p.a. simple interest rate.

EXERCISE 4 (A)

Choose the most appropriate option (a) (b) (c) or (d).

1. S.I on ₹ 3,500 for 3 years at 12% p.a. is

   (a) ₹ 1,200 (b) ₹ 1,260 (c) ₹ 2,260 (d) none of these

2. P = 5,000, R = 15, T = 4 ½ using I = PRT/100, I will be

   (a) ₹ 3,375 (b) ₹ 3,300 (c) ₹ 3,735 (d) none of these

3. If P = 5,000, T = 1, I = ₹ 300, R will be

   (a) 5% (b) 4% (c) 6% (d) none of these

4. If P = ₹ 4,500, A = ₹ 7,200, than Simple interest i.e. I will be

   (a) ₹ 2,000 (b) ₹ 3,000 (c) ₹ 2,500 (d) ₹ 2,700

5. P = ₹ 12,000, A = ₹ 16,500, T = 2 ½ years. Rate percent per annum simple interest will be

   (a) 15% (b) 12% (c) 10% (d) none of these

6 P = ₹ 10,000, I = ₹ 2,500, R = 12 ½% SI. The number of years T will be

   (a) 1 ½ years (b) 2 years (c) 3 years (d) none of these

7. P = ₹ 8,500, A = ₹ 10,200, R = 12 ½ % SI, t will be.

   (a) 1 yr. 7 mth. (b) 2 yrs. (c) 1 ½ yr. (d) none of these

8. The sum required to earn a monthly interest of ₹ 1,200 at 18% per annum SI is

   (a) ₹ 50,000 (b) ₹ 60,000 (c) ₹ 80,000 (d) none of these

9. A sum of money amount to ₹ 6,200 in 2 years and ₹ 7,400 in 3 years. The principal and rate of interest are

   (a) ₹ 3,800, 31.58% (b) ₹ 3,000, 20% (c) ₹ 3,500, 15% (d) none of these

10. A sum of money doubles itself in 10 years. The number of years it would triple itself is

   (a) 25 years. (b) 15 years. (c) 20 years (d) none of these

4.4.2 Compound Interest

We have learnt about the simple interest. We know that if the principal remains the same for the entire period or time then interest is called as simple interest. However in practice the method according to which banks, insurance corporations and other money lending and deposit taking companies calculate interest is different. To understand this method we consider an example :

Suppose you deposit ₹ 50,000 in ICICI bank for 2 years at 7% p.a. compounded annually. Interest will be calculated in the following way:

INTEREST FOR FIRST YEAR

$$I = Pit$$

$$= ₹\ 50{,}000 \times \frac{7}{100} \times 1 = ₹\ 3{,}500$$

INTEREST FOR SECOND YEAR

For calculating interest for second year principal would not be the initial deposit. Principal for calculating interest for second year will be the initial deposit plus interest for the first year. Therefore principal for calculating interest for second year would be

$$= ₹\ 50{,}000 + ₹\ 3{,}500$$

$$= ₹\ 53{,}500$$

$$\text{Interest for the second year} = ₹\ 53{,}500 \times \frac{7}{100} \times 1$$

$$= ₹\ 3{,}745$$

Total interest = Interest for first year + Interest for second year

$$= ₹\ (3{,}500 + 3{,}745)$$

$$= ₹\ 7{,}245$$

This interest is ₹ 245 more than the simple interest on ₹ 50,000 for two years at 7% p.a. As you must have noticed this excess in interest is due to the fact that the principal for the second year was more than the principal for first year. The interest calculated in this manner is called compound interest.

Thus we can define the compound interest as the interest that accrues when earnings for each specified period of time added to the principal thus increasing the principal base on which subsequent interest is computed.

Example 11: Saina deposited ₹ 1,00,000 in a nationalized bank for three years. If the rate of interest is 7% p.a., calculate the interest that bank has to pay to Saina after three years if interest is compounded annually. Also calculate the amount at the end of third year.

Solution: Principal for first year ₹ 1,00,000

$$\text{Interest for first year} = Pit$$

$$= 1{,}00{,}000 \times \frac{7}{100} \times 1$$

$$= ₹\ 7{,}000$$

Principal for the second year = Principal for first year + Interest for first year

$$= ₹\ 1{,}00{,}000 + ₹\ 7{,}000$$

$$= ₹\ 1{,}07{,}000$$

Interest for second year $= 1{,}07{,}000 \times \frac{7}{100} \times 1$

$= ₹\ 7{,}490$

Principal for the third year = Principal for second year + Interest for second year

$= 1{,}07{,}000 + 7{,}490$

$= 1{,}14{,}490$

Interest for the third year $= ₹\ 1{,}14{,}490 \times \frac{7}{100} \times 1$

$= ₹\ 8{,}014.30$

Compound interest at the end of third year

$= ₹\ (7{,}000 + 7{,}490 + 8{,}014.30)$

$= ₹\ 22{,}504.30$

Amount at the end of third year

= Principal (initial deposit) + compound interest

$= ₹\ (1{,}00{,}000 + 22{,}504.30)$

$= ₹\ 1{,}22{,}504.30$

Difference between simple interest and compound interest

Now we can summarize the main difference between simple interest and compound interest. The main difference between simple interest and compound interest is that in simple interest the principal remains constant throughout whereas in the case of compound interest principal goes on changing at the end of specified period. For a given principal, rate and time the compound interest is generally more than the simple interest.

4.4.3 Conversion period

In the example discussed above the interest was calculated on yearly basis i.e. the interest was compounded annually. However in practice it is not necessary that the interest be compounded annually. For example in banks the interest is often compounded twice a year (half yearly or semi annually) i.e. interest is calculated and added to the principal after every six months. In some financial institutions interest is compounded quarterly i.e. four times a year. The period at the end of which the interest is compounded is called conversion period. When the interest is calculated and added to the principal every six months the conversion period is six months. In this case number of conversion periods per year would be two. If the loan or deposit was for five years then the number of conversion period would be ten.

Typical conversion periods are given below:

Conversion period	Description	Number of conversion period in a year
1 day	Compounded daily	365
1 month	Compounded monthly	12
3 months	Compounded quarterly	4
6 months	Compounded semi annually	2
12 months	Compounded annually	1

4.4.4 Formula for compound interest

Taking the principal as P, the rate of interest per conversion period as i (in decimal), the number of conversion period as n, the accrued amount after n payment periods as A_n we have accrued amount at the end of first payment period

$A_1 = P + P\,i = P(1 + i)$;

at the end of second payment period

$A_2 = A_1 + A_1 i = A_1(1 + i)$

$= P(1 + i)(1 + i)$

$= P(1 + i)^2$;

at the end of third payment period

$A_3 = A_2 + A_2 i$

$= A_2(1 + i)$

$= P(1 + i)^2(1 + i)$

$= P(1 + i)^3$

$A_n = A_{n-1} + A_{n-1} i$

$= A_{n-1}(1 + i)$

$= P(1 + i)^{n-1}(1 + i)$

$= P(1 + i)^n$

Thus the accrued amount A_n on a principal P after n conversion periods at i (in decimal) rate of interest per conversion period is given by

$A_n = P(1 + i)^n$

$$\text{where, } i = \frac{\text{Annual rate of interest}}{\text{Number of conversion periods per year}}$$

Interest $= A_n - P = P(1 + i)^n - P$

$= P\left[(1+i)^n - 1\right]$

n is total conversions i.e. t x no. of conversions per year

Note : Computation of a shall be quite simple with a calculator. However compound interest table and tables for at various rates per annum with (a) annual compounding ; (b) monthly compounding and (c) daily compounding are available.

Example 12: ₹ 2,000 is invested at annual rate of interest of 10%. What is the amount after two years if compounding is done (a) Annually (b) Semi-annually (c) Quarterly (d) monthly.

Solution: (a) Compounding is done annually

Here principal P = ₹ 2,000; since the interest is compounded yearly the number of conversion periods n in 2 years are 2. Also the rate of interest per conversion period (1 year) i is 0.10

$A_n = P(1 + i)^n$

$A_2 = ₹\ 2{,}000\ (1 + 0.1)^2$

$= ₹\ 2{,}000 \times (1.1)^2$

$= ₹\ 2{,}000 \times 1.21$

$= ₹\ 2{,}420$

(b) For semiannual compounding

$n = 2 \times 2 = 4$

$i = \frac{0.1}{2} = 0.05$

$A_4 = 2{,}000\ (1+0.05)^4$

$= 2{,}000 \times 1.2155$

$= ₹\ 2{,}431$

(c) For quarterly compounding

$n = 4 \times 2 = 8$

$i = \frac{0.1}{4} = 0.025$

$A_8 = 2{,}000\ (1+ 0.025)^8$

$= 2{,}000 \times 1.2184$

$= ₹\ 2{,}436.80$

(d) For monthly compounding

$n = 12 \times 2 = 24, i = 0.1/12 = 0.00833$

$A_{24} = 2{,}000\ (1 + 0.00833)^{24}$

$= 2{,}000 \times 1.22029$

$= ₹\ 2{,}440.58$

Example 13: Determine the compound amount and compound interest on ₹ 1000 at 6% compounded semi-annually for 6 years. Given that $(1 + i)^n = 1.42576$ for i = 3% and n = 12.

Solution: $i = \frac{0.06}{2} = 0.03;\ n = 6 \times 2 = 12$

P = 1,000

Compound Amount (A_{12}) = $P(1 + i)^n$

= ₹ $1{,}000(1 + 0.03)^{12}$

= $1{,}000 \times 1.42576$

= ₹ 1,425.76

Compound Interest = ₹ (1,425.76 – 1,000)

= ₹ 425.76

Example 14: Compute the compound interest on ₹ 4,000 for 1½ years at 10% per annum compounded half- yearly.

Solution: Here principal P = ₹ 4,000. Since the interest is compounded half-yearly the number of conversion periods in 1½ years are 3. Also the rate of interest per conversion period (6 months) is 10% x 1/2 = 5% (0.05 in decimal).

Thus the amount A_n (in ₹) is given by

$$A_n = P(1 + i)^n$$

$$A_3 = 4{,}000(1 + 0.05)^3$$

$$= 4{,}630.50$$

The compound interest is therefore ₹ (4,630.50 - 4,000)

= ₹ 630.50

To find the Principal/Time/Rate

The Formula $A_n = P(1 + i)^n$ connects four variables A_n, P, i and n.

Similarly, C.I.(Compound Interest) = $P\left[(1+i)^n - 1\right]$ connects C.I., P, i and n. Whenever three out of these four variables are given the fourth can be found out by simple calculations.

Examples 15: On what sum will the compound interest at 5% per annum for two years compounded annually be ₹ 1,640?

Solution: Here the interest is compounded annually the number of conversion periods in two years are 2. Also the rate of interest per conversion period (1 year) is 5%.

n = 2 i = 0.05

We know

C.I. $= P\left[(1+i)^n - 1\right]$

➢ $1{,}640 = P\left[(1+0.05)^2 - 1\right]$

➢ $1{,}640 = P\,(1.1025 - 1)$

➢ $P = \dfrac{1{,}640}{0.1025} = 16{,}000$

Hence the required sum is ₹ 16,000.

Example 16: What annual rate of interest compounded annually doubles an investment in 7 years? Given that $2^{\frac{1}{7}} = 1.104090$

Solution: If the principal be P then $A_n = 2P$.

Since $A_n = P(1+ i)^n$

➢ $2P = P\,(1 + i)^7$

➢ $2^{1/7} = (1 + i)$

➢ $1.104090 = 1 + i$

➢ $i = 0.10409$

∴ Required rate of interest = 10.41% per annum

Example 17: In what time will ₹ 8,000 amount to ₹ 8,820 at 10% per annum interest compounded half-yearly?

Solution: Here interest rate per conversion period (i) = $\dfrac{10}{2}$ %

= 5% (= 0.05 in decimal)

Principal (P) = ₹ 8,000

Amount (A_n) = ₹ 8,820

We know

$A_n = P\,(I + i)^n$

➢ $8{,}820 = 8{,}000\,(1 + 0.05)^n$

➢ $\dfrac{8{,}820}{8{,}000} = (1.05)^n$

➢ $1.1025 = (1.05)^n$

➢ $(1.05)^2 = (1.05)^n$

➢ $n = 2$

Hence number of conversion period is 2 and the required time = n/2 = 2/2 = 1 year

Example 18: Find the rate percent per annum if ₹ 2,00,000 amount to ₹ 2,31,525 in 1½ year interest being compounded half-yearly.

Solution: Here P = ₹ 2,00,000

Number of conversion period (n) = 1½ × 2 = 3

Amount (A_3) = ₹ 2,31,525

We know that

$$A_3 = P(1 + i)^3$$

➢ $2{,}31{,}525 = 2{,}00{,}000\,(1 + i)^3$

➢ $\dfrac{2{,}31{,}525}{2{,}00{,}000} = (1 + i)^3$

➢ $1.157625 = (1 + i)^3$

➢ $(1.05)^3 = (1 + i)^3$

➢ $i = 0.05$

i is the Interest rate per conversion period (six months) = 0.05 = 5% &

Interest rate per annum = 5% × 2 = 10%

Example 19: A certain sum invested at 4% per annum compounded semi-annually amounts to ₹78,030 at the end of one year. Find the sum.

Solution: Here A_n = 78,030

n = 2 × 1 = 2

i = 4 × 1/2 % = 2% = 0.02

P(in ₹) = ?

We have

$$A_n = P(1 + i)^n$$

➢ $A_2 = P(1 + 0.02)^2$

➢ $78{,}030 = P\,(1.02)^2$

➢ $P = \dfrac{78{,}030}{(1.02)^2}$

$= 75{,}000$

Thus the sum invested is ₹ 75,000 at the begining of 1 year.

Example 20: ₹ 16,000 invested at 10% p.a. compounded semi-annually amounts to ₹ 18,522. Find the time period of investment.

Solution: Here P = ₹ 16,000

A_n = ₹ 18,522

i = 10 × 1/2 % = 5% = 0.05

$n = ?$

We have $A_n = P(1 + i)^n$

➢ $18{,}522 = 16{,}000(1+0.05)^n$

➢ $\frac{18{,}522}{16{,}000} = (1.05)^n$

➢ $(1.157625) = (1.05)^n$

➢ $(1.05)^3 = (1.05)^n$

➢ $n = 3$

Therefore time period of investment is three half years i.e. $1\frac{1}{2}$ years.

Example 21: A person opened an account on April, 2011 with a deposit of ₹ 800. The account paid 6% interest compounded quarterly. On October 1 2011 he closed the account and added enough additional money to invest in a 6 month time-deposit for ₹ 1,000, earning 6% compounded monthly.

(a) How much additional amount did the person invest on October 1?

(b) What was the maturity value of his time deposit on April 1 2012?

(c) How much total interest was earned?

Given that $(1 + i)^n$ is 1.03022500 for i=1½ % n = 2 and $(1+ i)^n$ is 1.03037751 for i = ½ % and n = 6.

Solution: (a) The initial investment earned interest for April-June and July- September quarter i.e. for two quarters. In this case i = 6/4 = 1½ % = 0.015, n $\left[n = \frac{6}{12} \times 4\right] = 2$

$$\text{and the compounded amount} = 800(1 + 0.015)^2$$
$$= 800 \times 1.03022500$$
$$= ₹\ 824.18$$

$$\text{The additional amount invested} = ₹\ (1{,}000 - 824.18)$$
$$= ₹\ 175.82$$

(b) In this case the time-deposit earned interest compounded monthly for six months.

Here $i = \frac{6}{12} = 1/2\ \% = (0.005)$ $n = 6$ and $P = ₹\ 1{,}000$

$= \frac{6}{12} \times 12$

Maturity value = $1{,}000(1+0.005)^6$

= 1,000×1.03037751

= ₹ 1,030.38

(c) Total interest earned = ₹ (24.18+30.38) = ₹ 54.56

4.5 EFFECTIVE RATE OF INTEREST

If interest is compounded more than once a year the effective interest rate for a year exceeds the per annum interest rate. Suppose you invest ₹ 10,000 for a year at the rate of 6% per annum compounded semi annually. Effective interest rate for a year will be more than 6% per annum since interest is being compounded more than once in a year. For computing effective rate of interest first we have to compute the interest. Let us compute the interest.

Interest for first six months = ₹ 10,000 × 6/100 × 6/12

= ₹ 300

Principal for calculation of interest for next six months

= Principal for first period one + Interest for first period

= ₹ (10,000 + 300)

= ₹ 10,300

Interest for next six months = ₹ 10,300 × 6/100 × 6/12 = ₹ 309

Total interest earned during the current year

= Interest for first six months + Interest for next six months

= ₹ (300 + 309) = ₹ 609

Interest of ₹ 609 can also be computed directly from the formula of compound interest.

We can compute effective rate of interest by following formula

I = PEt

Where I = Amount of interest

E = Effective rate of interest in decimal

t = Time period

P = Principal amount

Putting the values we have

$609 = 10{,}000 \times E \times 1$

$\Rightarrow E = \dfrac{609}{10{,}000}$

$= 0.0609$ or

$= 6.09\%$

Thus if we compound the interest more than once a year effective interest rate for the year will be more than actual interest rate per annum. But if interest is compounded annually effective interest rate for the year will be equal to actual interest rate per annum.

So effective interest rate can be defined as the equivalent annual rate of interest compounded annually if interest is compounded more than once in a year.

The effective interest rate can be computed directly by following formula:

$$E = (1 + i)^n - 1$$

Where E is the effective interest rate

i = actual interest rate in decimal

n = number of conversion period

Example 22: ₹ 5,000 is invested in a Term Deposit Scheme that fetches interest 6% per annum compounded quarterly. What will be the interest after one year? What is effective rate of interest?

Solution: We know that

$$I = P\left[(1+i)^n - 1\right]$$

Here P = ₹ 5,000

i = 6% p.a. = 0.06 p.a. or 0.015 per quarter

n = 4

and I = amount of compound interest

putting the values we have

$$I = ₹\,5{,}000\left[(1+0.015)^4 - 1\right]$$

$$= ₹\,5{,}000 \times 0.06136355$$

$$= ₹\,306.82$$

For effective rate of interest using I = PEt we find

$$306.82 = 5{,}000 \times E \times 1.$$

$$\Rightarrow E = \frac{306.82}{5000}$$

$$= 0.0613 \text{ or } 6.13\%$$

Note: We may arrive at the same result by using

$$E = (1+i)^n - 1$$

$$\Rightarrow E = (1 + 0.015)^4 - 1$$

$$= 1.0613 - 1$$

$$= .0613 \text{ or } 6.13\%$$

We may also note that effective rate of interest is not related to the amount of principal. It is related to the interest rate and frequency of compounding the interest.

Example 23: Find the amount of compound interest and effective rate of interest if an amount of ₹ 20,000 is deposited in a bank for one year at the rate of 8% per annum compounded semi annually.

Solution: We know that

$$I = P\left[(1+i)^n - 1\right]$$

here $P = ₹\ 20{,}000$

$i = 8\%$ p.a. $= 8/2\ \%$ semi annually $= 0.04$

$n = 2$

$$I = ₹\ 20{,}000\left[(1+0.04)^2 - 1\right]$$

$$= ₹\ 20{,}000 \times 0.0816$$

$$= ₹\ 1{,}632$$

Effective rate of interest:

We know that

$$I = PEt$$

➢ $1{,}632 = 20{,}000 \times E \times 1$

➢ $E = \dfrac{1632}{20000} = 0.0816$

$= 8.16\%$

Effective rate of interest can also be computed by following formula

$$E = (1 + i)^n - 1$$

$$= (1 + 0.04)^2 - 1$$

$$= 0.0816 \quad \text{or } 8.16\%$$

Example 24: Which is a better investment 3% per year compounded monthly or 3.2% per year simple interest? Given that $(1+0.0025)^{12} = 1.0304$.

Solution: $i = 3/12 = 0.25\% = 0.0025$

$n = 12$

$E = (1 + i)^n - 1$

$= (1 + 0.0025)^{12} - 1$

$= 1.0304 - 1 = 0.0304$

$= 3.04\%$

Effective rate of interest (E) being less than 3.2%, the simple interest 3.2% per year is the better investment.

EXERCISE 4 (B)

Choose the most appropriate option (a) (b) (c) or (d).

1. If P = ₹ 1,000, R = 5% p.a, n = 4; What is Amount and C.I. is

   (a) ₹ 1,215.50, ₹ 215.50 (b) ₹ 1,125, ₹ 125
   (c) ₹ 2,115, ₹ 115 (d) none of these

2. ₹ 100 will become after 20 years at 5% p.a compound interest of

   (a) ₹ 250 (b) ₹ 205 (c) ₹ 165.33 (d) none of these

3. The effective rate of interest corresponding to a nominal rate 3% p.a payable half yearly is

   (a) 3.2% p.a (b) 3.25% p.a (c) 3.0225% p.a (d) none of these

4. A machine is depreciated at the rate of 20% on reducing balance. The original cost of the machine was ₹ 1,00,000 and its ultimate scrap value was ₹ 30,000. The effective life of the machine is

   (a) 4.5 years (appx.) (b) 5.4 years (appx.)
   (c) 5 years (appx.) (d) none of these

5. If A = ₹ 1,000, n = 2 years, R = 6% p.a compound interest payable half-yearly, then principal (P) is

   (a) ₹ 888.50 (b) ₹ 885 (c) 800 (d) none of these

6. The population of a town increases every year by 2% of the population at the beginning of that year. The number of years by which the total increase of population be 40% is

   (a) 7 years (b) 10 years (c) 17 years (app) (d) none of these

7. The difference between C.I and S.I on a certain sum of money invested for 3 years at 6% p.a is ₹ 110.16. The principle is

   (a) ₹ 3,000 (b) ₹ 3,700 (c) ₹ 12,000 (d) ₹ 10,000

8. The useful life of a machine is estimated to be 10 years and cost ₹ 10,000. Rate of depreciation is 10% p.a. The scrap value at the end of its life is

   (a) ₹ 3,486.78 (b) ₹ 4,383 (c) ₹ 3,400 (d) none of these

9. The effective rate of interest corresponding a nominal rate of 7% p.a convertible quarterly is

   (a) 7% (b) 7.5% (c) 5% (d) 7.18%

10. The C.I on ₹ 16000 for 1 ½ years at 10% p.a payable half -yearly is

    (a) ₹ 2,222 (b) ₹ 2,522 (c) ₹ 2,500 (d) none of these

11. The C.I on ₹ 40000 at 10% p.a for 1 year when the interest is payable quarterly is

    (a) ₹ 4,000 (b) ₹ 4,100 (c) ₹ 4,152.51 (d) none of these

12. The difference between the S.I and the C.I on ₹ 2,400 for 2 years at 5% p.a is

    (a) ₹ 5 (b) ₹ 10 (c) ₹ 16 (d) ₹ 6

13. The annual birth and death rates per 1,000 are 39.4 and 19.4 respectively. The number of years in which the population will be doubled assuming there is no immigration or emigration is

    (a) 35 years. (b) 30 years. (c) 25 years (d) none of these

14. The C.I on ₹ 4,000 for 6 months at 12% p.a payable quarterly is

    (a) ₹ 243.60 (b) ₹ 240 (c) ₹ 243 (d) none of these

4.6 ANNUITY

In many cases you must have noted that your parents have to pay an equal amount of money regularly like every month or every year. For example payment of life insurance premium, rent of your house (if you stay in a rented house), payment of housing loan, vehicle loan etc. In all these cases they pay a constant amount of money regularly. Time period between two consecutive payments may be one month, one quarter or one year.

Sometimes some people received a fixed amount of money regularly like pension, rent of house etc. In all these cases annuity comes into the picture. When we pay (or receive) a fixed amount of money periodically over a specified time period we create an annuity.

Thus annuity can be defined as a sequence of periodic payments (or receipts) regularly over a specified period of time.

There is a special kind of annuity also, that is called Perpetuity. It is one where the receipt or payment takes place forever. Since the payment is forever we cannot compute a future value of perpetuity. However we can compute the present value of the perpetuity. We will discuss later about future value and present value of annuity.

To be called annuity a series of payments (or receipts) must have following features:

(1) Amount paid (or received) must be constant over the period of annuity and

(2) Time interval between two consecutive payments (or receipts) must be the same.

Consider following tables. Can payments/receipts shown in the table for five years be called annuity?

Table - 4.1

Year end	Payments/Receipts (₹)
I	5,000
II	6,000
III	4,000
IV	5,000
V	7,000

Table - 4.2

Year end	Payments/Receipts (₹)
I	5,000
II	5,000
III	–
IV	5,000
V	5,000

Table - 4.3

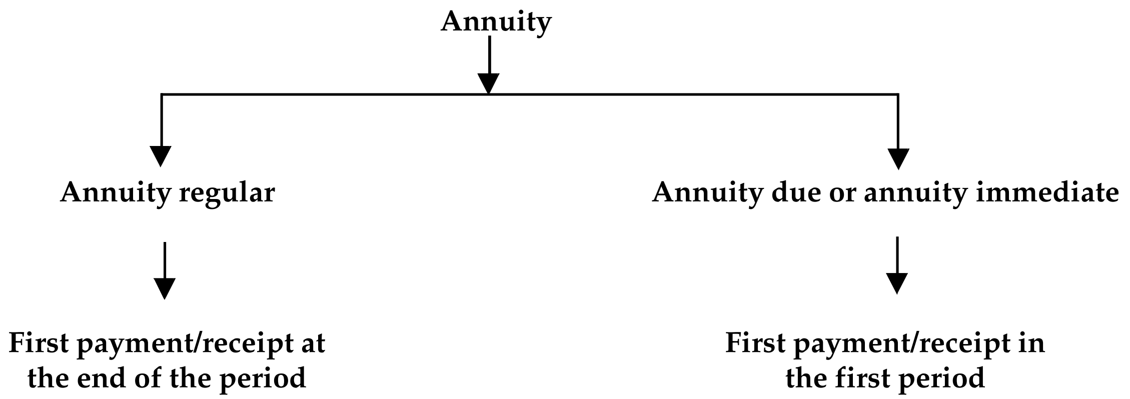
Year end	Payments/Receipts (₹)
I	5,000
II	5,000
III	5,000
IV	5,000
V	5,000

Payments/Receipts shown in table 4.1 cannot be called annuity. Payments/Receipts though have been made at regular intervals but amount paid are not constant over the period of five years.

Payments/receipts shown in table 4.2 cannot also be called annuity. Though amounts paid/ received are same in every year but time interval between different payments/receipts is not equal. You may note that time interval between second and third payment/receipt is two year and time interval between other consecutive payments/receipts (first and second third and fourth and fourth and fifth) is only one year. You may also note that for first two years the payments/ receipts can be called annuity.

Now consider table 4.3. You may note that all payments/receipts over the period of 5 years are constant and time interval between two consecutive payments/receipts is also same i.e. one year. Therefore payments/receipts as shown in table-4.3 can be called annuity.

4.6.1 Annuity regular and Annuity due/immediate



Annuity may be of two types:

(1) **Annuity regular:** In annuity regular first payment/receipt takes place at the end of first period. Consider following table:

Table - 4.4

Year end	Payments/Receipts (₹)
I	5,000
II	5,000
III	5,000
IV	5,000
V	5,000

We can see that first payment/receipts takes place at the end of first year therefore it is an annuity regular.

(2) **Annuity Due or Annuity Immediate:** When the first receipt or payment is made today (at the beginning of the annuity) it is called annuity due or annuity immediate. Consider following table:

Table - 4.5

In the beginning of	Payment/Receipt (₹)
I year	5,000
II year	5,000
III year	5,000
IV year	5,000
V year	5,000

We can see that first receipt or payment is made in the beginning of the first year. This type of annuity is called annuity due or annuity immediate.

4.7 FUTURE VALUE

Future value is the cash value of an investment at some time in the future. It is tomorrow's value of today's money compounded at the rate of interest. Suppose you invest ₹ 1,000 in a fixed deposit that pays you 7% per annum as interest. At the end of first year you will have ₹ 1,070. This consist of the original principal of ₹ 1,000 and the interest earned of ₹ 70. ₹ 1,070 is the future value of ₹ 1,000 invested for one year at 7%. We can say that ₹ 1000 today is worth ₹ 1070 in one year's time if the interest rate is 7%.

Now suppose you invested ₹ 1,000 for two years. How much would you have at the end of the second year. You had ₹ 1,070 at the end of the first year. If you reinvest it you end up having ₹ 1,070(1+0.07)= ₹ 1144.90 at the end of the second year. Thus ₹ 1,144.90 is the future value of ₹ 1,000 invested for two years at 7%. We can compute the future value of a single cash flow by applying the formula of compound interest.

We know that

$$A_n = P(1+i)^n$$

Where A = Accumulated amount

n = number of conversion period

i = rate of interest per conversion period in decimal

P = principal

Future value of a single cash flow can be computed by above formula. Replace A by future value (F) and P by single cash flow (C.F.) therefore

$$F = C.F.\ (1 + i)^n$$

Example 25: You invest ₹ 3000 in a two year investment that pays you 12% per annum. Calculate the future value of the investment.

Solution: We know

$$F = C.F.\ (1 + i)^n$$

where F = Future value

C.F. = Cash flow = ₹ 3,000

i = rate of interest = 0.12

n = time period = 2

$$F = ₹\ 3,000(1+0.12)^2$$

$$= ₹\ 3,000 \times 1.2544$$

$$= ₹\ 3,763.20$$

4.7.1 Future value of an annuity regular

Now we can discuss how do we calculate future value of an annuity.

Suppose a constant sum of ₹ 1 is deposited in a savings account at the end of each year for four years at 6% interest. This implies that ₹ 1 deposited at the end of the first year will grow for three years, ₹ 1 at the end of second year for 2 years, ₹ 1 at the end of the third year for one year and ₹ 1 at the end of the fourth year will not yield any interest. Using the concept of compound interest we can compute the future value of annuity. The compound value (compound amount) of ₹ 1 deposited in the first year will be

$$A_3 = ₹\ 1\ (1 + 0.06)^3$$

$$= ₹\ 1.191$$

The compound value of ₹ 1 deposited in the second year will be

$$A_2 = ₹\ 1\ (1 + 0.06)^2$$

$$= ₹\ 1.124$$

The compound value of ₹ 1 deposited in the third year will be

A_1 = ₹ 1 $(1 + 0.06)^1$

= ₹ 1.06

and the compound value of ₹ 1 deposited at the end of fourth year will remain ₹ 1.

The aggregate compound value of ₹ 1 deposited at the end of each year for four years would be:

₹ (1.191 + 1.124 + 1.060 + 1.00) = ₹ 4.375

This is the compound value of an annuity of ₹ 1 for four years at 6% rate of interest.

The above computation is summarized in the following table:

Table 4.6

End of year	Amount Deposit (₹)	Future value at the end of fourth year (₹)
0	–	–
1	1	$1\,(1 + 0.06)^3 = 1.191$
2	1	$1\,(1 + 0.06)^2 = 1.124$
3	1	$1\,(1 + 0.06)^1 = 1.060$
4	1	$1\,(1 + 0.06)^0 = 1$
	Future Value	4.375

The computation shown in the table can be expressed as follows:

$$A\,(4, i) = A\,(1 + i)^0 + A\,(1 + i) + A(1 + i)^2 + A(1 + i)^3$$

i.e. $A\,(4, i) = A\left[1+(1+i) +(1+i)^2+(1 + i)^3\right]$

In above equation A is annuity, A (4, i) is future value at the end of year four, i is the rate of interest shown in decimal.

We can extend above equation for n periods and rewrite as follows:

$$A\,(n, i) = A\,(1 + i)^0 + A\,(1 + i)^1 + \ldots\ldots\ldots + A\,(1 + i)^{n-2} + A\,(1 + i)^{n-1}$$

Here A = Re.1

Therefore

$$A\,(n, i) = 1\,(1 + i)^0 + 1\,(1 + i)^1 + \ldots\ldots\ldots + 1\,(1 + i)^{n-2} + 1\,(1 + i)^{n-1}$$

$$= 1 + (1 + i)^1 + \ldots\ldots\ldots + (1 + i)^{n-2} + (1 + i)^{n-1}$$

[a geometric series with first term 1 and common ratio (1+ i)]

$$= \frac{1.\left[1-(1+i)^n\right]}{1-(1+i)}$$

$$= \frac{1-(1+i)^n}{-i}$$

$$= \frac{(1+i)^n-1}{i}$$

If A be the periodic payments, the future value A(n, i) of the annuity is given by

$$A(n, i) \quad = \quad A\left[\frac{(1+i)^n-1}{i}\right]$$

Example 26: Find the future value of an annuity of ₹ 500 made annually for 7 years at interest rate of 14% compounded annually. Given that $(1.14)^7 = 2.5023$.

Solution: Here annual payment A = ₹ 500

n = 7

i = 14% = 0.14

Future value of the annuity

$$A(7, 0.14) = 500\left[\frac{(1+0.14)^7-1}{(0.14)}\right]$$

$$= \frac{500\times(2.5023-1)}{0.14}$$

= ₹ 5,365.35

Example 27: ₹ 200 is invested at the end of each month in an account paying interest 6% per year compounded monthly. What is the future value of this annuity after 10^{th} payment? Given that $(1.005)^{10} = 1.0511$

Solution: Here A = ₹ 200

n = 10

i = 6% per annum = 6/12% per month = 0.005

Future value of annuity after 10 months is given by

$$A(n, i) = A\left[\frac{(1+i)^n-1}{i}\right]$$

$$A(10, 0.005) = 200\left[\frac{(1+0.005)^{10}-1}{0.005}\right]$$

$$= 200\left[\frac{1.0511-1}{0.005}\right]$$

= 200 × 10.22

= ₹ 2,044

4.7.2 Future value of Annuity due or Annuity Immediate

As we know that in Annuity due or Annuity immediate first receipt or payment is made today. Annuity regular assumes that the first receipt or the first payment is made at the end of first period. The relationship between the value of an annuity due and an ordinary annuity in case of future value is:

Future value of an Annuity due/Annuity immediate = Future value of annuity regular x (1+i) where i is the interest rate in decimal.

Calculating the future value of the annuity due involves two steps.

Step-1 Calculate the future value as though it is an ordinary annuity.

Step-2 Multiply the result by (1+ i)

Example 28: Z invests ₹ 10,000 every year starting from today for next 10 years. Suppose interest rate is 8% per annum compounded annually. Calculate future value of the annuity. Given that $(1 + 0.08)^{10} = 2.15892500$.

Solution: Step-1: Calculate future value as though it is an ordinary annuity.

Future value of the annuity as if it is an ordinary annuity

$$= ₹\ 10{,}000\left[\frac{(1+0.08)^{10}-1}{0.08}\right]$$

= ₹ 10,000 × 14.4865625

= ₹ 1,44,865.625

Step-2: Multiply the result by (1 + i)

= ₹ 1,44,865.625 × (1+0.08)

= ₹ 1,56,454.875

4.8 PRESENT VALUE

We have read that future value is tomorrow's value of today's money compounded at some interest rate. We can say present value is today's value of tomorrow's money discounted at the interest rate. Future value and present value are related to each other in fact they are the reciprocal of each other. Let's go back to our fixed deposit example. You invested ₹ 1000 at 7% and get ₹ 1,070 at the end of the year. If ₹ 1,070 is the future value of today's ₹ 1000 at 7% then ₹ 1,000 is present value of tomorrow's ₹ 1,070 at 7%. We have also seen that if we invest ₹ 1,000 for two years at 7% per annum we will get ₹ 1,144.90 after two years. It means ₹ 1,144.90 is the future value of today's ₹ 1,000 at 7% and ₹ 1,000 is the present value of ₹ 1,144.90 where time period is two years and rate of interest is 7% per annum. We can get the present value of a cash flow (inflow or outflow) by applying compound interest formula.

The present value P of the amount A_n due at the end of n period at the rate of i per interest period may be obtained by solving for P the below given equation

$A_n = P(1 + i)^n$

i.e. $P = \frac{A_n}{(1+i)^n}$

- Computation of P may be simple if we make use of either the calculator or the present value table showing values of $\frac{1}{(1+i)^n}$ for various time periods/per annum interest rates.
- For positive i the factor $\frac{1}{(1+i)^n}$ is always less than 1 indicating thereby future amount has smaller present value.

Example 29: What is the present value of ₹ 1 to be received after two years compounded annually at 10% interest rate?

Solution: Here A_n = ₹1

i = 10% = 0.1

n= 2

Required present value $= \frac{A_n}{(1+i)^n}$

$= \frac{1}{(1+0.1)^2}$

$= \frac{1}{1.21} = 0.8264$

= ₹ 0.83

Thus ₹ 0.83 shall grow to ₹ 1 after 2 years at 10% interest rate compounded annually.

Example 30: Find the present value of ₹ 10,000 to be required after 5 years if the interest rate be 9%. Given that $(1.09)^5$=1.5386.

Solution: Here i = 0.09 = 9%

n = 5

A_n = 10,000

Required present value $= \dfrac{A_n}{(1+i)^n}$

$$= \frac{10,000}{(1+0.09)^5}$$

$$= \frac{10,000}{1.5386} = ₹\ 6,499.42$$

4.8.1 Present value of an Annuity regular: We have seen how compound interest technique can be used for computing the future value of an Annuity. We will now see how we compute present value of an annuity. We take an example, Suppose your mom promise you to give you ₹ 1,000 on every 31st December for the next five years. Suppose today is 1st January. How much money will you have after five years from now if you invest this gift of the next five years at 10%? For getting answer we will have to compute future value of this annuity.

But you don't want ₹ 1,000 to be given to you each year. You instead want a lump sum figure today. Will you get ₹ 5,000. The answer is no. The amount that she will give you today will be less than ₹ 5,000. For getting the answer we will have to compute the present value of this annuity. For getting present value of this annuity we will compute the present value of these amounts and then aggregate them. Consider following table:

Table 4.7

Year End	Gift Amount (₹)	Present Value $[A_n / (1 + i)^n]$		
I	1,000	$1,000/(1 + 0.1)$	=	909.091
II	1,000	$1,000/(1 + 0.1)^2$	=	826.446
III	1,000	$1,000/(1 + 0.1)^3$	=	751.315
IV	1,000	$1,000/(1 + 0.1)^4$	=	683.013
V	1,000	$1,000/(1 + 0.1)^5$	=	620.921
	Present Value		**=**	**3790.786**

Thus the present value of annuity of ₹ 1,000 for 5 years at 10% is ₹ 3,790.79

It means if you want lump sum payment today instead of ₹ 1,000 every year you will get ₹ 3,790.79.

The above computation can be written in formula form as below.

The present value (V) of an annuity (A) is the sum of the present values of the payments.

$$\therefore \quad V = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \frac{A}{(1+i)^4} + \frac{A}{(1+i)^5}$$

We can extend above equation for n periods and rewrite as follows:

$$V = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \dots\dots\dots + \frac{A}{(1+i)^{n-1}} + \frac{A}{(1+i)^n} \quad \dots\dots\dots\dots(1)$$

multiplying throughout by $\frac{1}{(1+i)}$ we get

$$\frac{V}{(1+i)} = \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \ldots\ldots\ldots\ldots + \frac{A}{(1+i)^n} + \frac{A}{(1+i)^{n+1}} \ldots\ldots\ldots\ldots(2)$$

subtracting (2) from (1) we get

$$V - \frac{V}{(1+i)} = \frac{A}{(1+i)^1} - \frac{A}{(1+i)^{n+1}}$$

Or $$V(1+i) - V = A - \frac{A}{(1+i)^n}$$

Or $$Vi = A\left[1 - \frac{1}{(1+i)^n}\right]$$

$$\therefore \quad V = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A.P(n, i)$$

Where, $$P(n, i) = \frac{(1+i)^n - 1}{i(1+i)^n}$$

Consequently $A = \frac{V}{P(n,i)}$ which is useful in problems of amortization.

A loan with fixed rate of interest is said to be amortized if entire principal and interest are paid over equal periods of time by way of sequence of equal payment.

$A = \frac{V}{P(n,i)}$ can be used to compute the amount of annuity if we have present value (V), n the number of time period and/the rate of interest in decimal.

Suppose your dad purchases a car for ₹ 5,50,000. He gets a loan of ₹ 5,00,000 at 15% p.a. from a Bank and balance 50,000 he pays at the time of purchase. Your dad has to pay whole amount of loan in 12 equal monthly instalments with interest starting from the end of first month.

Now we have to calculate how much money has to be paid at the end of every month. We can compute equal instalment by following formula

$$A = \frac{V}{P(n,i)}$$

Here V = ₹ 5,00,000

n = 12

$$i = \frac{0.15}{12} = 0.0125$$

$$P(n, i) = \frac{(1+i)^n - 1}{i(1+i)^n}$$

$$P(12, 0.0125) = \frac{(1+0.0125)^{12} - 1}{0.0125(1+0.0125)^{12}}$$

$$= \frac{1.16075452 - 1}{0.0125 \times 1.16075452}$$

$$= \frac{0.16075452}{0.01450943} = 11.079$$

$$\therefore \quad A = \frac{5,00,000}{11.079} = ₹\ 45130.43$$

Therefore your dad will have to pay 12 monthly instalments of ₹ 45,130.43.

Example 31: S borrows ₹ 5,00,000 to buy a house. If he pays equal instalments for 20 years and 10% interest on outstanding balance what will be the equal annual instalment?

Solution: We know

$$A = \frac{V}{P(n,i)}$$

Here V = ₹ 5,00,000

n = 20

i = 10% p.a.= 0.10

$$\therefore \quad A = \frac{V}{P(n,i)} = ₹\ \frac{5,00,000}{P(20,\ 0.10)}$$

$$= ₹\ \frac{5,00,000}{8.51356}$$ [P(20, 0.10) = 8.51356 from table 2(a)]

= ₹ 58,729.84

Example 32: ₹ 5,000 is paid every year for ten years to pay off a loan. What is the loan amount if interest rate be 14% per annum compounded annually?

Solution: V = A × P(n, i)

Here A = ₹ 5,000

n = 10

$i = 0.14$

$V = 5000 \times P(10, 0.14)$

$= 5000 \times 5.21611 \qquad = ₹\ 26,080.55$

Therefore the loan amount is ₹ 26,080.55

Note: Value of P(10, 0.14) can be seen from table 2(a) or it can be computed by formula derived in preceding paragraph.

Example 33: Y bought a TV costing ₹ 13,000 by making a down payment of ₹ 3000 and agreeing to make equal annual payment for four years. How much would be each payment if the interest on unpaid amount be 14% compounded annually?

Solution: In the present case we have present value of the annuity i.e. ₹ 10,000 (13,000-3,000) and we have to calculate equal annual payment over the period of four years.

We know that

$$V = A.P(n, i)$$

Here $n = 4$ and $i = 0.14$

$$A = \frac{V}{P(n, i)}$$

$$= \frac{10,000}{P(4,\ 0.14)}$$

$$= \frac{10,000}{2.91371} \text{ [from table 2(a)]}$$

$$= ₹\ 3,432.05$$

Therefore each payment would be ₹ 3,432.05

4.8.2 Present value of annuity due or annuity immediate

Present value of annuity due/immediate for n years is the same as an annuity regular for (n-1) years plus an initial receipt or payment in beginning of the period. Calculating the present value of annuity due involves two steps.

Step 1: Compute the present value of annuity as if it were a annuity regular for one period short.

Step 2: Add initial cash payment/receipt to the step 1 value.

Example 34: Suppose your mom decides to gift you ₹ 10,000 every year starting from today for the next five years. You deposit this amount in a bank as and when you receive and get 10% per annum interest rate compounded annually. What is the present value of this annuity?

Solution: It is an annuity immediate. For calculating value of the annuity immediate following steps will be followed:

Step 1: Present value of the annuity as if it were a regular annuity for one year less i.e. for four years

= ₹ 10,000 × P (4, 0.10)

= ₹ 10,000 × 3.16987

= ₹ 31,698.70

Step 2: Add initial cash deposit to the step 1 value

₹ (31,698.70+10,000) = ₹ 41,698.70

4.9 SINKING FUND

It is the fund credited for a specified purpose by way of sequence of periodic payments over a time period at a specified interest rate. Interest is compounded at the end of every period. Size of the sinking fund deposit is computed from A = P.A(n, i) where A is the amount to be saved, P the periodic payment, n the payment period.

Example 35: How much amount is required to be invested every year so as to accumulate ₹ 300000 at the end of 10 years if interest is compounded annually at 10%?

Solution: Here A = 3,00,000

n = 10

i = 0.1

Since A = P × A (n, i)

300000 = P.A.(10, 0.1)

= P × 15.9374248

$$\therefore \quad P = \frac{3,00,000}{15.9374248} = ₹\ 18,823.62$$

This value can also be calculated by the formula of future value of annuity regular.
We know that

$$A(n\ i) = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$300000 = A\left[\frac{(1+0.1)^{10} - 1}{0.1}\right]$$

$$300000 = A \times 15.9374248$$

$$A = \frac{3,00,000}{15.9374248}$$

$$= ₹\ 18,823.62$$

4.10 APPLICATIONS

4.10.1 Leasing

Leasing is a financial arrangement under which the owner of the asset (lessor) allows the user of the asset (lessee) to use the asset for a defined period of time(lease period) for a consideration (lease rental) payable over a given period of time. This is a kind of taking an asset on rent. How can we decide whether a lease agreement is favourable to lessor or lessee, it can be seen by following example.

Example 36: ABC Ltd. wants to lease out an asset costing ₹ 3,60,000 for a five year period. It has fixed a rental of ₹ 1,05,000 per annum payable annually starting from the end of first year. Suppose rate of interest is 14% per annum compounded annually on which money can be invested by the company. Is this agreement favourable to the company?

Solution: First we have to compute the present value of the annuity of ₹ 1,05,000 for five years at the interest rate of 14% p.a. compounded annually.

The present value V of the annuity is given by

$$V = A.P\,(n, i)$$

$$= 1{,}05{,}000 \times P(5, 0.14)$$

$$= 1{,}05{,}000 \times 3.43308 = ₹\ 3{,}60{,}473.40$$

which is greater than the initial cost of the asset and consequently leasing is favourable to the lessor.

Example 37: A company is considering proposal of purchasing a machine either by making full payment of ₹ 4,000 or by leasing it for four years at an annual rate of ₹ 1,250. Which course of action is preferable if the company can borrow money at 14% compounded annually?

Solution: The present value V of annuity is given by

$$V = A.P\,(n, i)$$

$$= 1{,}250 \times P\,(4, 0.14)$$

$$= 1{,}250 \times 2.91371 = ₹\ 3{,}642.11$$

which is less than the purchase price and consequently leasing is preferable.

4.10.2 Capital Expenditure (investment decision)

Capital expenditure means purchasing an asset (which results in outflows of money) today in anticipation of benefits (cash inflow) which would flow across the life of the investment. For taking investment decision we compare the present value of cash outflow and present value of cash inflows. If present value of cash inflows is greater than present value of cash outflows decision should be in the favour of investment. Let us see how do we take capital expenditure (investment) decision.

Example 38: A machine can be purchased for ₹ 50000. Machine will contribute ₹ 12000 per year for the next five years. Assume borrowing cost is 10% per annum compounded annually. Determine whether machine should be purchased or not.

Solution: The present value of annual contribution

$$V = A.P(n, i)$$
$$= 12{,}000 \times P(5, 0.10)$$
$$= 12{,}000 \times 3.79079$$
$$= ₹\ 45{,}489.48$$

which is less than the initial cost of the machine. Therefore machine must not be purchased.

Example 39: A machine with useful life of seven years costs ₹ 10,000 while another machine with useful life of five years costs ₹ 8,000. The first machine saves labour expenses of ₹ 1,900 annually and the second one saves labour expenses of ₹ 2,200 annually. Determine the preferred course of action. Assume cost of borrowing as 10% compounded per annum.

Solution: The present value of annual cost savings for the first machine

$$= ₹\ 1{,}900 \times P(7, 0.10)$$
$$= ₹\ 1{,}900 \times 4.86842$$
$$= ₹\ 9{,}249.99$$
$$= ₹\ 9{,}250$$

Cost of machine being ₹ 10,000 it costs more by ₹ 750 than it saves in terms of labour cost.

The present value of annual cost savings of the second machine

$$= ₹\ 2{,}200 \times P(5, 0.10)$$
$$= ₹\ 2{,}200 \times 3.79079$$
$$= ₹\ 8{,}339.74$$

Cost of the second machine being ₹ 8,000 effective savings in labour cost is ₹ 339.74. Hence the second machine is preferable.

4.10.3 Valuation of Bond

A bond is a debt security in which the issuer owes the holder a debt and is obliged to repay the principal and interest. Bonds are generally issued for a fixed term longer than one year.

The bond issuer enters into contract with bondholder to pay interest.

Example 40: An investor intends purchasing a three year ₹ 1,000 par value bond having nominal interest rate of 10%. At what price the bond may be purchased now if it matures at par and the investor requires a rate of return of 14%?

Solution: Present value of the bond

$$= \frac{100}{(1+0.14)^1} + \frac{100}{(1+0.14)^2} + \frac{100}{(1+0.14)^3} + \frac{1{,}000}{(1+0.14)^3}$$
$$= 100 \times 0.87719 + 100 \times 0.769467 + 100 \times 0.674\ 972 + 1{,}000 \times 0.674972$$
$$= 87.719 + 76.947 + 67.497 + 674.972$$
$$= 907.125$$

Thus the purchase value of the bond is ₹ 907.125

4.11 PERPETUITY

Perpetuity is an annuity in which the periodic payments or receipts begin on a fixed date and continue indefinitely or perpetually. Fixed coupon payments on permanently invested (irredeemable) sums of money are prime examples of perpetuities.

The formula for evaluating perpetuity is relatively straight forward. Two points which are important to understand in this regard are:.

(a) The value of the perpetuity is finite because receipts that are anticipated far in the future have extremely low present value (today's value of the future cash flows).

(b) Additionally, because the principal is never repaid, there is no present value for the principal.

Therefore, the price of perpetuity is simply the coupon amount over the appropriate discount rate or yield.

4.11.1 Calculation of multi period perpetuity:

The formula for determining the present value of multi-period perpetuity is as follows:

$$PVA\infty = \frac{R}{(1+i)^1} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + ...+ \frac{R}{(1+i)} = \sum_{n=1}^{\infty} \frac{R}{(1+i)^n} = \frac{R}{i}$$

Where:

R = the payment or receipt each period

i = the interest rate per payment or receipt period

Example 41: Ramesh wants to retire and receive ₹ 3,000 a month. He wants to pass this monthly payment to future generations after his death. He can earn an interest of 8% compounded annually. How much will he need to set aside to achieve his perpetuity goal?

Solution:

R = ₹ 3,000

i = 0.08/12 or 0.00667

Substituting these values in the above formula, we get

$$PVA = \frac{₹\ 3{,}000}{0.00667}$$

$$= ₹\ 4{,}49{,}775$$

If he wanted the payments to start today, he must increase the size of the funds to handle the first payment. This is achieved by depositing ₹ 4,52,775 (PV of normal perpetuity + perpetuity received in the beginning = 4,49,775 + 3,000) which provides the immediate payment of

₹ 3,000 and leaves ₹ 4,49,775 in the fund to provide the future ₹ 3,000 payments.

4.11.2 Calculation of Growing Perpetuity:

A stream of cash flows that grows at a constant rate forever is known as growing perpetuity.

The formula for determining the present value of growing perpetuity is as follows:

$$PVA=\frac{R}{(1+i)^1}+\frac{R(1+g)}{(1+i)^2}+\frac{R(1+g)^2}{(1+i)^3}+.....+\frac{R(1+g)^\infty}{(1+i)^\infty}$$

$$\sum_{n=1}^{\infty}\frac{R(1+g)^{n-1}}{(1+i)^n}=\frac{R}{i-g}$$

Example 42: Assuming that the discount rate is 7% per annum, how much would you pay to receive ₹ 50, growing at 5%, annually, forever?

Solution:

$$PVA=\frac{R}{i-g}=\frac{50}{0.07-0.05}=2,500$$

Where

R = Cash flow stream, i = interest rate or discount rate, g = growth rate in interest

Calculating Rate of Return:

1) Calculating the rate of return provides important information that can be used for future investments. For example, if you invested in a stock that showed a substantial gain after several months of performance, you may decide to purchase more of that stock. If the stock showed a continual loss, it may be wise to conduct research to find a better-performing stock.

2) Calculating the rate of return is that it allows you to gauge your investment and decision-making skills. Investments that create a gain or profit are great. However, if you continually make investments at a loss, then you may want to change your investment strategies. A great attribute of successful business people is knowing how and when to make investments, as is knowing when to change strategies. With a firm grasp of calculating the rate of return, you can manage and monitor your investments at various stages to determine the outcome of your investments. This leads to a higher level of confidence and the skills necessary to be a savvy investor.

Net Present Value Technique (NPV): The net present value technique is a discounted cash flow method that considers the time value of money in evaluating capital investments. An investment has cash flows throughout its life, and it is assumed that a rupee of cash flow in the early years of an investment is worth more than a rupee of cash flow in a later year.

The net present value method uses a specified discount rate to bring all subsequent net cash inflows after the initial investment to their present values (the time of the initial investment is year 0).

Determining Discount Rate

Theoretically, the discount rate or desired rate of return on an investment is the rate of return the firm would have earned by investing the same funds in the best available alternative investment that has the same risk. Determining the best alternative opportunity available is difficult in practical terms so rather than using the true opportunity cost, organizations often use an alternative measure for the desired rate of return. An organization may establish a minimum rate of return that all capital projects must meet; this minimum could be based on an industry average or the cost of other investment opportunities. Many organizations choose to use the overall cost of capital or Weighted Average Cost of Capital (WACC) that an organization has incurred in raising funds or expects to incur in raising the funds needed for an investment.

The net present value of a project is the amount, in current value of rupees, the investment earns after paying cost of capital in each period.

4.12 NET PRESENT VALUE

Net present value = Present value of net cash inflow – Total net initial investment

Since it might be possible that some additional investment may also be required during the life time of the project then appropriate formula shall be:

Net present value = Present value of cash inflow – Present value of cash outflow

The steps to calculate net present value are:-

1. Determine the net cash inflow in each year of the investment.
2. Select the desired rate of return or discounting rate or Weighted Average Cost of Capital.
3. Find the discount factor for each year based on the desired rate of return selected.
4. Determine the present values of the net cash flows by multiplying the cash flows by respective the discount factors of respective period called Present Value (PV) of Cash flows
5. Total the amounts of all PVs of Cash Flows

Decision Rule:

If NPV > 0 Accept the Proposal

If NPV < 0 Reject the Proposal

Example 43: Compute the net present value for a project with a net investment of ₹ 1,00,000 and net cash flows for year one is ₹ 55,000; for year two is ₹ 80,000 and for year three is ₹ 15,000. Further, the company's cost of capital is 10%?

[PVIF @ 10% for three years are 0.909, 0.826 and 0.751]

Solution:

Year	*Net Cash Flows*	*PVIF @ 10%*	*Discounted Cash Flows*
0	(1,00,000)	1.000	(1,00,000)
1	55,000	0.909	49,995
2	80,000	0.826	66,080
3	15,000	0.751	11,265
Net Present Value			27,340

Recommendation: Since the net present value of the project is positive, the company should accept the project.

4.13 NOMINAL RATE OF RETURN

The nominal rate is the stated interest rate. If a bank pays 5% annually on a savings account, then 5% is the nominal interest rate. So if you deposit ₹ 100 for 1 year, you will receive ₹ 5 in interest. However, that Rs. 5 will probably be worth less at the end of the year than it would have been at the beginning. This is because inflation lowers the value of money. As goods, services, and assets, such as real estate, rise in price.

The nominal interest rate is conceptually the simplest type of interest rate. It is quite simply the stated interest rate of a given bond or loan. It is also defined as a stated interest rate. This interest works according to the simple interest and does not take into account the compounding periods.

Real Rate of Return: The real interest rate is so named because it states the "real" rate that the lender or investor receives after inflation is factored in; that is, the interest rate that exceeds the inflation rate.

A comparison of real and nominal interest rates can therefore be summed up in this equation:

Nominal Rate of Return – Inflation = Real Rate of Return

Nominal Interest Rate = Real Interest Rate + Inflation

Effective Rate:

It is the actual equivalent annual rate of interest at which an investment grows in value when interest is credited more often than once a year. If interest is paid *m* times in a year it can be found by calculating:

$$\left(1+\frac{i}{m}\right)^{m}-1$$

The chief advantage to knowing the difference between nominal, real and effective rates is that it allows consumers to make better decisions about their loans and investments. A loan with frequent compounding periods will be more expensive than one that compounds annually. A bond that only pays a 1% real interest rate may not be worth investors' time if they seek to grow their assets over time. These rates effectively reveal the true return that will be posted by a fixed-income investment and the true cost of borrowing for an individual or business.

Effective and nominal interest rates allow banks to use the number that looks most advantageous to the consumer. When banks are charging interest, they advertise the nominal rate, which is lower and does not reflect how much interest the consumer would owe on the balance after a full year of compounding. On the other hand, with deposit accounts where banks are paying interest, they generally advertise the effective rate because it is higher than the nominal rate.

4.14 COMPOUND ANNUAL GROWTH RATE (CAGR)

Compound Annual Growth Rate (CAGR) is a business and investing specific term for the smoothed annualized gain of an investment over a given time periodit is not an accounting term, but remains widely used, particularly in growth industries or to compare the growth rates of two investments because CAGR dampens the effect of volatility of periodic returns that can render arithmetic means irrelevant. CAGR is often used to describe the growth over a period of time of some element of the business, for example revenue, units delivered, registered users, etc.

$$\text{CAGR}(t_0, t_n) = \left(\frac{V(t_n)}{V(t_0)}\right)^{\frac{1}{t_n - t_0}} - 1$$

Where $V(t_0)$ = Beginning Period ; $V(t_n)$ = End Period

Example: Suppose the revenues of a company for four years, V(t) in the above formula, have been

Year	2013	2014	2015	2016
Revenues	100	120	160	210

Calculate Compound annual Growth Rate.

Solution:

$t_n - t_0$ = 2016 - 2013 = 3

The CAGR revenues over the three-year period from the end of 2013 to the end of 20016 is

$$\text{CAGR}(0, 3) = \left(\frac{210}{100}\right)^{\frac{1}{3}} - 1 = 1.2774 - 1 = 27.74\,\%$$

Applications: These are some of the common CAGR applications:

- Calculating average returns of investment funds.
- Demonstrating and comparing the performance of investment advisors.
- Comparing the historical returns of stocks with bonds or with a savings account.
- Forecasting future values based on the CAGR of a data series.
- Analyzing and communicating the behavior, over a series of years, of different business measures such as sales, market share, costs, customer satisfaction, and performance.

EXERCISE 4 (C)

Choose the most appropriate option (a) (b) (c) or (d).

1. The present value of an annuity of ₹ 3000 for 15 years at 4.5% p.a CI is

   (a) ₹ 23,809.41 (b) ₹ 32,218.76 (c) ₹ 32,908.41 (d) none of these

2. The amount of an annuity certain of ₹ 150 for 12 years at 3.5% p.a C.I is

   (a) ₹ 2,190.28 (b) ₹ 1,290.28 (c) ₹ 2,180.28 (d) none of these

3. A loan of ₹ 10,000 is to be paid back in 30 equal instalments. The amount of each installment to cover the principal and at 4% p.a CI is

   (a) ₹ 587.87 (b) ₹ 587 (c) ₹ 578.30 (d) none of these

4. A = ₹ 1,200 n = 12 years i = 0.08, V = ?

   Using the formula $V = \frac{A}{i}\left[1 - \frac{1}{(1+i)^n}\right]$ value of v will be

   (a) ₹ 3,039 (b) ₹ 3,990 (c) ₹ 9,930 (d) 9,043.30

5. a = ₹ 100 n = 10, *i* = 5% find the FV of annuity

   Using the formula FV = a / {1 + i) ^{n} – 1}, FV is equal to

   (a) ₹ 1,258 (b) ₹ 2,581 (c) ₹ 1,528 (d) none of these

6. If the amount of an annuity after 25 years at 5% p.a C.I is ₹ 50,000 the annuity will be

   (a) ₹ 1,406.90 (b) ₹ 1,047.62 (c) ₹ 1,146.90 (d) none of these

7. Given annuity of ₹ 100 amounts to ₹ 3137.12 at 4.5% p.a C. I. The number of years will be

   (a) 25 years (appx.) (b) 20 years (appx.) (c) 22 years (d) none of these

8. A company borrows ₹ 10,000 on condition to repay it with compound interest at 5% p.a by annual installments of ₹ 1000 each. The number of years by which the debt will be clear is

   (a) 14.2 years (b) 10 years (c) 12 years (d) none of these

9. Mr. X borrowed ₹ 5,120 at 12 ½ % p.a C.I. At the end of 3 yrs, the money was repaid along with the interest accrued. The amount of interest paid by him is

   (a) ₹ 2,100 (b) ₹ 2,170 (c) ₹ 2,000 (d) none of these

10. Mr. Paul borrows ₹ 20,000 on condition to repay it with C.I. at 5% p.a in annual installments of ₹ 2000 each. The number of years for the debt to be paid off is

    (a) 10 years (b) 12 years (c) 11 years (d) 14.2 years

11. A person invests ₹ 500 at the end of each year with a bank which pays interest at 10% p. a C.I. annually. The amount standing to his credit one year after he has made his yearly investment for the 12th time is.

(a) ₹ 11,761.36 (b) ₹ 10,000 (c) ₹ 12,000 (d) none of these

12. The present value of annuity of ₹ 5,000 per annum for 12 years at 4% p.a C.I. annually is

(a) ₹ 46,000 (b) ₹ 46,850 (c) ₹ 15,000 (d) ₹ 46,925.40

13. A person desires to create a fund to be invested at 10% CI per annum to provide for a prize of ₹ 300 every year. Using V = a/I find V and V will be

(a) ₹ 2,000 (b) ₹ 2,500 (c) ₹ 3,000 (d) none of these

SUMMARY

- **Time value of money:** Time value of money means that the value of a unity of money is different in different time periods. The sum of money received in future is less valuable than it is today. In other words the present worth of money received after some time will be less than money received today.
- **Interest:** Interest is the price paid by a borrower for the use of lender's money. If you borrow (or lend) some money from (or to) a person for a particular period you would pay (or receive) more money than your initial borrowing (or lending).
- **Simple interest:** is the interest computed on the principal for the entire period of borrowing.

  $$I = Pit$$
  $$A = P + I$$
  $$I = A - P$$

  Here, A = Accumulated amount (final value of an investment)

  P = Principal (initial value of an investment)

  i = Annual interest rate in decimal.

  I = Amount of Interest

  t = Time in years
- **Compound interest** in the interest that accrues when earnings for each specified period of time added to the principal thus increasing the principal base on which subsequent interest is computed.

  Formula for compound interest:

  $$A_n = P(1 + i)^n$$

  where, i = Annual rate of interest

  n = Number of conversion periods per year

  $$\text{Interest} = A_n - P = P(1 + i)^n - P$$

  n is total conversions i.e. t x no. of conversions per year
- **Effective Rate of Interest:** The effective interest rate can be computed directly by following formula:

  $$E = (1 + i)^n - 1$$

Where E is the effective interest rate

i = actual interest rate in decimal

n = number of conversion period

- Future value of a single cash flow can be computed by above formula. Replace A by future value (F) and P by single cash flow (C.F.) therefore

$$F = C.F.\ (1 + i)^n$$

- Annuity can be defined as a sequence of periodic payments (or receipts) regularly over a specified period of time.

  Annuity may be of two types:

  (i) Annuity regular: In annuity regular first payment/receipt takes place at the end of first period.

  (ii) Annuity Due or Annuity Immediate: When the first receipt or payment is made today (at the beginning of the annuity) it is called annuity due or annuity immediate.

- If A be the periodic payments, the future value A(n, i) of the annuity is given by

$$A(n, i) = A\left[\frac{(1+i)^n - 1}{i}\right]$$

- Future value of an Annuity due/Annuity immediate = Future value of annuity regular x (1+i) where i is the interest rate in decimal.
- The present value P of the amount A_n due at the end of n period at the rate of i per interest period may be obtained by solving for P the below given equation

$$A_n = P(1 + i)^n$$

i.e. $$P = \frac{A_n}{(1+i)^n}$$

- **Present value of annuity due or annuity immediate:** Present value of annuity due/ immediate for n years is the same as an annuity regular for (n-1) years plus an initial receipt or payment in beginning of the period. Calculating the present value of annuity due involves two steps.

  Step 1: Compute the present value of annuity as if it were a annuity regular for one period short.

  Step 2: Add initial cash payment/receipt to the step 1 value.

- **Sinking Fund**: It is the fund credited for a specified purpose by way of sequence of periodic payments over a time period at a specified interest rate. Interest is compounded at the end of every period. Sizeof the sinking fund deposit is computed from $A = P \times A(n, i)$ where A is the amount to be saved P the periodic payment, n the payment period.

- **Annuity applications:**

  (a) Leasing: Leasing is a financial arrangement under which the owner of the asset (lessor) allows the user of the asset (lessee) to use the asset for a defined period of time (lease period) for a consideration (lease rental) payable over a given period of time. This is a kind of taking an asset on rent.

  (b) Capital Expenditure (investment decision): Capital expenditure means purchasing an asset (which results in outflows of money) today in anticipation of benefits (cash inflow) which would flow across the life of the investment.

  (c) Valuation of Bond: A bond is a debt security in which the issuer owes the holder a debt and is obliged to repay the principal and interest. Bonds are generally issued for a fixed term longer than one year.

MISCELLANEOUS PROBLEMS

EXERCISE 4 (D)

Choose the most appropriate option (a), (b), (c) or (d).

1. A = ₹ 5,200, R = 5% p.a., T = 6 years, P will be

   (a) ₹ 2,000 (b) ₹ 3,880 (c) ₹ 3,000 (d) none of these

2 If P = 1,000, n = 4 years., R = 5% p.a then C. I will be

   (a) ₹ 215.50 (b) ₹ 210 (c) ₹ 220 (d) none of these

3 The time in which a sum of money will be double at 5% p.a C.I is

   (a) ₹ 10 years (b) 12 years (c) 14.2 years (d) none of these

4. If A = ₹ 10,000, n = 18yrs., R = 4% p.a C.I, P will be

   (a) ₹ 4,000 (b) ₹ 4,900 (c) ₹ 4,500 (d) 4,936.30

5. The time by which a sum of money would treble it self at 8% p. a C. I is

   (a) 14.28 years (b) 14 years (c) 12 years (d) none of these

6. The present value of an annuity of ₹ 80 a years for 20 years at 5% p.a is

   (a) ₹ 997 (appx.) (b) ₹ 900 (c) ₹ 1,000 (d) none of these

7. A person bought a house paying ₹ 20,000 cash down and ₹ 4,000 at the end of each year for 25 yrs. at 5% p.a. C.I. The cash down price is

   (a) ₹ 75,000 (b) ₹ 76,000 (c) ₹ 76,375.50 (d) none of these.

8. A man purchased a house valued at ₹ 3,00,000. He paid ₹ 2,00,000 at the time of purchase and agreed to pay the balance with interest at 12% per annum compounded half yearly in 20 equal half yearly instalments. If the first instalment is paid after six months from the date of purchase then the amount of each instalment is

   [Given log 10.6 = 1.0253 and log 31.19 = 1.494]

   (a) ₹ 8,718.45 (b) ₹ 8,769.21 (c) ₹ 7,893.13 (d) none of these.

ANSWERS

Exercise 4(a)

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(d)	**5.**	(a)	**6.**	(b)
7.	(a)	**8.**	(c)	**9.**	(a)	**10.**	(c)				

Exercise 4(b)

1.	(a)	**2.**	(c)	**3.**	(c)	**4.**	(b)	**5.**	(a)	**6.**	(c)
7.	(d)	**8.**	(a)	**9.**	(d)	**10.**	(b)	**11.**	(c)	**12.**	(d)
13.	(a)	**14.**	(a)								

Exercise 4(c)

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(d)	**5.**	(a)	**6.**	(b)
7.	(b)	**8.**	(a)	**9.**	(b)	**10.**	(d)	**11.**	(a)	**12.**	(d)
13.	(c)										

Exercise 4(d)

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(d)	**5.**	(a)	**6.**	(a)
7.	(c)	**8.**	(a)								

ADDITIONAL QUESTION BANK

1. The difference between compound and simple interest at 5% per annum for 4 years on ₹ 20,000 is ₹ ________

   (a) 250 (b) 277 (c) 300 (d) 310

2. The compound interest on half-yearly rests on ₹ 10,000 the rate for the first and second years being 6% and for the third year 9% p.a. is ₹.____________.

   (a) 2,200 (b) 2,287 (c) 2,285 (d) None

3. The present value of ₹ 10,000 due in 2 years at 5% p.a. compound interest when the interest is paid on yearly basis is ₹ ________.

   (a) 9,070 (b) 9,000 (c) 9,061 (d) None

4. The present value of ₹ 10,000 due in 2 years at 5% p.a. compound interest when the interest is paid on half-yearly basis is ₹ ________.

   (a) 9,070 (b) 9,069 (c) 9,061 (d) None

5. Johnson left ₹ 1,00,000 with the direction that it should be divided in such a way that his minor sons Tom, Dick and Harry aged 9, 12 and 15 years should each receive equally after attaining the age 25 years. The rate of interest being 3.5%, how much each son receive after getting 25 years old?

(a) 50,000 (b) 51,994 (c) 52,000 (d) None

6. In how many years will a sum of money double at 5% p.a. compound interest?

(a) 15 years 3 months (b) 14 years 2 months
(c) 14 years 3 months (d) 15 years 2 months

7. In how many years a sum of money trebles at 5% p.a. compound interest payable on half-yearly basis?

(a) 18 years 7 months (b) 18 years 6 months
(c) 18 years 8 months (d) 22 years 3 months

8. A machine depreciates at 10% of its value at the beginning of a year. The cost and scrap value realized at the time of sale being ₹ 23,240 and ₹ 9,000 respectively. For how many years the machine was put to use?

(a) 7 years (b) 8 years (c) 9 years (d) 10 years

9. A machine worth ₹ 4,90,740 is depreciated at 15% on its opening value each year. When its value would reduce to ₹ 2,00,000?

(a) 4 years 6 months (b) 4 years 7 months
(c) 4 years 5 months (d) 5 years 7 months approximately

10. A machine worth ₹ 4,90,740 is depreciated at 15% of its opening value each year. When its value would reduce by 90%?

(a) 11 years 6 months (b) 11 years 7 months
(c) 11 years 8 months (d) 14 years 2 months approximately

11. Alibaba borrows ₹ 6 lakhs Housing Loan at 6% repayable in 20 annual installments commencing at the end of the first year. How much annual payment is necessary.

(a) ₹ 52,420 (b) ₹ 52,419 (c) ₹ 52,310 (d) ₹ 52,320

12. A sinking fund is created for redeming debentures worth ₹ 5 lakhs at the end of 25 years. How much provision needs to be made out of profits each year provided sinking fund investments can earn interest at 4% p.a.?

(a) ₹ 12,006 (b) ₹ 12,040 (c) ₹ 12,039 (d) ₹ 12,035

13. A machine costs ₹ 5,20,000 with an estimated life of 25 years. A sinking fund is created to replace it by a new model at 25% higher cost after 25 years with a scrap value realization of ₹ 25000. what amount should be set aside every year if the sinking fund investments accumulate at 3.5% compound interest p.a.?

(a) ₹ 16,000 (b) ₹ 16,500 (c) ₹ 16,050 (d) ₹ 16,005

14. Raja aged 40 wishes his wife Rani to have ₹ 40 lakhs at his death. If his expectation of life is another 30 years and he starts making equal annual investments commencing now at 3% compound interest p.a. how much should he invest annually?

(a) 84,448 (b) 84,450 (c) 84,449 (d) 84,080

15. Appu retires at 60 years receiving a pension of 14,400 a year paid in half-yearly installments for rest of his life after reckoning his life expectation to be 13 years and that interest at 4% p.a. is payable half-yearly. What single sum is equivalent to his pension?

(a) 1,45,000 (b) 1,44,900 (c) 1,44,800 (d) 1,44,700

ANSWERS

1.	(d)	**2.**	(d)	**3.**	(a)	**4.**	(c)	**5.**	(d)	**6.**	(b)
7.	(d)	**8.**	(c)	**9.**	(d)	**10.**	(d)	**11.**	(c)	**12.**	(a)
13.	(c)	**14.**	(d)	**15.**	(b)						

Note : In most cases Loan payment dues are made at the end of the period only, i.e Annuity Regular . Questions based on Annuity applications can be solved using the Annuity Regular method . For knowledge purpose students may try another method *viz.*, Annuity Due.

NOTES

CHAPTER F

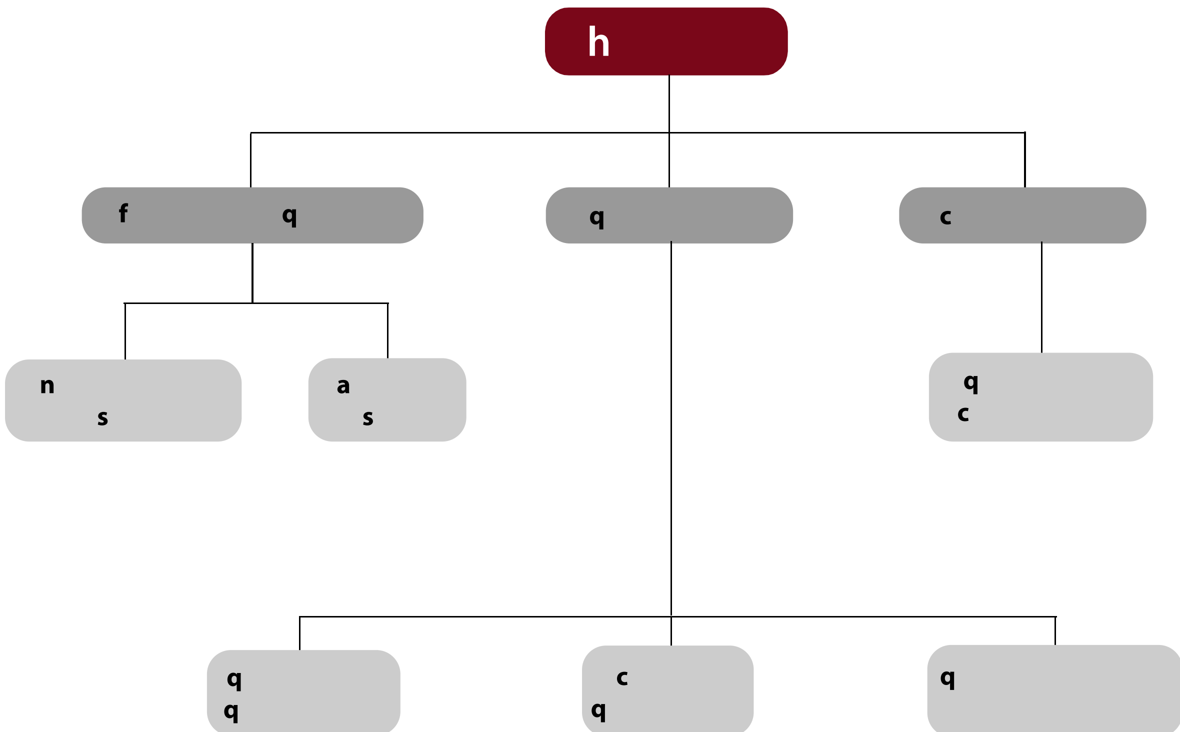
BASIC CONCEPTS OF PERMUTATIONS AND COMBINATIONS

LEARNING OBJECTIVES

After reading this Chapter a student will be able to understand —

- difference between permutation and combination for the purpose of arranging different objects;
- number of permutations and combinations when r objects are chosen out of n different objects.
- meaning and computational techniques of circular permutation and permutation with restrictions.

CHAPTER OVERVIEW



5.1 INTRODUCTION

In this chapter we will learn problem of arranging and grouping of certain things, taking particular number of things at a time. It should be noted that (a, b) and (b, a) are two different arrangements, but they represent the same group. In case of arrangements, the sequence or order of things is also taken into account.

The manager of a large bank has a difficult task of filling two important positions from a group of five equally qualified employees. Since none of them has had actual experience, he decides to allow each of them to work for one month in each of the positions before he makes the decision. How long can the bank operate before the positions are filled by permanent appointments?

Solution to above - cited situation requires an efficient counting of the possible ways in which the desired outcomes can be obtained. A listing of all possible outcomes may be desirable, but is likely to be very tedious and subject to errors of duplication or omission. We need to devise certain techniques which will help us to cope with such problems. The techniques of permutation and combination will help in tackling problems such as above.

FUNDAMENTAL PRINCIPLES OF COUNTING

(a) **Multiplication Rule:** If certain thing may be done in 'm' different ways and when it has been done, a second thing can be done in 'n ' different ways then total number of ways of doing both things simultaneously = m × n.

Eg. if one can going to school by 5 different buses and then come back by 4 different buses then total number of ways of going to and coming back from school = 5 × 4 = 20.

(b) **Addition Rule :** It there are two alternative jobs which can be done in 'm' ways and in 'n' ways respectively then either of two jobs can be done in (m + n) ways.

Eg. if one wants to go school by bus where there are 5 buses or to by auto where there are 4 autos, then total number of ways of going school = 5 + 4 = 9.

Note :- 1)

AND ⇒ Multiply
OR ⇒ Add

2) The above fundamental principles may be generalised, wherever necessary.

5.2 THE FACTORIAL

Definition: The factorial n, written as n! or $\underline{|n}$, represents the product of all integers from 1 to n both inclusive. To make the notation meaningful, when n = o, we define o! or $\underline{|0}$ = 1.

Thus, n! = n (n – 1) (n – 2) ….. …3.2.1

Example 1: Find 5!, 4! and 6!

Solution: 5! = 5 × 4 × 3 × 2 × 1 = 120; 4! = 4 × 3 × 2 × 1 = 24; 6! = 6 × 5 × 4 × 3 × 2 × 1 = 720.

Example 2: Find 9 ! / 6 ! ; 10 ! / 7 !.

Solution: $\frac{9!}{6!} = \frac{9\times8\times7\times6!}{6!} = 9\times8\times7 = 504\,;\ \frac{10!}{7!} = \frac{10\times9\times8\times7!}{7!}$ = 10 × 9 × 8 =720

Example 3: Find x if 1/9 ! + 1/10 ! = x/11 !

Solution: 1/9! (1 + 1/10) = x/11 × 10 × 9! or, 11/10 = x/11 × 10 i.e., x = 121

Example 4: Find n if $\underline{|n+1} = 30\underline{|n-1}$

Solution: $\underline{|n+1} = 30\underline{|n-1} \Rightarrow (n+1).n\,\underline{|n-1} = 30\,\underline{|n-1}$

or, $n^2 + n = 30$ or, $n^2 + n - 30$ or, $n^2 + 6n - 5n - 30 = 0$ or, $(n + 6)(n - 5) = 0$

either n = 5 or n = –6. (Not possible) $\therefore$ n = 5.

5.3 PERMUTATIONS

A group of persons want themselves to be photographed. They approach the photographer and request him to take as many different photographs as possible with persons standing in different positions amongst themselves. The photographer wants to calculate how many films does he need to exhaust all possibilities? How can he calculate the number?

In the situations such as above, we can use permutations to find out the exact number of films.

Definition: The ways of arranging or selecting smaller or equal number of persons or objects from a group of persons or collection of objects with due regard being paid to the order of arrangement or selection, are called permutations.

Let us explain, how the idea of permutation will help the photographer. Suppose the group consists of Mr. Suresh, Mr. Ramesh and Mr. Mahesh. Then how many films does the photographer need? He has to arrange three persons amongst three places with due regard to order. Then the various possibilities are (Suresh, Mahesh, Ramesh), (Suresh, Ramesh, Mahesh), (Ramesh, Suresh, Mahesh), (Ramesh, Mahesh, Suresh), (Mahesh, Ramesh, Suresh) and (Mahesh, Suresh, Ramesh). Thus there are six possibilities. Therefore he needs six films. Each one of these possibilities is called a permutation of three persons taken at a time.

This may also be exhibited as follows :

Alternative	Place 1	Place2	Place 3
1	Suresh	Mahesh	Ramesh
2	Suresh	Ramesh	Mahesh
3	Ramesh	Suresh	Mahesh
4	Ramesh	Mahesh	Suresh
5	Mahesh	Ramesh	Suresh
6	Mahesh	Suresh	Ramesh

with this example as a base, we can introduce a general formula to find the number of permutations.

Number of Permutations when r objects are chosen out of n different objects. (Denoted by ^nP_r or $_nP_r$ or $P_{(n,\,r)}$) :

Let us consider the problem of finding the number of ways in which the first *r* rankings are secured by n students in a class. As any one of the n students can secure the first rank, the number of ways in which the first rank is secured is n.

Now consider the second rank. There are (n – 1) students left and the second rank can be secured by any one of them. Thus the different possibilities are (n – 1) ways. Now, applying fundamental principle, we can see that the first two ranks can be secured in n (n – 1) ways by these n students.

After calculating for two ranks, we find that the third rank can be secured by any one of the remaining (n – 2) students. Thus, by applying the generalized fundamental principle, the first three ranks can be secured in n (n – 1) (n – 2) ways .

Continuing in this way we can visualise that the number of ways are reduced by one as the rank is increased by one. Therefore, again, by applying the generalised fundamental principle for r different rankings, we calculate the number of ways in which the first r ranks are secured by n students as

$$^nP_r = n\,\{(n-1)\ldots\left(n-\overline{r-1}\right)\}$$

$$= n\,(n-1)\ldots(n-r+1)$$

Theorem : The number of permutations of n things when r are chosen at a time

$$^nP_r = n\,(n-1)(n-2)\ldots(n-r+1)$$

where the product has exactly r factors.

5.4 RESULTS

1 Number of permutations of n different things taken all n things at a time is given by

$$^nP_n = n\,(n-1)\,(n-2)\ldots.(n-n+1)$$

$$= n\,(n-1)\,(n-2)\ldots..2.1 = n! \qquad \ldots(1)$$

2. ^nP_r using factorial notation.

$$^nP_r = n.\,(n-1)\,(n-2)\ldots..(n-r+1)$$

$$= n\,(n-1)\,(n-2)\ldots..(n-r+1) \times \frac{(n-r)\,(n-r-1)\,2.1}{1.2\ldots(n-r-1)\,(n-r)}$$

$$= n!/(n-r)! \qquad \ldots(2)$$

Thus

$$^nP_r = \frac{n!}{(n-r)!}$$

3. Justification for 0! = 1. Now applying r = n in the formula for ^nP_r, we get

$^nP_n = n!/ (n - n)! = n!/0!$

But from Result 1 we find that $^nP_n = n!$. Therefore, by applying this

we derive, $0! = n! / n! = 1$

Example 1: Evaluate each of $^5P_3$, $^{10}P_2$, $^{11}P_5$.

Solution: $^5P_3 = 5\times4\times (5-3+1) = 5 \times 4 \times 3 = 60$,

$^{10}P_2 = 10 \times .... \times (10-2+1) = 10 \times 9 = 90$,

$^{11}P_5 = 11! / (11 - 5)! = 11 \times 10 \times 9 \times 8 \times 7 \times 6! / 6! = 11 \times 10 \times 9 \times 8 \times 7 = 55440$.

Example 2: How many three letters words can be formed using the letters of the words (a) SQUARE and (b) HEXAGON?

(Any arrangement of letters is called a word even though it may or may not have any meaning or pronunciation).

Solution:

(a) Since the word 'SQUARE' consists of 6 different letters, the number of permutations of choosing 3 letters out of six equals $^6P_3 = 6 \times 5 \times 4 = 120$.

(b) Since the word 'HEXAGON' contains 7 different letters, the number of permutations is $^7P_3 = 7 \times 6 \times 5 = 210$.

Example 3: In how many different ways can five persons stand in a line for a group photograph?

Solution: Here we know that the order is important. Hence, this is the number of permutations of five things taken all at a time. Therefore, this equals

$^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Example 4: First, second and third prizes are to be awarded at an engineering fair in which 13 exhibits have been entered. In how many different ways can the prizes be awarded?

Solution: Here again, order of selection is important and repetitions are not meaningful as no exhibit can receive more than one prize. Hence , the answer is the number of permutations of 13 things taken three at a time. Therefore, we find $^{13}P_3 = 13!/10! = 13\times12\times11 = 1,716$ ways.

Example 5: In how many different ways can 3 students be associated with 4 chartered accountants, assuming that each chartered accountant can take at most one student?

Solution: This equals the number of permutations of choosing 3 persons out of 4. Hence , the answer is $^4P_3 = 4\times3\times2 = 24$.

Example 6: If six times the number permutations of n things taken 3 at a time is equal to seven times the number of permutations of (n – 1) things taken 3 at a time, find n.

Solution: We are given that $6 \times {}^nP_3 = 7 \times {}^{n-1}P_3$ and we have to solve this equality to find the value of n. Therefore,

$$6\frac{\lfloor n}{\lfloor n-3} = 7\frac{\lfloor n-1}{\lfloor n-4}$$

or, $6 n (n - 1) (n - 2) = 7 (n - 1) (n - 2) (n - 3)$

or, $6 n = 7 (n - 3)$

or, $6n = 7n - 21$

or, $n = 21$

Therefore, the value of n equals 21.

Example 7: Compute the sum of 4 digit numbers which can be formed with the four digits 1, 3, 5, 7, if each digit is used only once in each arrangement.

Solution: The number of arrangements of 4 different digits taken 4 at a time is given by $^4P_4 = 4! = 24$. All the four digits will occur equal number of times at each of the positions, namely ones, tens, hundreds, thousands.

Thus, each digit will occur 24 / 4 = 6 times in each of the positions. The sum of digits in one's position will be 6 × (1 + 3 + 5 + 7) = 96. Similar is the case in ten's, hundred's and thousand's places. Therefore, the sum will be 96 + 96 × 10 + 96 × 100 + 96 × 1000 = 1,06,656.

Example 8: Find n if $^nP_3 = 60$.

Solution: $^nP_3 = \frac{n!}{(n-3)!} = 60$ (given)

i.e., n (n–1) (n–2) = 60 = 5 × 4 × 3

Therefore, n = 5.

Example 9: If $^{56}P_{r+6} : {}^{54}P_{r+3} = 30{,}800 : 1$, find r.

Solution: We know $^nP_r = \frac{n!}{(n-r)!}$;

$$\therefore {}^{56}P_{r+6} = \frac{56!}{\{56-(r+6)\}!} = \frac{56!}{(50-r)!}$$

Similarly, $$^{54}P_{r+3} = \frac{54!}{\{54-(r+3)\}!} = \frac{54!}{(51-r)!}$$

Thus, $$\frac{^{56}P_{r+6}}{^{54}P_{r+3}} = \frac{56!}{(50-r!)} \text{x} \frac{(51-r)!}{54!}$$

$$\frac{56 \times 55 \times 54!}{(50-r)!} \times \frac{(51-r)(50-r)!}{54!} = \frac{56 \times 55 \times (51-r)}{1}$$

But we are given the ratio as 30800 **:** 1 ; therefore

$$\frac{56 \times 55 \times (51-r)}{1} = \frac{30{,}800}{1}$$

$$\text{or, } (51-r) = \frac{30{,}800}{56 \times 55} = 10, \quad \therefore r = 41$$

Example 10: Prove the following

$(n + 1)! - n! = \Rightarrow n.n!$

Solution: By applying the simple properties of factorial, we have

$(n + 1)! - n! = (n+1)\, n! - n! = n!.\,(n+1-1) = n.\, n!$

Example 11: In how many different ways can a club with 10 members select a President, Secretary and Treasurer, if no member can hold two offices and each member is eligible for any office?

Solution: The answer is the number of permutations of 10 persons chosen three at a time. Therefore, it is $^{10}p_3 = 10\times9\times8=720$.

Example 12: When Jhon arrives in New York, he has eight shops to see, but he has time only to visit six of them. In how many different ways can he arrange his schedule in New York?

Solution: He can arrange his schedule in $^8P_6 = 8\times 7 \times 6 \times 5 \times 4 \times 3 = 20{,}160$ ways.

Example 13: When Dr. Ram arrives in his dispensary, he finds 12 patients waiting to see him. If he can see only one patient at a time, find the number of ways, he can schedule his patients (a) if they all want their turn, and (b) if 3 leave in disgust before Dr. Ram gets around to seeing them.

Solution: (a) There are 12 patients and all 12 wait to see the doctor. Therefore the number of ways = $^{12}P_{12} = 12! = 479{,}001{,}600$

(b) There are 12–3 = 9 patients. They can be seen $^{12}P_9 = 79{,}833{,}600$ ways.

EXERCISE 5 (A)

Choose the most appropriate option (a) (b) (c) or (d)

1. $^4P_3$ is evaluated as
   a) 43 b) 34 c) 24 d) None of these
2. $^4P_4$ is equal to
   a) 1 b) 24 c) 0 d) none of these
3. $\underline{|7}$ is equal to
   a) 5040 b) 4050 c) 5050 d) none of these
4. $\underline{|0}$ is a symbol equal to
   a) 0 b) 1 c) Infinity d) none of these
5. In ^nP_r, n is always
   a) an integer b) a fraction c) a positive integer d) none of these
6. In ^nP_r, the restriction is
   a) $n > r$ b) $n \geq r$ c) $n \leq r$ d) none of these
7. In $^nP_r = n\,(n-1)\,(n-2)\,\ldots\ldots\ldots\ldots\ldots(n-r+1)$, the number of factors is
   a) n b) r–1 c) n–r d) r
8. ^nP_r can also written as
   a) $\dfrac{\underline{|n}}{\underline{|n-r}}$ b) $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ c) $\dfrac{\underline{|r}}{\underline{|n-r}}$ d) none of these
9 If $^nP_4 = 12 \times {}^nP_2$, the n is equal to
   a) –1 b) 6 c) 5 d) none of these

10. If . ${}^{n}P_3 : {}^{n}P_2 = 3 : 1$, then n is equal to

a) 7 b) 4 c) 5 d) none of these

11. ${}^{m+n}P_2 = 56$, ${}^{m-n}P_2 = 30$ then

a) m =6, n = 2 b) m = 7, n= 1 c) m=4,n=4 d) none of these

12. if ${}^{5}P_r = 60$, then the value of r is

a) 3 b) 2 c) 4 d) none of these

13. If ${}^{n_1+n_2}P_2 = 132$, ${}^{n_1-n_2}P_2 = 30$ then,

a) $n_1=6, n_2=6$ b) $n_1 = 10, n_2 = 2$ c) $n_1 = 9, n_2 = 3$ d) none of these

14. The number of ways the letters of the word `COMPUTER' can be rearranged is

a) 40,320 b) 40,319 c) 40,318 d) none of these

15. The number of arrangements of the letters in the word `FAILURE', so that vowels are always coming together is

a) 576 b) 575 c) 570 d) none of these

16. 10 examination papers are arranged in such a way that the best and worst papers never come together. The number of arrangements is

a) $9\underline{|8}$ b) $\underline{|10}$ c) $8\underline{|9}$ d) none of these

17. n articles are arranged in such a way that 2 particular articles never come together. The number of such arrangements is

a) $(n-2)\,\underline{|n-1}$ b) $(n-1)\,\underline{|n-2}$ c) $\underline{|n}$ d) none of these

18. If 12 school teams are participating in a quiz contest, then the number of ways the first, second and third positions may be won is

a) 1,230 b) 1,320 c) 3,210 d) none of these

19. The sum of all 4 digit number containing the digits 2, 4, 6, 8, without repetitions is

a) 1,33,330 b) 1,22,220 c) 2,13,330 d) 1,33,320

20 The number of 4 digit numbers greater than 5,000 can be formed out of the digits 3,4,5,6 and 7(No. digit is repeated). The number of such is

a) 72 b) 27 c) 70 d) none of these

21. 4 digit numbers to be formed out of the figures 0, 1, 2, 3, 4 (no digit is repeated) then number of such numbers is

(a) 120 (b) 20 (c) 96. (d) none of these

22. The number of ways the letters of the word `TRIANGLE' to be arranged so that the word 'angle' will be always present is

(a) 20 (b) 60 (c) 24 (d) 32

23. If the letters word 'DAUGHTER' are to be arranged so that vowels occupy the odd places, then number of different words are

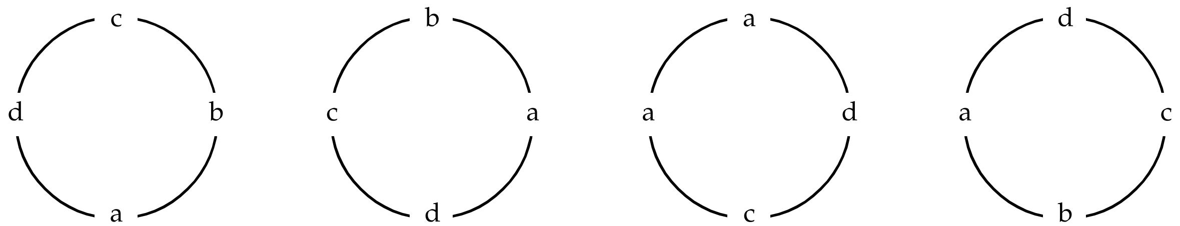
(a) 2,880 (b) 676 (c) 625 (d) 576

5.5 CIRCULAR PERMUTATIONS

So for we have discussed arrangements of objects or things in a row which may be termed as linear permutation. But if we arrange the objects along a closed curve viz., a circle, the permutations are known as circular permutations.

The number of circular permutations of n different things chosen at a time is (n–1)!.

Proof : Let any one of the permutations of n different things taken. Then consider the rearrangement of this permutation by putting the last thing as the first thing. Even though this is a different permutation in the ordinary sense, it will not be different in all *n* things are arranged in a circle. Similarly, we can consider shifting the last two things to the front and so on. Specially, it can be better understood, if we consider *a,b,c,d*. If we place *a,b,c,d* in order, then we also get *abcd*, *dabc, cdab, bcda* as four ordinary permutations. These four words in circular case are one and same thing. See below circles.



abcd dabc cdab bcda

Thus we find in above illustration that four ordinary permutations equals one in circular.

Therefore, *n* ordinary permutations equal one circular permutation.

Hence there are ^nP_n/ n ways in which all the *n* things can be arranged in a circle. This equals *(n–1)!*.

Example 1: In how many ways can 4 persons sit at a round table for a group discussions?

Solution: The answer can be get from the formula for circular permutations. The answer is (4–1)! = 3! = 6 ways.

NOTE : These arrangements are such that every person has got the same two neighbours. The only change is that right side neighbour and vice-versa.

Thus the number of ways of arranging n persons along a round table so that no person has the same two neighbours is $=\frac{1}{2}\underline{|n-1}$

Similarly, in forming a necklace or a garland there is no distinction between a clockwise and anti clockwise direction because we can simply turn it over so that clockwise becomes anti clockwise and vice versa. **Hence, the number of necklaces formed with n beads of different colours** $=\frac{1}{2}\underline{|n-1}$

5.6 PERMUTATION WITH RESTRICTIONS

In many arrangements there may be number of restrictions. in such cases, we are to arrange or select the objects or persons as per the restrictions imposed. The total number of arrangements in all cases, can be found out by the application of fundamental principle.

Theorem 1. Number of permutations of n distinct objects taken r at a time when a particular object is not taken in any arrangement is $^{n-1}p_r$.

Proof : Since a particular object is always to be excluded, we have to place n – 1 objects at r places. Clearly this can be done in $^{n-1}p_r$ ways.

Theorem 2. Number of permutations of r objects out of n distinct objects when a particular object is always included in any arrangement is r. $^{n-1}p_{r-1}$

Proof : If the particular object is placed at first place, remaining r – 1 places can be filled from n – 1 distinct objects in $^{n-1}p_{r-1}$ ways. Similarly, by placing the particular object in 2nd, 3rd, ….., r^{th} place, we find that in each case the number of permutations is $^{n-1}p_{r-1}$. This the total number of arrangements in which a particular object always occurs is r. $^{n-1}p_{r-1}$

The following examples will enlighten further:

Example 1: How many arrangements can be made out of the letters of the word `DRAUGHT', the vowels never beings separated?

Solution: The word `DRAUGHT' consists of 7 letters of which 5 are consonants and two are vowels. In the arrangement we are to take all the 7 letters but the restriction is that the two vowels should not be separated.

We can view the two vowels as one letter. The two vowels A and U in this one letter can be arranged in 2! = 2 ways. (i) AU or (ii) UA. Further, we can arrange the six letters : 5 consonants and one letter (compound letter consisting of two vowels). The total number of ways of arranging them is $^6P_6 = 6! = 720$ ways.

Hence, by the fundamental principle, the total number of arrangements of the letters of the word DRAUGHT, the vowels never being separated = 2 × 720 = 1440 ways.

Example 2: Show that the number of ways in which n books can be arranged on a shelf so that two particular books are not together.The number is $(n–2).(n–1)!$

Solution: We first find the total number of arrangements in which all n books can be arranged on the shelf without any restriction. The number is, $^nP_n = n!$ ….. (1)

Then we find the total number of arrangements in which the two particular books are together.

The books can be together in $^2P_2 = 2! = 2$ ways. Now we consider those two books which are kept together as one composite book and with the rest of the $(n–2)$ books from $(n–1)$ books which are to be arranged on the shelf; the number of arrangements = $^{n-1}P_{n-1} = (n–1)$!. Hence by the Fundamental Principle, the total number of arrangements on which the two particular books are together equals = $2 \times (n–1)!$ …….(2)

the required number of arrangements of n books on a shelf so that two particular books are not together

= (1) – (2)

= $n! - 2 \times (n-1)!$

= $n.(n-1)! - 2 . (n-1)!$

= $(n-1)! . (n-2)$

Example 3: There are 6 books on Economics, 3 on Mathematics and 2 on Accountancy. In how many ways can these be placed on a shelf if the books on the same subject are to be together?

Solution: Consider one such arrangement. 6 Economics books can be arranged among themselves in 6! Ways, 3 Mathematics books can be arranged in 3! Ways and the 2 books on Accountancy can be arranged in 2! ways. Consider the books on each subject as one unit. Now there are three units. These 3 units can be arranged in 3! Ways.

Total number of arrangements = $3! \times 6! \times 3! \times 2!$

= 51,840.

Example 4: How many different numbers can be formed by using any three out of five digits 1, 2, 3, 4, 5, no digit being repeated in any number?

How many of these will (i) begin with a specified digit? (ii) begin with a specified digit and end with another specified digit?

Solution: Here we have 5 different digits and we have to find out the number of permutations of them 3 at a time. Required number is $^5P_3 = 5.4.3 = 60$.

(i) If the numbers begin with a specified digit, then we have to find

the number of Permutations of the remaining 4 digits taken 2 at a time. Thus, desired number is $^4P_2 = 4.3 = 12$.

(ii) Here two digits are fixed; first and last; hence, we are left with the choice of finding the number of permutations of 3 things taken one at a time i.e., $^3P_1 = 3$.

Example 5: How many four digit numbers can be formed out of the digits 1,2,3,5,7,8,9, if no digit is repeated in any number? How many of these will be greater than 3000?

Solution: We are given 7 different digits and a four-digit number is to be formed using any 4 of these digits. This is same as the permutations of 7 different things taken 4 at a time.

Hence, the number of four-digit numbers that can be formed = $^7P_4 = 7 \times 6 \times 5 \times 4 \times = 840$ ways.

Next, there is the restriction that the four-digit numbers so formed must be greater than 3,000. Thus, it will be so if the first digit-that in the thousand's position, is one of the five digits 3, 5, 7, 8, 9. Hence, the first digit can be chosen in 5 different ways; when this is done, the rest of the 3 digits are to be chosen from the rest of the 6 digits without any restriction and this can be done in $^6P_3$ ways.

Hence, by the Fundamental principle, we have the number of four-digit numbers greater than 3,000 that can be formed by taking 4 digits from the given 7 digits = $5 \times {}^6P_3 = 5 \times 6 \times 5 \times 4 = 5 \times 120$ = 600.

Example 6: Find the total number of numbers greater than 2000 that can be formed with the digits 1, 2, 3, 4, 5 no digit being repeated in any number.

Solution: All the 5 digit numbers that can be formed with the given 5 digits are greater than 2000. This can be done in

$^5P_5 = 5! = 120$ ways ....(1)

The four digited numbers that can be formed with any four of the given 5 digits are greater than 2000 if the first digit, i.e.,the digit in the thousand's position is one of the four digits 2, 3, 4, 5. this can be done in $^4P_1 = 4$ ways. When this is done, the rest of the 3 digits are to be chosen from the rest of 5–1 = 4 digits. This can be done in $^4P_3 = 4 \times 3 \times 2 = 24$ ways.

Therefore, by the Fundamental principle, the number of four-digit numbers greater than 2000

$= 4 \times 24 = 96$ …. (2)

Adding (1) and (2), we find the total number greater than 2000 to be 120 + 96 = 216.

Example 7: There are 6 students of whom 2 are Indians, 2 Americans, and the remaining 2 are Russians. They have to stand in a row for a photograph so that the two Indians are together, the two Americans are together and so also the two Russians. Find the number of ways in which they can do so.

Solution: The two Indians can stand together in $^2P_2 = 2! = 2$ ways. So is the case with the two Americans and the two Russians.

Now these 3 groups of 2 each can stand in a row in $^3P_3 = 3 \times 2 = 6$ ways. Hence by the generalized fundamental principle, the total number of ways in which they can stand for a photograph under given conditions is

$6 \times 2 \times 2 \times 2 = 48$

Example 8: A family of 4 brothers and three sisters is to be arranged for a photograph in one row. In how many ways can they be seated if (i) all the sisters sit together, (ii) no two sisters sit together?

Solution:

(i) Consider the sisters as one unit and each brother as one unit. 4 brothers and 3 sisters make 5 units which can be arranged in 5! ways. Again 3 sisters may be arranged amongst themselves in 3! Ways

Therefore, total number of ways in which all the sisters sit together = 5!×3! = 720 ways.

(ii) In this case, each sister must sit on each side of the brothers. There are 5 such positions as indicated below by upward arrows :

↑ B1 ↑ B2 ↑ B3 ↑ B4 ↑

4 brothers may be arranged among themselves in 4! ways. For each of these arrangements 3 sisters can sit in the 5 places in $^5P_3$ ways.

Thus the total number of ways = $^5P_3 \times 4! = 60 \times 24 = 1{,}440$

Example 9: In how many ways can 8 persons be seated at a round table? In how many cases will 2 particular persons sit together?

Solution: This is in form of circular permutation. Hence the number of ways in which eight persons can be seated at a round table is $(n-1)! = (8-1)! = 7! = 5040$ ways.

Consider the two particular persons as one person. Then the group of 8 persons becomes a group of 7 (with the restriction that the two particular persons be together) and seven persons can be arranged in a circular in 6! Ways.

Hence, by the fundamental principle, we have, the total number of cases in which 2 particular persons sit together in a circular arrangement of 8 persons = 2! × 6! = 2 × 6 × 5 × 4 × 3 ×2×1 = 1,440.

Example 10: Six boys and five girls are to be seated for a photograph in a row such that no two girls sit together and no two boys sit together. Find the number of ways in which this can be done.

Solution: Suppose that we have 11 chairs in a row and we want the 6 boys and 5 girls to be seated such that no two girls and no two boys are together. If we number the chairs from left to right, the arrangement will be possible if and only if boys occupy the odd places and girls occupy the even places in the row. The six odd places from 1 to 11 may filled in by 6 boys in ${}^6P_6$ ways. Similarly, the five even places from 2 to 10 may be filled in by 5 girls in ${}^5P_5$ ways.

Hence, by the fundamental principle, the total number of required arrangements = ${}^6P_6 \times {}^5P_5$ = 6! × 5! = 720 × 120 = 86,400.

EXERCISE 5 (B)

Choose the most appropriate option (a) (b) (c) or (d)

1 The number of ways in which 7 girls form a ring is
(a) 700 (b) 710 (c) 720 (d) none of these

2. The number of ways in which 7 boys sit in a round table so that two particular boys may sit together is
(a) 240 (b) 200 (c) 120 (d) none of these

3. If 50 different jewels can be set to form a necklace then the number of ways is
(a) $\frac{1}{2}\underline{|50}$ (b) $\frac{1}{2}\underline{|49}$ (c) $\underline{|49}$ (d) none of these

4. 3 ladies and 3 gents can be seated at a round table so that any two and only two of the ladies sit together. The number of ways is
(a) 70 (b) 27 (c) 72 (d) none of these

5. The number of ways in which the letters of the word `DOGMATIC' can be arranged is
(a) 40,319 (b) 40,320 (c) 40,321 (d) none of these

6. The number of arrangements of 10 different things taken 4 at a time in which one particular thing always occurs is
(a) 2015 (b) 2016 (c) 2014 (d) none of these

7. The number of permutations of 10 different things taken 4 at a time in which one particular thing never occurs is
(a) 3,020 (b) 3,025 (c) 3,024 (d) none of these

8. Mr. X and Mr. Y enter into a railway compartment having six vacant seats. The number of ways in which they can occupy the seats is
(a) 25 (b) 31 (c) 32 (d) 30

9. The number of numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5, 6, 7 is
(a) 210 (b) 200 (c) 110 (d) none of these

10. The number of numbers lying between 10 and 1000 can be formed with the digits 2,3,4,0,8,9 is
(a) 124 (b) 120 (c) 125 (d) none of these

11. In a group of boys the number of arrangement of 4 boys is 12 times the number of arrangements of 2 boys. The number of boys in the group is
(a) 10 (b) 8 (c) 6 (d) none of these

12. The value of $\sum_{r=1}^{10} r.{}^{r}P_{r}$ is
(a) ${}^{11}P_{11}$ (b) ${}^{11}P_{11} - 1$ (c) ${}^{11}P_{11} + 1$ (d) none of these

13. The total number of 9 digit numbers of different digits is
(a) $10\underline{|9}$ (b) $8\underline{|9}$ (c) $9\underline{|9}$ (d) none of these

14. The number of ways in which 6 men can be arranged in a row so that the particular 3 men sit together, is
(a) ${}^{4}P_{4}$ (b) ${}^{4}P_{4} \times {}^{3}P_{3}$ (c) $(\underline{|3})^2$ (d) none of these

15. There are 5 speakers A, B, C, D and E. The number of ways in which A will speak always before B is
(a) 24 (b) $\underline{|4} \times \underline{|2}$ (c) $\underline{|5}$ (d) none of these

16. There are 10 trains plying between Calcutta and Delhi. The number of ways in which a person can go from Calcutta to Delhi and return by a different train is
(a) 99 (b) 90 (c) 80 (d) none of these

17. The number of ways in which 8 sweats of different sizes can be distributed among 8 persons of different ages so that the largest sweat always goes to be younger assuming that each one of then gets a sweat is
(a) $\underline{|8}$ (b) 5040 (c) 5039 (d) none of these

18. The number of arrangements in which the letters of the word `MONDAY' be arranged so that the words thus formed begin with M and do not end with N is
(a) 720 (b) 120 (c) 96 (d) none of these

19. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is
(a) $\underline{|7} / \underline{|3}$ (b) $\underline{|6} \times \underline{|7} / \underline{|3}$ (c) 35 (d) none of these

20. The number of ways in which the letters of the word `MOBILE' be arranged so that consonants always occupy the odd places is
(a) 36 (b) 63 (c) 30 (d) none of these.

21. 5 persons are sitting in a round table in such way that Tallest Person is always on the right–side of the shortest person; the number of such arrangements is
(a) 6 (b) 8 (c) 24 (d) none of these

5.7 COMBINATIONS

We have studied about permutations in the earlier section. There we have said that while arranging, we should pay due regard to order. There are situations in which order is not important. For example, consider selection of 5 clerks from 20 applicants. We will not be concerned about the order in which they are selected. In this situation, how to find the number of ways of selection? The idea of combination applies here.

Definition : The number of ways in which smaller or equal number of things are arranged or selected from a collection of things where the order of selection or arrangement is not important, are called combinations.

The selection of a poker hand which is a combination of five cards selected from 52 cards is an example of combination of 5 things out of 52 things.

Number of combinations of n different things taken r at a time. (denoted by ${}^{n}C_{r}$ C(n,r), $C_{n,r}$)

Let ${}^{n}C_{r}$ denote the required number of combinations. Consider any one of those combinations. It will contain r things. Here we are not paying attention to order of selection. Had we paid attention to this, we will have permutations or r items taken r at a time. In other words, every combination of r things will have ${}^{r}P_{r}$ permutations amongst them. Therefore, ${}^{n}C_{r}$ combinations will give rise to ${}^{n}C_{r}\cdot{}^{r}P_{r}$ permutations of r things selected from n things. From the earlier section, we can say that ${}^{n}C_{r}\cdot{}^{r}P_{r} = {}^{n}P_{r}$ as ${}^{n}P_{r}$ denotes the number of permutations of r things chosen out of n things.

Since, ${}^{n}C_{r}\cdot{}^{r}P_{r} = {}^{n}P_{r}$,

$${}^{n}C_{r} = {}^{n}P_{r}/{}^{r}P_{r} = n!/(n-r)! \div r!/(r-r)!$$

$$= n!/(n-r)! \times 0!/r!$$

$$= n!/r!(n-r)!$$

$$\therefore {}^{n}C_{r} = n!/r!(n-r)!$$

Remarks: Using the above formula, we get

(i) ${}^{n}C_{o} = n!/0!(n-0)! = n!/n! = 1$. [As $0! = 1$]

${}^{n}C_{n} = n!/n!(n-n)! = n!/n!\,0! = 1$ [Applying the formula for ${}^{n}C_{r}$ with $r = n$]

Example 1: Find the number of different poker hands in a pack of 52 playing cards.

Solution: This is the number of combinations of 52 cards taken five at a time. Now applying the formula,

$^{52}C_5$ = 52!/5! (52 – 5)! = 52!/5! 47! = $\dfrac{52\times51\times50\times49\times48\times47!}{5\times4\times3\times2\times1\times47!}$

= 2,598,960

Example 2: Let S be the collection of eight points in the plane with no three points on the straight line. Find the number of triangles that have points of S as vertices.

Solution: Every choice of three points out of S determines a unique triangle. The order of the points selected is unimportant as whatever be the order, we will get the same triangle. Hence, the desired number is the number of combinations of eight things taken three at a time. Therefore, we get

$^8C_3$ = 8!/3!5! = 8×7×6/3×2×1 = 56 choices.

Example 3: A committee is to be formed of 3 persons out of 12. Find the number of ways of forming such a committee.

Solution: We want to find out the number of combinations of 12 things taken 3 at a time and this is given by

$^{12}C_3$ = 12!/3!(12 – 3)! [by the definition of ^nC_r]

= 12!/3!9! = 12×11×10×9!/3!9! = 12×11×10/3×2 = 220

Example 4: A committee of 7 members is to be chosen from 6 Chartered Accountants, 4 Economists and 5 Cost Accountants. In how many ways can this be done if in the committee, there must be at least one member from each group and at least 3 Chartered Accountants?

Solution: The various methods of selecting the persons from the various groups are shown below:

Committee of 7 members			
	C.A.s	**Economists**	**Cost Accountants**
Method 1	3	2	2
Method 2	4	2	1
Method 3	4	1	2
Method 4	5	1	1
Method 5	3	3	1
Method 6	3	1	3

Number of ways of choosing the committee members by

Method 1 = $^6C_3\times{}^4C_2\times{}^5C_2 = \dfrac{6\times5\times4}{3\times2\times1}\times\dfrac{4\times3}{2\times1}\times\dfrac{5\times4}{2\times1}$ =20×6×10=1,200.

Method 2 = $^6C_4\times{}^4C_2\times{}^5C_1 = \dfrac{6\times5}{2\times1}\times\dfrac{4\times3}{2\times1}\times\dfrac{5}{1}$ = 15×6×5 = 450

Method 3 = $^6C_4\times{}^4C_1\times{}^5C_2 = \dfrac{6\times5}{2\times1}\times4\times\dfrac{5\times4}{2\times1}$ = 15×4×10 = 600.

Method 4 = $^6C_5 \times {}^4C_1 \times {}^5C_1 = 6 \times 4 \times 5 = 120$.

Method 5 = $^6C_3 \times {}^4C_3 \times {}^5C_1 = \frac{6\times5\times4}{3\times2\times1} \times \frac{4\times3\times2}{3\times2\times1} \times 5 \quad = 20\times4\times5 = 400$.

Method 6 = $^6C_3 \times {}^4C_1 \times {}^5C_3 = \frac{6\times5\times4}{3\times2\times1} \times 4 \times \frac{5\times4}{2\times1} \quad = 20\times4\times10 = 800$.

Therefore, total number of ways = 1,200 + 450 + 600 + 120 + 400 + 800 = 3,570

Example 5: A person has 12 friends of whom 8 are relatives. In how many ways can he invite 7 guests such that 5 of them are relatives?

Solution: Of the 12 friends, 8 are relatives and the remaining 4 are not relatives. He has to invite 5 relatives and 2 friends as his guests. 5 relatives can be chosen out of 8 in $^8C_5$ ways; 2 friends can be chosen out of 4 in $^4C_2$ ways.

Hence, by the fundamental principle, the number of ways in which he can invite 7 guests such that 5 of them are relatives and 2 are friends.

$= {}^8C_5 \times {}^4C_2$

$= \{8! / 5!\,(8-5)!\} \times \{4! / 2!\,(4-2)!\} = [(8 \times 7 \times 6 \times 5!)/5! \times 3!] \times \frac{4\times3\times2\times!}{2!\,2!} = 8\times7\times6$

$= 336.$

Example 6: A Company wishes to simultaneously promote two of its 6 department heads to assistant managers. In how many ways these promotions can take place?

Solution: This is a problem of combination. Hence, the promotions can be done in

$^6C_2 = 6\times5 / 2 = 15$ ways

Example 7: A building contractor needs three helpers and ten men apply. In how many ways can these selections take place?

Solution: There is no regard for order in this problem. Hence, the contractor can select in any of $^{10}C_3$ ways i.e.,

$(10 \times 9 \times 8) / (3 \times 2 \times 1) = 120$ ways.

Example 8: In each case, find n:

Solution: (a) $4.\,{}^nC_2 = {}^{n+2}C_3$ (b) $^{n+2}C_n = 45$.

(a) We are given that $4.\,{}^nC_2 = {}^{n+2}C_3$. Now applying the formula,

$$4\times\frac{n!}{2!(n-2)!} = \frac{(n+2)!}{3!(n+2-3)!}$$

or, $$\frac{4\times n.(n-1)(n-2)!}{2!(n-2)!} = \frac{(n+2)\,(n+1)\,.\,n\,.\,(n-1)!}{3!\,(n-1)!}$$

$$4n(n-1)/2 = (n+2)(n+1)n/3!$$

or, $4n(n-1) / 2 = (n+2)(n+1)n / 3\times2\times1$

or, $12(n-1)=(n+2)(n+1)$

or, $12n-12 = n^2 + 3n + 2$

or, $n^2 - 9n + 14 = 0.$

or, $n^2 - 2n - 7n + 14 = 0.$

or, $(n-2)(n-7) = 0$

∴ n=2 or 7.

(b) We are given that $^{n+2}C_n = 45$. Applying the formula,

$(n+2)!/\{n!(n+2-n)!\} = 45$

or, $(n+2)(n+1)\,n! / n!\,2! = 45$

or, $(n+1)(n+2) = 45\times2! = 90$

or, $n^2+3n-88 = 0$

or, $n^2+11n-8n-88 = 0$

or, $(n+11)(n-8) = 0$

Thus, n equals either – 11 or 8. But negative value is not possible. Therefore we conclude that n=8.

Example 9: A box contains 7 red, 6 white and 4 blue balls. How many selections of three balls can be made so that (a) all three are red (b) none is red (c) one is of each colour?

Solution: (a) All three balls will be of red colour if they are taken out of 7 red balls and this can be done in

$^7C_3 = 7! / 3!(7-3)!$

$= 7! / 3!4! = 7\times6\times5\times4! / (3\times2\times4!) = 7\times6\times5 / (3\times2) = 35$ ways

Hence, 35 selections (groups) will be there such that all three balls are red.

(b) None of the three will be red if these are chosen from (6 white and 4 blue balls) 10 balls and this can be done in

$^{10}C_3 = 10!/\{3!(10-3)!\} = 10! / 3!7!$

$= 10\times9\times8\times7! / (3\times2\times1\times7!) = 10\times9\times8 / (3\times2) = 120$ ways.

Hence, the selections (or groups) of three such that none is a red ball are 120 in number.

One red ball can be chosen from 7 balls in $^7C_1 = 7$ ways. One white ball can be chosen from 6 white balls in $^6C_1$ ways. One blue ball can be chosen from 4 blue balls in $^4C_1 = 4$ ways. Hence, by generalized fundamental principle, the number of groups of three balls such that one is of each colour = 7×6×4 = 168 ways.

Example 10: If $^{10}P_r = 6{,}04{,}800$ and $^{10}C_r = 120$; find the value of r,

Solution: We know that $^nC_r \cdot {}^rP_r = {}^nP_r$. We will us this equality to find r.

$^{10}P_r = {}^{10}C_r .r!$

or, 6,04,800 =120 ×r!

or, r! = 6,04,800 ÷ 120 = 5,040

But r! = 5040 = 7×6×4×3×2×1 = 7!

Therefore, r=7.

Properties of ^nC_r :

1. $^nC_r = {}^nC_{n-r}$

   We have ^nC_r = n! / {r!(n–r)!} and $^nC_{n-r}$ = n! / [(n–r)! {n–(n–r)}!] = n! / {(n–r)!(n–n+r)!}

   Thus $^nC_{n-r}$ = n! / {(n–r)! (n–n+r)!} = n! / {(n–r)!r!} = ^nC_r

2. $^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

   By definition,

   $^nC_{r-1} + {}^nC_r$ = n! / {(r–1)! (n–r+1)!} + n! / {r!(n–r)!}

   But r! = r×(r–1)! and (n–r+1)! = (n–r+1) × (n–r)!. Substituting these in above, we get

$$^nC_{r-1} + {}^nC_r = n!\left\{\frac{1}{(r-1)!(n-r+1)(n-r)!} + \frac{1}{r(r-1)!\,(n-r)!}\right\}$$

   = {n! / (r–1)! (n–r)!} {(1 / n–r+1) + (1/r) }

   = {n! / (r–1)! (n–r)!} {(r+n–r+1) / r(n–r+1) }

   = (n+1) n! / {r . (r–1)! (n–r)! . (n–r+1)}

   = (n+1)! / {r!(n+1–r)!} = $^{n+1}C_r$

3. $\mathbf{^nC_o}$ = n!/{0! (n–0)!} = n! / n! =**1**.

4. $\mathbf{^nC_n}$ = n!/{n! (n–n)!} = n! / n! . 0! = **1**.

Note

(a) ^nC_r has a meaning only when r and n are integers $0 \le r \le n$ and $^nC_{n-r}$ has a meaning only when $0 \le n - r \le n$.

(b) ^nC_r and $^nC_{n-r}$ are called complementary combinations, for if we form a group of r things out of n different things, (n–r) remaining things which are not included in this group form another group of rejected things. The number of groups of n different things, taken r at a time should be equal to the number of groups of n different things taken (n–r) at a time.

Example 11: Find r if $^{18}C_r = {}^{18}C_{r+2}$

Solution: As $^nC_r = {}^nC_{n-r}$, we have $^{18}C_r = {}^{18}C_{18-r}$

But it is given, $^{18}C_r = {}^{18}C_{r+2}$

$\therefore\ ^{18}C_{18-r} = {}^{18}C_{r+2}$

or, 18 – r = r+2

Solving, we get

2r = 18 – 2 = 16 i.e., r=8.

Example 12: Prove that

$^{n}C_{r} = {}^{n-2}C_{r-2} + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_{r}$

Solution: R.H.S $= {}^{n-2}C_{r-2} + {}^{n-2}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-2}C_{r}$

$= {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ [using Property 2 listed earlier]

$= {}^{(n-1)+1}C_{r}$ [using Property 2 again]

$= {}^{n}C_{r}$ = L.H.S.

Hence, the result

Example 13: If $^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.

Solution: We have $^{n}C_{r} = n! / \{r!(n-r)!\}$ Now, substituting for n and r, we get

$^{28}C_{2r} = 28! / \{(2r)!(28 - 2r)!\}$,

$^{24}C_{2r-4} = 24! / [(2r-4)!\{24 - (2r-4)\}!] = 24! / \{(2r-4)!(28-2r)!\}$

We are given that $^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$. Now we calculate,

$$\frac{^{28}C_{2r}}{^{24}C_{2r-4}} = \frac{28!}{(2r)!(28-2r)!} \times \frac{(2r-4)!(28-2r)!}{24!}$$

$$= \frac{28\times27\times26\times25\times24!}{(2r)(2r\text{-}1)(2r\text{-}2)(2r\text{-}3)(2r\text{-}4)!(28\text{-}2r)!} \times \frac{(2r\text{-}4)!(28\text{-}2r)!}{24!}$$

$$= \frac{28\times27\times26\times25}{(2r)(2r-1)(2r-2)(2r-3)} = \frac{225}{11}$$

or, $(2r)(2r-1)(2r-2)(2r-3) = \dfrac{11\times28\times27\times26\times25}{225}$

$= 11\times28\times3\times26$

$= 11\times7\times4\times3\times13\times2$

$= 11\times12\times13\times14$

$= 14\times13\times12\times11$

$\therefore$ 2r= 14 i.e., r = 7

Example 14: Find x if $^{12}C_{5} + 2\,{}^{12}C_{4} + {}^{12}C_{3} = {}^{14}C_{x}$

Solution: L.H.S $= {}^{12}C_{5} + 2\,{}^{12}C_{4} + {}^{12}C_{3}$

$= {}^{12}C_{5} + {}^{12}C_{4} + {}^{12}C_{4} + {}^{12}C_{3}$

$= {}^{13}C_{5} + {}^{13}C_{4}$

$= {}^{14}C_{5}$

Also $^{n}C_{r} = {}^{n}C_{n-r}$. Therefore $^{14}C_{5} = {}^{14}C_{14-5} = {}^{14}C_{9}$

Hence, L.H.S = ${}^{14}C_5 = {}^{14}C_9 = {}^{14}C_x$ = R.H.S by the given equality

This implies, either x = 5 or x = 9.

Example 15 : Prove by reasoning that

(i) ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

(ii) ${}^nP_r = {}^{n-1}P_r + r{}^{n-1}P_{r-1}$

Solution: (i) ${}^{n+1}C_r$ stands for the number of combinations of (n+1) things taken r at a time. As a specified thing can either be included in any combination or excluded from it, the total number of combinations which can be combinations or (n+1) things taken r at a time is the sum of :

(a) combinations of (n+1) things taken r at time in which one specified thing is always included and

(b) the number of combinations of (n+1) things taken r at time from which the specified thing is always excluded.

Now, in case (a), when a specified thing is always included , we have to find the number of ways of selecting the remaining (r–1) things out of the remaining n things which is ${}^nC_{r-1.}$

Again, in case (b), since that specified thing is always excluded, we have to find the number of ways of selecting r things out of the remaining n things, which is ${}^nC_r$.

Thus, ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$

(i) We divide ${}^nP_r$ i.e., the number of permutations of n things take r at a time into two groups:

(a) those which contain a specified thing

(b) those which do not contain a specified thing.

In (a) we fix the particular thing in any one of the r places which can be done in r ways and then fill up the remaining (r–1) places out of (n–1) things which give rise to ${}^{n-1}P_{r-1}$ ways. Thus, the number of permutations in case (a) = r × ${}^{n-1}P_{r-1.}$

In case (b), one thing is to be excluded; therefore, r places are to be filled out of (n–1) things. Therefore, number of permutations = ${}^{n-1}P_r$

Thus, total number of permutations = ${}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

i.e., ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

5.8 STANDARD RESULTS

We now proceed to examine some standard results in permutations and combinations. These results have special application and hence are dealt with separately.

I. Permutations when some of the things are alike, taken all at a time

The number of ways *p* in which n things may be arranged among themselves, taking them all at a time, when n_1 of the things are exactly alike of one kind , n_2 of the things are exactly alike of another kind, n_3 of the things are exactly alike of the third kind, and the rest all are different is given by,

$$p = \frac{n!}{n_1!n_2!n_3!}$$

Proof : Let there be n things. Suppose n_1 of them are exactly alike of one kind; n_2 of them are exactly alike of another kind; n_3 of them are exactly alike of a third kind; let the rest $(n-n_1-n_2-n_3)$ be all different.

Let p be the required permutations; then if the n things, all exactly alike of one kind were replaced by n_1 different things different from any of the rest in any of the p permutations without altering the position of any of the remaining things, we could form $n_1!$ new permutations. Hence, we should obtain $p \times n_1!$ permutations.

Similarly if n_2 things exactly alike of another kind were replaced by n_2 different things different form any of the rest, the number of permutations would be $p \times n_1! \times n_2!$

Similarly, if n_3 things exactly alike of a third kind were replaced by n_3 different things different from any of the rest, the number of permutations would be $p \times n_1! \times n_2! \times n_3! = n!$

But now because of these changes all the n things are different and therefore, the possible number of permutations when all of them are taken is n!.

Hence, $p \times n_1! \times n_2!\, n_3! = n!$

i.e., $p = \frac{n!}{n_1!n_2!n_3!}$

which is the required number of permutations. This results may be extended to cases where there are different number of groups of alike things.

II. Permutations when each thing may be repeated once, twice,…upto r times in any arrangement.

Result: The number of permutations of n things taken r at time when each thing may be repeated r times in any arrangement is n^r.

Proof: There are n different things and any of these may be chosen as the first thing. Hence, there are n ways of choosing the first thing.

When this is done, we are again left with n different things and any of these may be chosen as the second (as the same thing can be chosen again.)

Hence, by the fundamental principle, the two things can be chosen in $n \times n = n^2$ number of ways.

Proceeding in this manner, and noting that at each stage we are to chose a thing from n different things, the total number of ways in which r things can be chosen is obviously equal to $n \times n \times$ ………to r terms = n^r.

III. Combinations of n different things taking some or all of n things at a time

Result : The total number of ways in which it is possible to form groups by taking some or all of n things $(2^n - 1)$.

In symbols, $\sum_{r=1}^{n} {}^{n}C_r = 2^n - 1$

Proof : Each of the n different things may be dealt with in two ways; it may either be taken or left. Hence, by the generalised fundamental principle, the total number of ways of dealing with n things :

$2 \times 2 \times 2 \times \ldots\ldots 2$, n times i.e., 2^n

But this include the case in which all the things are left, and therefore, rejecting this case, the total number of ways of forming a group by taking some or all of n different things is $2^n - 1$.

IV. Combinations of n things taken some or all at a time when n_1 of the things are alike of one kind, n_2 of the things are alike of another kind n_3 of the things are alike of a third kind. etc.

Result : The total, number of ways in which it is possible to make groups by taking some or all out of n $(=n_1 + n_2 + n_3 + \ldots)$ things, where n_1 things are alike of one kind and so on, is given by

$\{ (n_1 + 1) (n_2 + 1) (n_3 + 1)\ldots\} -1$

Proof : The n_1 things all alike of one kind may be dealt with in $(n_1 + 1)$ ways. We may take 0, 1, 2,….n, of them. Similarly n_2 things all alike of a second kind may be dealt with in $(n_2 +1)$ ways and n_3 things all alike of a third kind may de dealt with in $(n_3 +1)$ ways.

Proceeding in this way and using the generalised fundamental principle, the total number of ways of dealing with all n $(= n_1 + n_2 + n_3 +\ldots)$ things, where n_1, things are alike of one kind and so on, is given by

$(n_1 + 1) (n_2 + 1) (n_3 + 1)\ldots$

But this includes the case in which none of the things are taken. Hence, rejecting this case, total number of ways is $\{(n_1 + 1) (n_2 + 1) (n_3 + 1)\ldots\} -1\}$

V. The notion of Independence in Combinations

Many applications of Combinations involve the selection of subsets from two or more independent sets of objects or things. If the combination of a subset having r_1 objects form a set having n_1 objects does not affect the combination of a subset having r_2 objects from a different set having n_2 objects, we call the combinations as being independent. Whenever such combinations are independent, any subset of the first set of objects can be combined with each subset of the second set of the object to form a bigger combination. The total number of such combinations can be found by applying the generalised fundamental principle.

Result : The combinations of selecting r_1 things from a set having n_1 objects and r_2 things from a set having n_2 objects where combination of r_1 things, r_2 things are independent is given by

$$^{n_1}C_{r_1} \times {}^{n_2}C_{r_2}$$

Note : This result can be extended to more than two sets of objects by a similar reasoning.

Example 1: How many different permutations are possible from the letters of the word `CALCULUS'?

Solution: The word `CALCULUS' consists of 8 letters of which 2 are C and 2 are L, 2 are U and the rest are A and S. Hence , by result (I), the number of different permutations from the letters of the word `CALCULUS' taken all at a time

$$= \frac{8!}{2!2!2!1!1!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 2 \times 2} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5{,}040$$

Example 2: In how many ways can 17 billiard balls be arranged , if 7 of them are black, 6 red and 4 white?

Solution: We have, the required number of different arrangements:

$$\frac{17!}{7!\,6!\,4!} = 40{,}84{,}080$$

Example 3: An examination paper with 10 questions consists of 6 questions in Algebra and 4 questions in Geometry. At least one question from each section is to be attempted. In how many ways can this be done?

Solution: A student must answer atleast one question from each section and he may answer all questions from each section.

Consider Section I : Algebra. There are 6 questions and he may answer a question or may not answer it. These are the two alternatives associated with each of the six questions. Hence, by the generalised fundamental principle, he can deal with two questions in 2 × 2 ….6 factors = 2^6 number of ways. But this includes the possibility of none of the question from Algebra being attempted. This cannot be so, as he must attempt at least one question from this section. Hence, excluding this case, the number of ways in which Section I can be dealt with is (2^6 –1).

Similarly, the number of ways in which Section II can be dealt with is (2^4 –1).

Hence, by the Fundamental Principle, the examination paper can be attempted in (2^6 –1) (2^4 –1) number of ways.

Example 4: A man has 5 friends. In how many ways can he invite one or more of his friends to dinner?

Solution: By result, (III) of this section, as he has to select one or more of his 5 friends, he can do so in $2^5 - 1 = 31$ ways.

Note : This can also be done in the way, outlines below. He can invite his friends one by one, in twos, in threes, etc. and hence the number of ways.

$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= 5 + 10 + 10 + 5 + 1 = 31$ ways.

Example 5: There are 7 men and 3 ladies. Find the number of ways in which a committee of 6 can be formed of them if the committee is to include atleast two ladies?

Solution: The committee of six must include at least 2 ladies, i.e., two or more ladies. As there are only 3 ladies, the following possibilities arise:

The committee of 6 consists of (i) 4 men and 2 ladies (ii) 3 men and 3 ladies.

The number of ways for (i) = ${}^{7}C_{4} \times {}^{3}C_{2} = 35 \times 3 = 105$;

The number of ways for (ii) = ${}^{7}C_{3} \times {}^{3}C_{3} = 35 \times 1 = 35$.

Hence the total number of ways of forming a committee so as to include at least two ladies = 105 +35 = 140.

Example 6: Find the number of ways of selecting 4 letters from the word `EXAMINATION'.

Solution: There are 11 letters in the word of which A, I, N are repeated twice.

Thus we have 11 letters of 8 different kinds (A, A), (I, I), (N, N), E, X, M, T, O.

The group of four selected letters may take any of the following forms:

(i) Two alike and other two alike

(ii) Two alike and other two different

(iii) All four different

In case (i), the number of ways = ${}^{3}C_{2} = 3$.

In case (ii), the number of ways = ${}^{3}C_{1} \times {}^{7}C_{2} = 3 \times 21 = 63$.

In case (iii), the number of ways = ${}^{8}C_{4} = \dfrac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$

Hence , the required number of ways = 3 + 63 + 70 = 136 ways

SUMMARY

- **Fundamental principles of counting**

  (a) **Multiplication Rule:** If certain thing may be done in 'm' different ways and when it has been done, a second thing can be done in 'n ' different ways then total number of ways of doing both things simultaneously = m × n.

  (b) **Addition Rule :** It there are two alternative jobs which can be done in 'm' ways and in 'n' ways respectively then either of two jobs can be done in (m + n) ways.

- **Factorial:** The factorial n, written as n! or $\lfloor n$, represents the product of all integers from 1 to n both inclusive. To make the notation meaningful, when n = o, we define o! or $\lfloor o$ = 1.

  Thus, n! = n (n – 1) (n – 2) ….. …3.2.1

- **Permutations:** The ways of arranging or selecting smaller or equal number of persons or objects from a group of persons or collection of objects with due regard being paid to the order of arrangement or selection, are called permutations.

The number of permutations of n things chosen r at a time is given by

$$^nP_r = n\,(n-1)(n-2)\ldots(n-r+1)$$

where the product has exactly r factors.

- **Circular Permutations:** (a) n ordinary permutations equal one circular permutation.

  Hence there are $^nP_n/n$ ways in which all the n things can be arranged in a circle. This equals $(n-1)!$.

  (b) the number of necklaces formed with n beads of different colours = $=\frac{1}{2}\underline{|n-1}$.

- (a) Number of permutations of n distinct objects taken r at a time when a particular object is not taken in any arrangement is $^{n-1}p_r$.

  (b) Number of permutations of r objects out of n distinct objects when a particular object is always included in any arrangement is $r.\,^{n-1}p_{r-1}$

- **Combinations:** The number of ways in which smaller or equal number of things are arranged or selected from a collection of things where the order of selection or arrangement is not important, are called combinations.

  $^nC_r = n!/r!\,(n-r)!$

  $^nC_r = {^nC_{n-r}}$

  $^n\mathbf{C}_0 = n!/\{0!\,(n-0)!\} = n!\,/\,n! = \mathbf{1}$.

  $^n\mathbf{C}_n = n!/\{n!\,(n-n)!\} = n!\,/\,n!\,.\,0! = \mathbf{1}$.

(a) ^nC_r has a meaning only when r and n are integers $0 \le r \le n$ and $^nC_{n-r}$ has a meaning only when $0 \le n-r \le n$.

(i) $^{n+1}C_r = {^nC_r} + {^nC_{r-1}}$

(ii) $^nP_r = {^{n-1}P_r} + r\,^{n-1}P_{r-1}$

- Permutations when some of the things are alike, taken all at a time

$$p = \frac{n!}{n_1!\,n_2!\,n_3!}$$

- Permutations when each thing may be repeated once, twice,...upto r times in any arrangement = n!.

- The total number of ways in which it is possible to form groups by taking some or all of n things $(2^n - 1)$.

- The total, number of ways in which it is possible to make groups by taking some or all out of n $(=n_1 + n_2 + n_3 + \ldots)$ things, where n_1 things are alike of one kind and so on, is given by

  $\{(n_1 + 1)(n_2 + 1)(n_3 + 1)\ldots\} - 1$

- The combinations of selecting r_1 things from a set having n_1 objects and r_2 things from a set having n_2 objects where combination of r_1 things, r_2 things are independent is given by

  $^{n_1}C_{r_1} \times {}^{n_2}C_{r_2}$

EXERCISE 5 (C)

Choose the most appropriate option (a, b, c or d)

1. The value of $^{12}C_4 + {}^{12}C_3$ is
   (a) 715 (b) 710 (C) 716 (d) none of these
2. If $^np_r = 336$ and $^nC_r = 56$, then n and r will be
   (a) (3, 2) (b) (8, 3) (c) (7, 4) (d) none of these
3. If $^{18}C_r = {}^{18}C_{r+2}$, the value of $^rC_5$ is
   (a) 55 (b) 50 (c) 56 (d) none of these
4. If $^nC_{r-1} = 56$, $^nC_r = 28$ and $^nC_{r+1} = 8$, then r is equal to
   (a) 8 (b) 6 (c) 5 (d) none of these
5. A person has 8 friends. The number of ways in which he may invite one or more of them to a dinner is.
   (a) 250 (b) 255 (c) 200 (d) none of these
6. The number of ways in which a person can chose one or more of the four electrical appliances : T.V, Refrigerator, Washing Machine and a cooler is
   (a) 15 (b) 25 (c) 24 (d) none of these
7. If $^nc_{10} = {}^nc_{14}$, then $^{25}c_n$ is
   (a) 24 (b) 25 (c) 1 (d) none of these
8. Out of 7 gents and 4 ladies a committee of 5 is to be formed. The number of committees such that each committee includes at least one lady is
   (a) 400 (b) 440 (c) 441 (d) none of these
9. If $^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, then the value of r is
   (a) 7 (b) 5 (c) 6 (d) none of these
10. The number of diagonals in a decagon is
    (a) 30 (b) 35 (c) 45 (d) none of these
    Hint: The number of diagonals in a polygon of n sides is $\frac{1}{2}n(n-3)$.
11. There are 12 points in a plane of which 5 are collinear. The number of triangles is
    (a) 200 (b) 211 (c) 210 (d) none of these
12. The number of straight lines obtained by joining 16 points on a plane, no three of them being on the same line is
    (a) 120 (b) 110 (c) 210 (d) none of these

13. At an election there are 5 candidates and 3 members are to be elected. A voter is entitled to vote for any number of candidates not greater than the number to be elected. The number of ways a voter choose to vote is

    (a) 20 (b) 22 (c) 25 (d) none of these

14. Every two persons shakes hands with each other in a party and the total number of hand shakes is 66. The number of guests in the party is

    (a) 11 (b) 12 (c) 13 (d) 14

15. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is

    (a) 6 (b) 18 (c) 12 (d) 9

16. The number of ways in which 12 students can be equally divided into three groups is

    (a) 5775 (b) 7575 (c) 7755 (d) none of these

17. The number of ways in which 15 mangoes can be equally divided among 3 students is

    (a) $\underline{|15} / \underline{|(5)}^4$ (b) $\underline{|15} / \underline{|(5)}^3$ (c) $\underline{|15} / \underline{|(5)}^2$ (d) none of these

18. 8 points are marked on the circumference of a circle. The number of chords obtained by joining these in pairs is

    (a) 25 (b) 27 (c) 28 (d) none of these

19. A committee of 3 ladies and 4 gents is to be formed out of 8 ladies and 7 gents. Mrs. X refuses to serve in a committee in which Mr. Y is a member. The number of such committees is

    (a) 1530 (b) 1500 (c) 1520 (d) 1540

20. If ${}^{500}C_{92} = {}^{499}C_{92} + {}^{n}C_{91}$ then n is

    (a) 501 (b) 500 (c) 502 (d) 499

21. The Supreme Court has given a 6 to 3 decision upholding a lower court; the number of ways it can give a majority decision reversing the lower court is

    (a) 256 (b) 276 (c) 245 (d) 226.

22. Five bulbs of which three are defective are to be tried in two bulb points in a dark room. Number of trials the room shall be lighted is

    (a) 6 (b) 8 (c) 5 (d) 7.

MISCELLANEOUS EXERCISE

EXERCISE 5(D)

Choose the appropriate option a,b,c or d

1. The letters of the words `CALCUTTA' and `AMERICA' are arranged in all possible ways. The ratio of the number of their arrangements is

    (a) 1:2 (b) 2:1 (c) 2:2 (d) none of these

2. The ways of selecting 4 letters from the word `EXAMINATION' is

(a) 136 (b) 130 (c) 125 (d) none of these

3. The number of different words that can be formed with 12 consonants and 5 vowels by taking 4 consonants and 3 vowels in each word is

(a) $^{12}c_4 \times {}^5c_3$ (b) $^{17}c_7$ (c) $4950 \times \underline{|7}!$ (d) none of these

4. Eight guests have to be seated 4 on each side of a long rectangular table.2 particular guests desire to sit on one side of the table and 3 on the other side. The number of ways in which the sitting arrangements can be made is

(a) 1732 (b) 1728 (c) 1730 (d) 1278.

5 A question paper contains 6 questions, each having an alternative.

The number of ways an examine can answer one or more questions is

(a) 720 (b) 728 (c) 729 (d) none of these

6. $^{51}c_{31}$ is equal to

(a) $^{51}c_{20}$ (b) $2.^{50}c_{20}$ (c) $2.^{45}c_{15}$ (d) none of these

7. The number of words that can be made by rearranging the letters of the word APURNA so that vowels and consonants appear alternate is

(a) 18 (b) 35 (c) 36 (d) none of these

8. The number of arrangement of the letters of the word `COMMERCE' is

(a) $\underline{|8}$ (b) $\underline{|8} / (\underline{|2}\underline{|2}\underline{|2})$ (c) 7 ! (d) none of these

9. A candidate is required to answer 6 out of 12 questions which are divided into two groups containing 6 questions in each group. He is not permitted to attempt not more than four from any group. The number of choices are.

(a) 750 (b) 850 (c) 800 (d) none of these

10. The results of 8 matches (Win, Loss or Draw) are to be predicted. The number of different forecasts containing exactly 6 correct results is

(a) 316 (b) 214 (c) 112 (d) none of these

11. The number of ways in which 8 different beads be strung on a necklace is

(a) 2500 (b) 2520 (c) 2250 (d) none of these

12. The number of different factors the number 75,600 has is

(a) 120 (b) 121 (c) 119 (d) none of these

13. The number of 4 digit numbers formed with the digits 1, 1, 2, 2, 3, 4 is

(a) 100 (b) 101 (c) 201 (d) none of these

14. The number of ways a person can contribute to a fund out of 1 ten-rupee note, 1 five-rupee note, 1 two-rupee and 1 one rupee note is

(a) 15 (b) 25 (c) 10 (d) none of these

15. The number of ways in which 9 things can be divided into twice groups containing 2,3, and 4 things respectively is

    (a) 1250 (b) 1260 (c) 1200 (d) none of these

16. ${}^{(n-1)}P_r + r.{}^{(n-1)}P_{(r-1)}$ is equal to

    (a) ${}^nC_r$ (b) $\underline{|n} / (\underline{|r}\ \underline{|n-r})$ (c) ${}^np_r$ (d) none of these

17. $\underline{|2n}$ can be written as

    (a) $2^n \{1.3.5....(2n-1)\}\underline{|n}$ (b) $2^n \underline{|n}$ (c) {1.3.5…..(2n –1)} (d) none of these

18. The number of even numbers greater than 300 can be formed with the digits 1, 2, 3, 4, 5 without repetion is

    (a) 110 (b) 112 (c) 111 (d) none of these

19. 5 letters are written and there are five letter-boxes. The number of ways the letters can be dropped into the boxes, are in each

    (a) 119 (b) 120 (c) 121 (d) none of these

20. ${}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + ...+ {}^nC_n$ equals

    (a) $2^n - 1$ (b) 2^n (c) $2^n + 1$ (d) none of these

ANSWERS

Exercise 5(A)

1.	(c)	**2.**	(b)	**3.**	(a)	**4.**	(b)	**5.**	(a)	**6.**	(b)	**7.**	(d)	**8.**	(a)
9.	(b)	**10.**	(c)	**11.**	(b)	**12.**	(a)	**13.**	(c)	**14.**	(b)	**15.**	(a)	**16.**	(c)
17.	(a)	**18.**	(b)	**19.**	(d)	**20.**	(a)	**21**	(c)	**22**	(c)	**23**	(a)		

Exercise 5 (B)

1.	(c)	**2.**	(a)	**3.**	(b)	**4.**	(c)	**5.**	(b)	**6.**	(b)	**7.**	(c)	**8.**	(d)
9.	(a)	**10.**	(c)	**11.**	(c)	**12.**	(b)	**13.**	(c)	**14.**	(b)	**15.**	(a)	**16.**	(b)
17.	(b)	**18.**	(c)	**19.**	(c)	**20.**	(a)	**21**	(a)						

Exercise 5 (C)

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(b)	**5.**	(b)	**6.**	(a)	**7.**	(b)	**8.**	(c)
9.	(a)	**10.**	(b)	**11.**	(c)	**12.**	(a)	**13.**	(c)	**14.**	(b)	**15.**	(b)	**16.**	(a)
17.	(b)	**18.**	(c)	**19.**	(d)	**20.**	(d)	**21.**	(a)	**22.**	(d)				

Exercise 5 (D)

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(b)	**5.**	(b)	**6.**	(a)	**7.**	(c)	**8.**	(b)&(c)
9.	(b)	**10.**	(c)	**11.**	(b)	**12.**	(c)	**13.**	(d)	**14.**	(a)	**15.**	(b)	**16.**	(c)
17.	(a)	**18.**	(c)	**19.**	(b)	**20.**	(a)								

ADDITIONAL QUESTION BANK

1. There are 6 routes for journey from station A to station B. In how many ways you may go from A to B and return if for returning you make a choice of any of the routes?

   (a) 6 (b) 12 (c) 36 (d) 30

2. As per question No.(1) if you decided to take the same route you may do it in _______ number of ways.

   (a) 6 (b) 12 (c) 36 (d) 30

3. As per question No.(1) if you decided not to take the same route you may do it in _______ number of ways.

   (a) 6 (b) 12 (c) 36 (d) 30

4. How many telephones connections may be allotted with 8 digits form the numbers 0,1,2 …….9?

   (a) 10^8 (b) 10! (c) ${}^{10}C_8$ (d) ${}^{10}P_8$

5. In how many different ways 3 rings of a lock can not combine when each ring has digits 0,1,2……9 leading to unsuccessful events?

   (a) 999 (b) 10^3 (c) 10! (d) 997

6. A dealer provides you Maruti Car & Van in 2 body patterns and 5 different colours. How many choices are open to you?

   (a) 2 (b) 7 (c) 20 (d) 10

7. 3 persons go into a railway carriage having 8 seats. In how many ways they may occupy the seats?

   (a) ${}^8P_3$ (b) ${}^8C_3$ (c) ${}^8C_5$ (d) None

8. Find how many five-letter words can be formed out of the word "LOGARITHMS" (the words may not convey any meaning)

   (a) ${}^{10}P_5$ (b) ${}^{10}C_5$ (c) ${}^9C_4$ (d) None

9. How many 4 digits numbers greater than 7000 can be formed out of the digits 3,5,7,8,9?

   (a) 24 (b) 48 (c) 72 (d) 50

10. In how many ways 5 Sanskrit 3 English and 3 Hindi books be arranged keeping the books of the same language together?

    (a) 5! × 3! × 3! × 3! (b) 5! × 3! × 3! (c) ${}^5P_3$ (d) None

11. In how many ways can 6 boys and 6 girls be seated around a table so that no 2 boys are adjacent?

    (a) $4! \times 5!$ (b) $5! \times 6!$ (c) ${}^6P_6$ (d) $5 \times {}^6P_6$

12. In how many ways can 4 Americans and 4 English men be seated at a round table so that no 2 Americans may be together?

    (a) $4! \times 3!$ (b) ${}^4P_4$ (c) $3 \times {}^4P_4$ (d) ${}^4C_4$

13. The chief ministers of 17 states meet to discuss the hike in oil price at a round table. In how many ways they seat themselves if the Kerala and Bengal chief ministers choose to sit together?

    (a) $15! \times 2!$ (b) $17! \times 2!$ (c) $16! \times 2!$ (d) None

14. The number of permutation of the word `ACCOUNTANT' is

    (a) $10! \div (2!)^4$ (b) $10! \div (2!)^3$ (c) $10!$ (d) None

15. The number of permutation of the word `ENGINEERING' is

    (a) $11! \div [(3!)^2(2!)^2]$ (b) $11!$ (c) $11! \div [(3!)(2!)]$ (d) None

16. The number of arrangements that can be made with the word `ASSASSINATION' is

    (a) $13! \div [3! \times 4! \times (2!)^2]$ (b) $13! \div [3! \times 4! \times 2!]$ (c) $13!$ (d) None

17. How many numbers higher than a million can be formed with the digits 0,4,4,5,5,5,3?

    (a) 420 (b) 360 (c) $7!$ (d) None

18. The number of permutation of the word `ALLAHABAD' is

    (a) $9! \div (4! \times 2!)$ (b) $9! \div 4!$ (c) $9!$ (d) None

19. In how many ways the vowels of the word `ALLAHABAD' will occupy the even places?

    (a) 120 (b) 60 (c) 30 (d) None

20. How many arrangements can be made with the letter of the word `MATHEMATICS'?

    (a) $11! \div (2!)^3$ (b) $11! \div (2!)^2$ (c) $11!$ (d) None

21. In how many ways of the word `MATHEMATICS' be arranged so that the vowels occur together?

    (a) $11! \div (2!)^3$ (b) $(8! \times 4!) \div (2!)^3$ (c) $12! \div (2!)^3$ (d) None

22. In how many ways can the letters of the word `ARRANGE' be arranged?

    (a) 1,200 (b) 1,250 (c) 1,260 (d) 1,300

23. In how many ways the word `ARRANGE' be arranged such that the 2 'R's come together?

(a) 400 (b) 440 (c) 360 (d) None

24. In how many ways the word `ARRANGE' be arranged such that the 2 'R's do not come together?

(a) 1,000 (b) 900 (c) 800 (d) None

25. In how many ways the word `ARRANGE' be arranged such that the 2 'R's and 2 'A's come together?

(a) 120 (b) 130 (c) 140 (d) None

26. If $^{n}P_{4} = 12,\ ^{n}P_{2}$ the value of n is

(a) 12 (b) 6 (c) -1 (d) both 6 -1

27. If $4.{}^{n}P_{3} = 5.{}^{n-1}P_{3}$ the value of n is

(a) 12 (b) 13 (c) 14 (d) 15

28. $^{n}P_{r} \div {}^{n-1}P_{r-1}$ is

(a) n (b) $n!$ (c) $(n–1)!$ (d) $^{n}C_{n}$

29. The total number of numbers less than 1000 and divisible by 5 formed with 0,1,2,…..9 such that each digit does not occur more than once in each number is

(a) 150 (b) 152 (c) 154 (d) None

30. The number of ways in which 8 examination papers be arranged so that the best and worst papers never come together is

(a) $8! – 2 \times 7!$ (b) $8! – 7!$ (c) $8!$ (d) None

31. In how many ways can 4 boys and 3 girls stand in a row so that no two girls are together?

(a) $5! \times 4! \div 3!$ (b) $^{5}P_{3} \times 3$ (c) $^{5}P_{3} \times 2$ (d) None

32. In how many ways can 3 boys and 4 girls be arranged in a row so that all the three boys are together?

(a) $4! \times 3!$ (b) $5! \times 3!$ (c) $7!$ (d) None

33. How many six digit numbers can be formed out of 4 5 …..9 no digits being repeated?

(a) $6! – 5!$ (b) $6!$ (c) $6! + 5!$ (d) None

34. In terms of question No.(33) how many of them are not divisible by 5?

(a) 6! – 5! (b) 6! (c) 6! + 5! (d) None

35. In how many ways the word `FAILURE' can be arranged so that the consonants occupy only the odd positions?

(a) 4! (b) $(4!)^2$ (c) 7! ÷ 3! (d) None

36. In how many ways can the word `STRANGE' be arranged so that the vowels are never separated?

(a) 6! × 2! (b) 7! (c) 7! ÷ 2! (d) None

37. In how many ways can the word `STRANGE' be arranged so that the vowels never come together?

(a) 7! – 6! × 2! (b) 7! – 6! (c) $^7P_6$ (d) None

38. In how many ways can the word `STRANGE' be arranged so that the vowels ocupy only the odd places?

(a) $^5P_5$ (b) $^5P_5 \times {}^4P_4$ (c) $^5P_5 \times {}^4P_2$ (d) None

39. How many four digits number can be formed by using 1,2, ……..7?

(a) $^7P_4$ (b) $^7P_3$ (c) $^7C_4$ (d) None

40. How many four digits numbers can be formed by using 1,2, …..7 which are grater than 3400?

(a) 500 (b) 550 (c) 560 (d) None

41. In how many ways it is possible to write the word `ZENITH' in a dictionary?

(a) $^6P_6$ (b) $^6C_6$ (c) $^6P_0$ (d) None

42. In terms of question No.(41) what is the rank or order of the word `ZENITH' in the dictionary?

(a) 613 (b) 615 (c) 616 (d) 618

43. If $^{n-1}P_3 \div {}^{n+1}P_3 = \frac{5}{12}$ the value of n is

(a) 8 (b) 4 (c) 5 (d) 2

44. If $^{n+3}P_6 \div {}^{n+2}P_4 = 14$ the value of n is

(a) 8 (b) 4 (c) 5 (d) 2

45. If $^7P_n \div {}^7P_{n-3} = 60$ the value of n is

(a) 8 (b) 4 (c) 5 (d) 2

46. There are 4 routes for going from Dumdum to Sealdah and 5 routes for going from Sealdah to Chandni. In how many different ways can you go from Dumdum to Chandni via Sealdah?

(a) 9 (b) 1 (c) 20 (d) None

47. In how many ways can 5 people occupy 8 vacant chairs?

(a) 5,720 (b) 6,720 (c) 7,720 (d) None

48. If there are 50 stations on a railway line how many different kinds of single first class tickets may be printed to enable a passenger to travel from one station to other?

(a) 2,500 (b) 2,450 (c) 2,400 (d) None

49. How many six digits numbers can be formed with the digits 9, 5, 3, 1, 7, 0?

(a) 600 (b) 720 (c) 120 (d) None

50. In terms of question No.(49) how many numbers will have 0's in ten's place?

(a) 600 (b) 720 (c) 120 (d) None

51. How many words can be formed with the letters of the word `SUNDAY'?

(a) 6! (b) 5! (c) 4! (d) None

52. How many words can be formed beginning with 'N' with the letters of the word `SUNDAY'?

(a) 6! (b) 5! (c) 4! (d) None

53. How many words can be formed beginning with 'N' and ending in 'A' with the letters of the word `SUNDAY'?

(a) 6! (b) 5! (c) 4! (d) None

54. How many different arrangements can be made with the letters of the word `MONDAY'?

(a) 6! (b) 8! (c) 4! (d) None

55. How many different arrangements can be made with the letters of the word `ORIENTAL'?

(a) 6! (b) 8! (c) 4! (d) None

56. How many different arrangements can be made beginning with 'A' and ending in 'N' with the letters of the word `MONDAY'?

(a) 6! (b) 8! (c) 4! (d) None

57. How many different arrangements can be made beginning with 'A' and ending with 'N' with the letters of the word `ORIENTAL'?

(a) 6! (b) 8! (c) 4! (d) None

58. In how many ways can a consonant and a vowel be chosen out of the letters of the word `LOGARITHM'?

(a) 18 (b) 15 (c) 3 (d) None

59. In how many ways can a consonant and a vowel be chosen out of the letters of the word `EQUATION'?

(a) 18 (b) 15 (c) 3 (d) None

60. How many different words can be formed with the letters of the word `TRIANGLE'?

(a) 8! (b) 7! (c) 6! (d) $2! \times 6!$

61. How many different words can be formed beginning with 'T' of the word `TRIANGLE'?

(a) 8! (b) 7! (c) 6! (d) $2! \times 6!$

62. How many different words can be formed beginning with 'E' of the letters of the word `TRIANGLE'?

(a) 8! (b) 7! (c) 6! (d) $2! \times 6!$

63. In question No. (60) how many of them will begin with 'T' and end with 'E'?

(a) 8! (b) 7! (c) 6! (d) $2! \times 6!$

64. In question No.(60) how many of them have 'T' and 'E' in the end places?

(a) 8! (b) 7! (c) 6! (d) $2! \times 6!$

65. In question No.(60) how many of them have consonants never together?

(a) $8! - 4! \times 5!$ (b) ${}^6P_3 \times 5!$ (c) $2! \times 5! \times 3!$ (d) ${}^4P_3 \times 5!$

66. In question No.(60) how many of them have arrangements that no two vowels are together?

(a) $8! - 4! \times 5!$ (b) ${}^6P_3 \times 5!$ (c) $2! \times 5! \times 3!$ (d) ${}^4P_3 \times 5!$

67. In question No.(60) how many of them have arrangements that consonants and vowels are always together?

(a) $8! - 4! \times 5!$ (b) ${}^6P_3 \times 5!$ (c) $2! \times 5! \times 3!$ (d) ${}^4P_3 \times 5!$

68. In question No.(60) how many of them have arrangements that vowels occupy odd places?

(a) $8! - 4! \times 5!$ (b) ${}^6P_3 \times 5!$ (c) $2! \times 5! \times 3!$ (d) ${}^4P_3 \times 5!$

69. In question No.(60) how many of them have arrangements that the relative positions of the vowels and consonants remain unchanged?

(a) $8! - 4! \times 5!$ (b) ${}^6P_3 \times 5!$ (c) $2! \times 5! \times 3!$ (d) $5! \times 3!$

70. In how many ways the letters of the word `FAILURE' can be arranged with the condition that the four vowels are always together?

(a) $(4!)^2$ (b) 4! (c) 7! (d) None

71. In how many ways n books can be arranged so that two particular books are not together?

(a) $(n-2) \times (n-1)!$ (b) $n \times n!$ (c) $(n-2) \times (n-2)!$ (d) None

72. In how many ways can 3 books on Mathematics and 5 books on English be placed so that books on the same subject always remain together?

(a) 1,440 (b) 240 (c) 480 (d) 144

73. 6 papers are set in an examination out of which two are mathematical. In how many ways can the papers be arranged so that 2 mathematical papers are together?

(a) 1,440 (b) 240 (c) 480 (d) 144

74. In question No.(73) will your answer be different if 2 mathematical papers are not consecutive?

(a) 1,440 (b) 240 (c) 480 (d) 144

75. The number of ways the letters of the word `SIGNAL' can be arranged such that the vowels occupy only odd positions is________.

(a) 1,440 (b) 240 (c) 480 (d) 144

76. In how many ways can be letters of the word `VIOLENT' be arranged so that the vowels occupy even places only?

(a) 1,440 (b) 240 (c) 480 (d) 144

77. How many numbers between 1000 and 10000 can be formed with 1, 2, …..9?

(a) 3,024 (b) 60 (c) 78 (d) None

78. How many numbers between 3000 and 4000 can be formed with 1, 2, …..6?

(a) 3,024 (b) 60 (c) 78 (d) None

79. How many numbers greater than 23,000 can be formed with 1, 2, …..5?

(a) 3,024 (b) 60 (c) 78 (d) None

80. If you have 5 copies of one book, 4 copies of each of two books, 6 copies each of three books and single copy of 8 books you may arrange it in ________number of ways.

(a) $\frac{39!}{5!\times(4!)^2\times(6!)^3}$ (b) $\frac{39!}{5!\times8!\times(4!)^2\times(6!)^3}$ (c) $\frac{39!}{5!\times8!\times4!\times(6!)^2}$ (d) $\frac{39!}{5!\times8!\times4!\times6!}$

81. How many arrangements can be made out of the letters of the word "PERMUTATION"?

(a) $\frac{1}{2}\,{}^{11}P_{11}$ (b) ${}^{11}P_{11}$ (c) ${}^{11}C_{11}$ (d) None

82. How many numbers greater than a million can be formed with the digits: One 0 Two 1 One 3 and Three 7?

(a) 360 (b) 240 (c) 840 (d) 20

83. How many arrangements can be made out of the letters of the word `INTERFERENCE' so that no two consonant are together?

(a) 360 (b) 240 (c) 840 (d) 20

84. How many different words can be formed with the letter of the word "HARYANA"?

(a) 360 (b) 240 (c) 840 (d) 20

85. In question No.(84) how many arrangements are possible keeping 'H' and 'N' together?

(a) 360 (b) 240 (c) 840 (d) 20

86. In question No.(84) how many arrangements are possible beginning with 'H' and ending with 'N'?

(a) 360 (b) 240 (c) 840 (d) 20

87. A computer has 5 terminals and each terminal is capable of four distinct positions including the positions of rest what is the total number of signals that can be made?

(a) 20 (b) 1,020 (c) 1,023 (d) None

88. In how many ways can 9 letters be posted in 4 letter boxes?

(a) 4^9 (b) 4^5 (c) ${}^9P_4$ (d) ${}^9C_4$

89. In how many ways can 8 beads of different colour be strung on a ring?

(a) 7! ÷ 2 (b) 7! (c) 8! (d) 8! ÷ 2

90. In how many ways can 8 boys form a ring?

(a) 7! ÷ 2 (b) 7! (c) 8! (d) 8! ÷ 2

91. In how many ways 6 men can sit at a round table so that all shall not have the same neighbours in any two occasions?

(a) 5! ÷ 2 (b) 5! (c) $(7!)^2$ (d) 7!

92. In how many ways 6 men and 6 women sit at a round table so that no two men are together?

(a) 5! ÷ 2 (b) 5! (c) 5! 6! (d) 7!

93. In how many ways 4 men and 3 women are arranged at a round table if the women never sit together?

(a) 6 × 6! (b) 6! (c) 7! (d) None

94. In how many ways 4 men and 3 women are arranged at a round table if the women always sit together?

(a) 6 × 6! (b) 6! (c) 7! (d) None

95. A family comprised of an old man, 6 adults and 4 children is to be seated is a row with the condition that the children would occupy both the ends and never occupy either side of the old man. How many sitting arrangements are possible?

(a) 4! × 5! × 7! (b) 4! × 5! × 6! (c) 2! × 4! × 5! × 6! (d) None

96. The total number of sitting arrangements of 7 persons in a row if 3 persons sit together in a particular order is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

97. The total number of sitting arrangements of 7 persons in a row if 3 persons sit together in any order is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

98. The total number of sitting arrangements of 7 persons in a row if two persons occupy the end seats is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

99. The total number of sitting arrangements of 7 persons in a row if one person occupies the middle seat is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

100. If all the permutations of the letters of the word `CHALK' are written in a dictionary the rank of this word will be ____________.

(a) 30 (b) 31 (c) 32 (d) None

101. In a ration shop queue 2 boys, 2 girls, and 2 men are standing in such a way that the boys the girls and the men are together each. The total number of ways of arranging the queue is ______.

(a) 42 (b) 48 (c) 24 (d) None

102. If you have to make a choice of 7 questions out of 10 questions set, you can do it in _______ number of ways.

(a) ${}^{10}C_7$ (b) ${}^{10}P_7$ (c) $7! \times {}^{10}C_7$ (d) None

103. From 6 boys and 4 girls 5 are to be seated. If there must be exactly 2 girls the number of ways of selection is ______.

(a) 240 (b) 120 (c) 60 (d) None

104. In your office 4 posts have fallen vacant. In how many ways a selection out of 31 candidates can be made if one candidate is always included?

(a) ${}^{30}C_3$ (b) ${}^{30}C_4$ (c) ${}^{31}C_3$ (d) ${}^{31}C_4$

105. In question No.(104) would your answer be different if one candidate is always excluded?

(a) ${}^{30}C_3$ (b) ${}^{30}C_4$ (c) ${}^{31}C_3$ (d) ${}^{31}C_4$

106. Out of 8 different balls taken three at a time without taking the same three together more than once for how many number of times you can select a particular ball?

(a) ${}^{7}C_2$ (b) ${}^{8}C_3$ (c) ${}^{7}P_2$ (d) ${}^{8}P_3$

107. In question No.(106) for how many number of times you can select any ball?

(a) ${}^{7}C_2$ (b) ${}^{8}C_3$ (c) ${}^{7}P_2$ (d) ${}^{8}P_3$

108. In your college Union Election you have to choose candidates. Out of 5 candidates 3 are to be elected and you are entitled to vote for any number of candidates but not exceeding the number to be elected. You can do it in _________ ways.

(a) 25 (b) 5 (c) 10 (d) None

109. In a paper from 2 groups of 5 questions each you have to answer any 6 questions attempting at least 2 questions from each group. This is possible in ________ number of ways.

(a) 50 (b) 100 (c) 200 (d) None

110. Out of 10 consonants and 4 vowels how many words can be formed each containing 6 consonant and 3 vowels?

(a) ${}^{10}C_6 \times {}^4C_3$ (b) ${}^{10}C_6 \times {}^4C_3 \times 9!$ (c) ${}^{10}C_6 \times {}^4C_3 \times 10!$ (d) None

111. A boat's crew consist of 8 men, 3 of whom can row only on one side and 2 only on the other. The number of ways in which the crew can be arranged is _________.

(a) ${}^3C_1 \times (4!)^2$ (b) ${}^3C_1 \times 4!$ (c) ${}^3C_1$ (d) None

112. A party of 6 is to be formed from 10 men and 7 women so as to include 3 men and 3 women. In how many ways the party can be formed if two particular women refuse to join it?

(a) 4,200 (b) 600 (c) 3,600 (d) None

113. You are selecting a cricket team of first 11 players out of 16 including 4 bowlers and 2 wicket-keepers. In how many ways you can do it so that the team contains exactly 3 bowlers and 1 wicket-keeper?

(a) 960 (b) 840 (c) 420 (d) 252

114. In question No.(113) would your answer be different if the team contains at least 3 bowlers and at least 1 wicket-keeper?

(a) 2,472 (b) 960 (c) 840 (d) 420

115. A team of 12 men is to be formed out of *n* persons. Then the number of times 2 men 'A' and 'B' are together is ___________.

(a) ${}^nC_{12}$ (b) ${}^{n-1}C_{11}$ (c) ${}^{n-2}C_{10}$ (d) None

116. In question No.(115) the number of times 3 men 'C' 'D' and 'E' are together is _____.

(a) ${}^nC_{12}$ (b) ${}^{n-1}C_{11}$ (c) ${}^{n-2}C_{10}$ (d) ${}^{n-2}C_{10}$

117. In question No.(115) it is found that 'A' and 'B' are three times as often together as 'C' 'D' and 'E' are. Then the value of *n* is ____________.

(a) 32 (b) 23 (c) 9 (d) None

118. The number of combinations that can be made by taking 4 letters of the word `COMBINATION' is _______.

(a) 70 (b) 63 (c) 3 (d) 136

119. If ${}^{18}C_n = {}^{18}C_{n+2}$ then the value of *n* is __________

(a) 0 (b) –2 (c) 8 (d) None

120. If $^{n}C_{6} \div {}^{n-2}C_{3} = \frac{91}{4}$ then the value of n is __________

(a) 15 (b) 14 (c) 13 (d) None

121. In order to pass PE-II examination minimum marks have to be secured in each of 7 subjects. In how many ways can a pupil fail?

(a) 128 (b) 64 (c) 127 (d) 63

122. In how many ways you can answer one or more questions out of 6 questions each having an alternative?

(a) 728 (b) 729 (c) 128 (d) 129

123. There are 12 points in a plane no 3 of which are collinear except that 6 points which are collinear. The number of different straight lines is _________.

(a) 50 (b) 51 (c) 52 (d) None

124. In question No.(123) the number of different triangles formed by joining the straight lines is ________.

(a) 220 (b) 20 (c) 200 (d) None

125. A committee is to be formed of 2 teachers and 3 students out of 10 teachers and 20 students. The numbers of ways in which this can be done is ______.

(a) $^{10}C_{2} \times {}^{20}C_{3}$ (b) $^{9}C_{1} \times {}^{20}C_{3}$ (c) $^{10}C_{2} \times {}^{19}C_{3}$ (d) None

126. In question No.(125) if a particular teacher is included the number of ways in which this can be done is _________.

(a) $^{10}C_{2} \times {}^{20}C_{3}$ (b) $^{9}C_{1} \times {}^{20}C_{3}$ (c) $^{10}C_{2} \times {}^{19}C_{3}$ (d) None

127. In question No.(125) if a particular student is excluded the number of ways in which this can be done is _________.

(a) $^{10}C_{2} \times {}^{20}C_{3}$ (b) $^{9}C_{1} \times {}^{20}C_{3}$ (c) $^{10}C_{2} \times {}^{19}C_{3}$ (d) None

128. In how many ways 21 red balls and 19 blue balls can be arranged in a row so that no two blue balls are together?

(a) 1540 (b) 1520 (c) 1560 (d) None

129. In forming a committee of 5 out of 5 males and 6 females how many choices you have to make so that there are 3 males and 2 females?

(a) 150 (b) 200 (c) 1 (d) 461

130. In question No.(129) how many choices you have to make if there are 2 males?

(a) 150 (b) 200 (c) 1 (d) 461

131. In question No.(129) how many choices you have to make if there is no female?

(a) 150 (b) 200 (c) 1 (d) 461

132. In question No.(129) how many choices you have to make if there is at least one female?

(a) 150 (b) 200 (c) 1 (d) 461

133. In question No.(129) how many choices you have to make if there are not more than 3 males?

(a) 200 (b) 1 (c) 461 (d) 431

134. From 7 men and 4 women a committee of 5 is to be formed. In how many ways can this be done to include at least one woman?

(a) 441 (b) 440 (c) 420 (d) None

135. You have to make a choice of 4 balls out of one red one blue and ten white balls. The number of ways this can be done to always include the red ball is ___________.

(a) ${}^{11}C_3$ (b) ${}^{10}C_3$ (c) ${}^{10}C_4$ (d) None

136. In question No.(135) the number of ways in which this can be done to include the red ball but exclude the blue ball always is _______.

(a) ${}^{11}C_3$ (b) ${}^{10}C_3$ (c) ${}^{10}C_4$ (d) None

137. In question No.(135) the number of ways in which this can be done to exclude both the red and blues ball is _______.

(a) ${}^{11}C_3$ (b) ${}^{10}C_3$ (c) ${}^{10}C_4$ (d) None

138. Out of 6 members belonging to party 'A' and 4 to party 'B' in how many ways a committee of 5 can be selected so that members of party 'A' are in a majority?

(a) 180 (b) 186 (c) 185 (d) 184

139. A question paper divided into 2 groups consisting of 3 and 4 questions respectively carries the note "it is not required to answer all the questions. One question must be answered from each group". In how many ways you can select the questions?

(a) 10 (b) 11 (c) 12 (d) 13

140. The number of words which can be formed with 2 different consonants and 1 vowel out of 7 different consonants and 3 different vowels the vowel to lie between 2 consonants is ______.

(a) $3 \times 7 \times 6$ (b) $2 \times 3 \times 7 \times 6$ (c) $2 \times 3 \times 7$ (d) None

141. How many combinations can be formed of 8 counters marked 1 2 …8 taking 4 at a time there being at least one odd and even numbered counter in each combination?

(a) 68 (b) 66 (c) 64 (d) 62

142. Find the number of ways in which a selection of 4 letters can be made from the word `MATHEMATICS'.

(a) 130 (b) 132 (c) 134 (d) 136

143. Find the number of ways in which an arrangement of 4 letters can be made from the word `MATHEMATICS'.

(a) 1680 (b) 756 (c) 18 (d) 2,454

144. In a cross word puzzle 20 words are to be guessed of which 8 words have each an alternative solution. The number of possible solution is ________.

(a) $(2\times8)^2$ (b) $^{20}C_{16}$ (c) $^{20}C_{8}$ (d) None

ANSWERS

1.	(c)	**19.**	(b)	**37.**	(a)	**55.**	(b)	**73.**	(b)	**91.**	(a)	**109.**	(c)	**127.**	(c)
2.	(a)	**20.**	(a)	**38.**	(c)	**56.**	(c)	**74.**	(c)	**92.**	(c)	**110.**	(b)	**128.**	(a)
3.	(d)	**21.**	(b)	**39.**	(a)	**57.**	(a)	**75.**	(d)	**93.**	(d)	**111.**	(a)	**129.**	(a)
4.	(a)	**22.**	(c)	**40.**	(c)	**58.**	(a)	**76.**	(d)	**94.**	(d)	**112.**	(c)	**130.**	(b)
5.	(a)	**23.**	(c)	**41.**	(a)	**59.**	(b)	**77.**	(a)	**95.**	(d)	**113.**	(a)	**131.**	(c)
6.	(c)	**24.**	(b)	**42.**	(c)	**60.**	(a)	**78.**	(b)	**96.**	(a)	**114.**	(a)	**132.**	(d)
7.	(a)	**25.**	(a)	**43.**	(a)	**61.**	(b)	**79.**	(d)	**97.**	(b)	**115.**	(c)	**133.**	(d)
8.	(a)	**26.**	(b)	**44.**	(b)	**62.**	(b)	**80.**	(a)	**98.**	(c)	**116.**	(d)	**134.**	(a)
9.	(c)	**27.**	(d)	**45.**	(c)	**63.**	(c)	**81.**	(a)	**99.**	(b)	**117.**	(a)	**135.**	(a)
10.	(a)	**28.**	(a)	**46.**	(c)	**64.**	(d)	**82.**	(a)	**100.**	(c)	**118.**	(d)	**136.**	(b)
11.	(b)	**29.**	(c)	**47.**	(b)	**65.**	(a)	**83.**	(d)	**101.**	(b)	**119.**	(c)	**137.**	(c)
12.	(a)	**30.**	(a)	**48.**	(b)	**66.**	(b)	**84.**	(c)	**102.**	(a)	**120.**	(d)	**138.**	(b)
13.	(a)	**31.**	(a)	**49.**	(a)	**67.**	(c)	**85.**	(b)	**103.**	(b)	**121.**	(c)	**139.**	(c)
14.	(a)	**32.**	(b)	**50.**	(c)	**68.**	(d)	**86.**	(d)	**104.**	(a)	**122.**	(a)	**140.**	(a)
15.	(a)	**33.**	(b)	**51.**	(a)	**69.**	(d)	**87.**	(c)	**105.**	(b)	**123.**	(c)	**141.**	(a)
16.	(a)	**34.**	(a)	**52.**	(b)	**70.**	(a)	**88.**	(a)	**106.**	(a)	**124.**	(c)	**142.**	(d)
17.	(b)	**35.**	(b)	**53.**	(c)	**71.**	(a)	**89.**	(a)	**107.**	(b)	**125.**	(a)	**143.**	(d)
18.	(a)	**36.**	(a)	**54.**	(a)	**72.**	(a)	**90.**	(b)	**108.**	(a)	**126.**	(b)	**144.**	(a)

NOTES

NOTES

CHAPTER 1

SEQUENCE AND SERIES-ARITHMETIC AND GEOMETRIC PROGRESSIONS

LEARNING OBJECTIVES

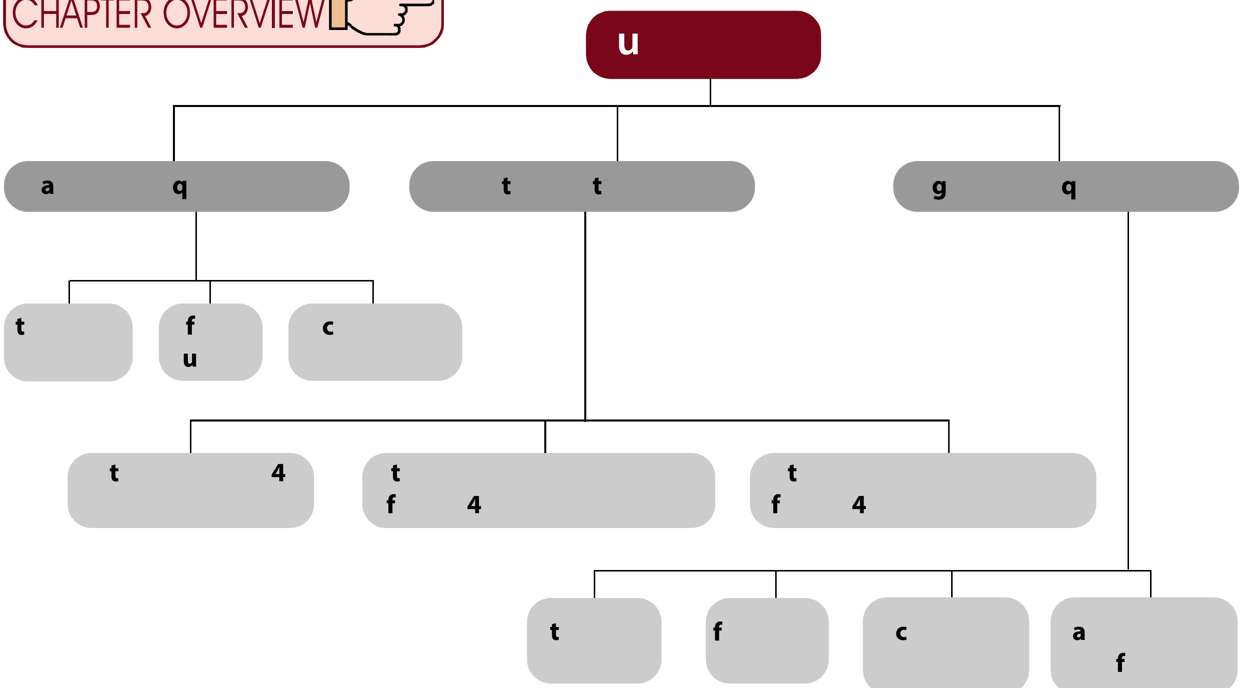
Often students will come across a sequence of numbers which are having a common difference, i.e., difference between the two consecutive pairs are the same. Also another very common sequence of numbers which are having common ratio, i.e., ratio of two consecutive pairs are the same. Could you guess what these special type of sequences are termed in mathematics?

Read this chapter to understand that these two special type of sequences are called Arithmetic Progression and Geometric Progression respectively. Further learn how to find out an element of these special sequences and how to find sum of these sequences.

These sequences will be useful for understanding various formulae of accounting and finance.

The topics of sequence, series, A.P. G.P. find useful applications in commercial problems among others; viz., to find interest earned through compound interest, depreciations after certain amount of time and total sum earned on recurring deposits, etc.

CHAPTER OVERVIEW



6.1 SEQUENCE

Let us consider the following collection of numbers-

(1) 28 , 2, 25, 27, ————————

(2) 2 , 7, 11, 19, 31, 51, ————

(3) 1, 2, 3, 4, 5, 6, ——————

(4) 20, 18, 16, 14, 12, 10, ————

In (1) the nos. are not arranged in a particular order. In (2) the nos. are in ascending order but they do not obey any rule or law. It is, therefore, not possible to indicate the number next to 51.

In (3) we find that by adding 1 to any number, we get the next one. Here the number next to 6 is 6 + 1 = 7.

In (4) if we subtract 2 from any number we get the nos. that follows. Here the number next to 10 is 10 –2 = 8.

Under these circumstances, we say, the numbers in the collections (1) and (2) do not form sequences whereas the numbers in the collections (3) & (4) form sequences.

Thus a sequence may be defined as follows:—

An ordered collection of numbers a_1, a_2, a_3, a_4, ..., a_n, ... is a sequence if according to some definite rule or law, there is a definite value of a_n called the term or element of the sequence, corresponding to any value of the natural number n.

Clearly, a_1 is the 1st term of the sequence , a_2 is the 2nd term, ..., a_n is the nth term.

In the nth term a_n, by putting n = 1, 2 ,3 ,......... successively , we get a_1, a_2, a_3, a_4, ...

Thus it is clear that the nth term of a sequence is a function of the positive integer n. The nth term is also called the general term of the sequence. To specify a sequence, nth term must be known, otherwise it may lead to confusion. A sequence may be finite or infinite.

If the number of elements in a sequence is finite, the sequence is called *finite sequence*; while if the number of elements is unending, the sequence is *infinite*.

A finite sequence a_1, a_2, a_3, a_4, ..., a_n is denoted by $\{a_i\}_{i=1}^{n}$ and an infinite sequence a_1, a_2, a_3, a_4, ..., a_n, ... is denoted by $\{a_n\}_{n=1}^{\infty}$ or simply by $\{a_n\}$ where a_n is the nth element of the sequence.

Example :

1) The sequence { 1/n } is 1, 1/2, 1/3, 1/4……
2) The sequence $\{(-1)^n n\}$ is –1, 2, –3, 4, –5,…..
3) The sequence { n } is 1, 2, 3,…
4) The sequence { n / (n + 1) } is 1/2, 2/3, 3/4, 4/5 …….
5) A sequence of even positive integers is 2, 4, 6, ...
6) A sequence of odd positive integers is 1, 3, 5, 7, ...

All the above are infinite sequences.

Example:

1) A sequence of even positive integers within 12 i.e., is 2, 4, 6, 8, 10.
2) A sequence of odd positive integers within 11 i.e., is 1, 3, 5, 7, 9.

All the above are finite sequences.

6.2 SERIES

An expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots\ldots$ which is the sum of the elements of the sequence $\{a_n\}$ is called a *series*. If the series contains a finite number of elements, it is called a *finite series*, otherwise called *an infinite series*.

If $S_n = u_1 + u_2 + u_3 + u_4 + \ldots + u_n$, then S_n is called the sum to n terms (or the sum of the first n terms) of the series and the term sum is denoted by the Greek letter Σ.

Thus, $S_n = \sum_{r=1}^{n} u_r$ or simply by Σu_n.

ILLUSTRATIONS:

(i) $1 + 3 + 5 + 7 + \ldots\ldots$ is a series in which 1st term = 1, 2nd term = 3 , and so on.

(ii) $2 - 4 + 8 - 16 + \ldots\ldots$ is also a series in which 1st term = 2, 2nd term = –4 , and so on.

6.3 ARITHMETIC PROGRESSION (A.P.)

A sequence $a_1, a_2, a_3, \ldots, a_n$ is called an Arithmetic Progression (A.P.) when $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$. That means A. P. is a sequence in which each term is obtained by adding a constant d to the preceding term. This constant 'd' is called the *common difference* of the A.P. If 3 numbers a, b, c are in A.P., we say

$b - a = c - b$ or $a + c = 2b$; b is called the arithmetic mean between a and c.

Example: **1)** 2,5,8,11,14,17,…… is an A.P. in which d = 3 is the common diference.

2) 15,13,11,9,7,5,3,1,–1, is an A.P. in which –2 is the common difference.

Solution: In (1) 2nd term = 5 , 1st term = 2, 3rd term = 8,

so 2nd term – 1st term = 5 – 2 = 3, 3rd term – 2nd term = 8 – 5 = 3

Here the difference between a term and the preceding term is same that is always constant. This constant is called common difference.

Now in generel an A.P. series can be written as

$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$

where 'a' is the 1st term and 'd' is the common difference.

Thus 1st term $(t_1) = a = a + (1 - 1)d$

2nd term $(t_2) = a + d = a + (2 - 1)d$

3rd term $(t_3) = a + 2d = a + (3 - 1)d$

4th term (t_4) = a + 3d = a + (4 – 1) d

…………………………………………….

nth term (t_n) = a + (n – 1) d, where n is the position number of the term.

Using this formula we can get

50th term (= t_{50}) = a+ (50 – 1) d = a + 49d

Example 1: Find the 7th term of the A.P. 8, 5, 2, –1, –4,…..

Solution: Here a = 8, d = 5 – 8 = –3

Now t_7 = 8 + (7 – 1) d

= 8 + (7 – 1) (– 3)

= 8 + 6 (– 3)

= 8 – 18

= – 10

Example 2: Which term of the AP $\frac{3}{\sqrt{7}}, \frac{4}{\sqrt{7}}, \frac{5}{\sqrt{7}}$ ....is $\frac{17}{\sqrt{7}}$?

Solution: $a = \frac{3}{\sqrt{7}}$, $d = \frac{4}{\sqrt{7}} - \frac{3}{\sqrt{7}} = \frac{1}{\sqrt{7}}$, $t_n = \frac{17}{\sqrt{7}}$

We may write

$$\frac{17}{\sqrt{7}} = \frac{3}{\sqrt{7}} + (n-1) \times \frac{1}{\sqrt{7}}$$

or, 17 = 3 + (n – 1)

or, n = 17 – 2 = 15

Hence, 15th term of the A.P. is $\frac{17}{\sqrt{7}}$.

Example 3: If 5th and 12th terms of an A.P. are 14 and 35 respectively, find the A.P.

Solution: Let a be the first term & d be the common difference of A.P.

t_5 = a + 4d = 14

t_{12} = a + 11d = 35

On solving the above two equations,

7d = 21 = i.e., d = 3

and a = 14 – (4 × 3) = 14 – 12 = 2

Hence, the required A.P. is 2, 5, 8, 11, 14,...............

Example 4: Divide 69 into three parts which are in A.P. and are such that the product of the first two parts is 483.

Solution: Given that the three parts are in A.P., let the three parts which are in A.P. be a – d, a, a + d.........

Thus a – d + a + a + d = 69

or $3a = 69$

or $a = 23$

So the three parts are 23 – d, 23, 23 + d

Since the product of first two parts is 483, therefore, we have

$23(23 - d) = 483$

or $23 - d = 483 / 23 = 21$

or $d = 23 - 21 = 2$

Hence, the three parts which are in A.P. are

$23 - 2 = 21, 23, 23 + 2 = 25$

Hence the three parts are 21, 23, 25.

Example 5: Find the arithmetic mean between 4 and 10.

Solution: We know that the A.M. of a & b is = (a + b) /2

Hence, The A. M between 4 & 10 = (4 + 10) /2 = 7

Example 6: Insert 4 arithmetic means between 4 and 324.

4, –, –, –, –, 324

Solution: Here a= 4, d = ? n = 2 + 4 = 6, $t_n = 324$

Now $t_n = a + (n - 1) d$

or $324 = 4 + (6 - 1) d$

or $320 = 5d$ i.e., = i.e., $d = 320 / 5 = 64$

So the 1^{st} AM = 4 + 64 = 68

2^{nd} AM = 68 + 64 = 132

3^{rd} AM = 132 + 64 = 196

4^{th} AM = 196 + 64 = 260

Sum of the first n terms

Let S be the Sum, a be the 1^{st} term and ℓ the last term of an A.P. If the number of term is n, then $t_n = \ell$. Let d be the common difference of the A.P.

Now $S = a + (a + d) + (a + 2d) + .. + (\ell - 2d) + (\ell - d) + \ell$

Again $S = \ell + (\ell - d) + (\ell - 2d) + ... + (a + 2d) + (a + d) + a$

On adding the above, we have

$2S = (a + \ell) + (a + \ell) + (a + \ell) + \ldots\ldots + (a + \ell)$

$= n(a + \ell)$

or $S = n(a + \ell) / 2$

Note: The above formula may be used to determine the sum of n terms of an A.P. when the first term a and the last term is given.

Now $\ell = t_n = a + (n - 1) d$

$$\therefore \; S = \frac{n\{a + a + (n-1)d\}}{2}$$

or $$S = \frac{n}{2}\{2a + (n-1)d\}$$

Note: The above formula may be used when the first term a, common difference d and the number of terms of an A.P. are given.

Sum of 1st n natural or counting numbers

$S = 1 + 2 + 3 + \ldots\ldots + \ldots\ldots (n - 2) + (n - 1) + n$

Again $S = n + (n - 1) + (n - 2) + \ldots\ldots\ldots + 3 + 2 + 1$

On adding the above, we get

$2S = (n + 1) + (n + 1) + \ldots\ldots$ to n terms

or $2S = n(n + 1)$

$S = n(n + 1)/2$

Then Sum of first n natural number is $n(n + 1) / 2$

i.e. $1 + 2 + 3 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$.

Sum of 1st n odd number

$S = 1 + 3 + 5 + \ldots\ldots + (2n - 1)$

Sum of first n odd number

$S = 1 + 3 + 5 + \ldots\ldots + (2n - 1)$

Since $S = n\{2a + (n - 1) d\} / 2$, we find

$$S = \frac{n}{2}\{2.1 + (n - 1) 2\} = \frac{n}{2}(2n) = n^2$$

or $S = n^2$

Then sum of first, n odd numbers is n^2, i.e. $1 + 3 + 5 + \ldots\ldots + (2n - 1) = n^2$

Sum of the Squares of the first n natural nos.

Let $S = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$

We know $m^3 - (m-1)^3 = 3m^2 - 3m + 1$

We put m = 1, 2, 3,……,n

$1^3 - 0 = 3.1^2 - 3.1 + 1$

$2^3 - 1^3 = 3.2^2 - 3.2 + 1$

$3^3 - 2^3 = 3.3^2 - 3.3 + 1$

……………………………..

$+ n^3 - (n-1)^3 = 3n^2 - 3.n + 1$

Adding both sides term by term,

$n^3 = 3S - 3n(n+1)/2 + n$

or $2n^3 = 6S - 3n^2 - 3n + 2n$

or $6S = 2n^3 + 3n^2 + n$

or $6S = n(2n^2 + 3n + 1)$

or $6S = n(n+1)(2n+1)$

$S = n(n+1)(2n+1)/6$

Thus sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$

i.e. $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Similarly, sum of the cubes of first n natural numbers can be found out as $\left\{\frac{n(n+1)}{2}\right\}^2$ by taking the identity

$m^4 - (m-1)^4 = 4m^3 - 6m^2 + 4m - 1$ and putting m = 1, 2, 3,…., n.

Thus

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

EXERCISE 6 (A)

Choose the most appropriate option (a), (b) , (c) or (d).

1. The nth element of the sequence 1, 3, 5, 7,………is
   (a) n (b) $2n - 1$ (c) $2n + 1$ (d) none of these
2. The nth element of the sequence –1, 2, –4, 8 ….. is
   (a) $(-1)^n 2^{n-1}$ (b) 2^{n-1} (c) 2^n (d) none of these
3. $\sum_{i=4}^{7} \sqrt{2i-1}$ can be written as
   (a) $\sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13}$ (b) $2\sqrt{7} + 2\sqrt{9} + 2\sqrt{11} + 2\sqrt{13}$
   (c) $2\sqrt{7} + 2\sqrt{9} + 2\sqrt{11} + 2\sqrt{13}$ (d) none of these.

4. The sum to ∞ of the series –5, 25, –125 , 625, ….. can be written as

(a) $\sum_{k=1}^{\infty}(-5)^k$ (b) $\sum_{k=1}^{\infty}5^k$ (c) $\sum_{k=1}^{\infty}-5^k$ (d) none of these

5. The first three terms of sequence when nth term t_n is $n^2 - 2n$ are
(a) –1, 0, 3 (b) 1, 0, 2 (c) –1, 0, –3 (d) none of these

6. Which term of the progression –1, –3, –5, …. is –39
(a) 21st (b) 20th (c) 19th (d) none of these

7. The value of x such that 8x + 4, 6x – 2, 2x + 7 will form an AP is
(a) 15 (b) 2 (c) 15/2 (d) none of the these

8. The mth term of an A. P. is n and nth term is m. The rth term of it is
(a) m + n +r (b) n + m – 2r (c) m + n + r/2 (d) m + n – r

9. The number of the terms of the series $10 + 9\frac{2}{3} + 9\frac{1}{3} + 9 + \ldots$ will amount to 155 is

(a) 30 (b) 31 (c) 32 (d) none of these

10. The nth term of the series whose sum to n terms is $5n^2 + 2n$ is
(a) 3n – 10 (b) 10n – 2 (c) 10n – 3 (d) none of these

11. The 20th term of the progression 1, 4, 7, 10................is
(a) 58 (b) 52 (c) 50 (d) none of these

12. The last term of the series 5, 7, 9,….. to 21 terms is
(a) 44 (b) 43 (c) 45 (d) none of these

13. The last term of the A.P. 0.6, 1.2, 1.8,… to 13 terms is
(a) 8.7 (b) 7.8 (c) 7.7 (d) none of these

14. The sum of the series 9, 5, 1,…. to 100 terms is
(a) –18,900 (b) 18,900 (c) 19,900 (d) none of these

15. The two arithmetic means between –6 and 14 is

(a) 2/3, 1/3 (b) $2/3, 7\frac{1}{3}$ (c) $-2/3, -7\frac{1}{3}$ (d) none of these

16. The sum of three integers in AP is 15 and their product is 80. The integers are
(a) 2, 8, 5 (b) 8, 2, 5 (c) 2, 5, 8 (d) 8, 5, 2

17. The sum of n terms of an AP is $3n^2 + 5n$. The series is
(a) 8, 14, 20, 26 (b) 8, 22, 42, 68 (c) 22, 68, 114, ... (d) none of these

18. The number of numbers between 74 and 25,556 divisible by 5 is
(a) 5,090 (b) 5,097 (c) 5,095 (d) none of these

19. The pth term of an AP is (3p – 1)/6. The sum of the first n terms of the AP is
(a) n (3n + 1) (b) n(3n + 1)/12 (c) n/12 (3n – 1) (d) none of these

20. The arithmetic mean between 33 and 77 is
(a) 50 (b) 45 (c) 55 (d) none of these

21. The 4 arithmetic means between –2 and 23 are
    (a) 3, 13, 8, 18 (b) 18, 3, 8, 13 (c) 3, 8, 13, 18 (d) none of these
22. The first term of an A.P is 14 and the sums of the first five terms and the first ten terms are equal in magnitude but opposite in sign. The 3rd term of the AP is
    (a) $6\frac{4}{11}$ (b) 6 (c) 4/11 (d) none of these
23. The sum of a certain number of terms of an AP series –8, –6, –4, …… is 52. The number of terms is
    (a) 12 (b) 13 (c) 11 (d) none of these
24. The first and the last term of an AP are –4 and 146. The sum of the terms is 7171. The number of terms is
    (a) 101 (b) 100 (c) 99 (d) none of these
25. The sum of the series 3 ½ + 7 + 10 ½ + 14 + …. to 17 terms is
    (a) 530 (b) 535 (c) 535 ½ (d) none of these

6.4 GEOMETRIC PROGRESSION (G.P.)

If in a sequence of terms each term is constant multiple of the proceeding term, then the sequence is called a Geometric Progression (G.P). The constant multiplier is called the *common ratio*

Examples: 1) In 5, 15, 45, 135,….. common ratio is 15/5 = 3

2) In 1, 1/2, 1/4, 1/9 … common ratio is (1/2) /1 = 1/2

3) In 2, –6, 18, –54, …. common ratio is (–6) / 2 = –3

Illustrations: Consider the following series :–

(i) 1 + 4 + 16 + 64 + …………….

Here second term / first term = 4/1 = 4; third term / second term = 16/4 = 4

fourth term/third term = 64/16 = 4 and so on.

Thus, we find that, in the entire series, the ratio of any term and the term preceding it, is a constant.

(ii) 1/3 – 1/9 + 1/27 – 1/81 + ………….

Here second term / 1st term = (–1/9) / (1/3) = –1/3

third term / second term = (1/27) / (–1/9) = –1/3

fourth term / third term = (–1/81) / (1/27) = –1/3 and so on.

Here also, in the entire series, the ratio of any term and the term preceding one is constant.

The above mentioned series are known as **Geometric Series.**

Let us consider the sequence a, ar, ar^2, ar^3, ….

1st term = a, 2nd term = ar = ar^{2-1}, 3rd term = $ar^2 = ar^{3-1}$, 4th term = $ar^3 = ar^{4-1}$, …..

Similarly nth term of GP $t_n = ar^{n-1}$

Thus, common ratio = $\dfrac{\text{Any term}}{\text{Preceding term}} = \dfrac{t_n}{t_{n-1}}$

$= ar^{n-1}/ar^{n-2} = r$

Thus, general term of a G.P is given by ar^{n-1} and the general form of G.P. is

$a + ar + ar^2 + ar^3 + \ldots\ldots\ \ldots.$

For example, $r = \dfrac{t_2}{t_1} = \dfrac{ar}{a}$

So $r = \dfrac{t_2}{t_1} = \dfrac{t_3}{t_2} = \dfrac{t_4}{t_3} = \ldots.$

Example 1: If a, ar, ar^2, ar^3, …. be in G.P. Find the common ratio.

Solution: 1st term = a, 2nd term = ar

Ratio of any term to its preceding term = ar/a = r = common ratio.

Example 2: Which term of the progression 1, 2, 4, 8,… is 256?

Solution: $a = 1, r = 2/1 = 2, n = ?\ t_n = 256$

$t_n = ar^{n-1}$

or $256 = 1 \times 2^{n-1}$ i.e., $2^8 = 2^{n-1}$ or, $n - 1 = 8$ i.e., $n = 9$

Thus 9th term of the G. P. is 256

6.5 GEOMETRIC MEAN

If a, b, c are in G.P we get $b/a = c/b \Rightarrow b^2 = ac$, b is called the geometric mean between a and c

Example 1: Insert 3 geometric means between 1/9 and 9.

Solution: 1/9, –, –, –, 9

$a = 1/9, r = ?, n = 2 + 3 = 5, t_n = 9$

we know $t_n = ar^{n-1}$

or $1/9 \times r^{5-1} = 9$

or $r^4 = 81 = 3^4 \Rightarrow r = 3$

Thus 1st G. M = 1/9 × 3 = 1/3

2nd G. M = 1/3 × 3 = 1

3rd G. M = 1× 3 = 3

Example 2: Find the G.P where 4^{th} term is 8 and 8^{th} term is 128/625

Solution: Let a be the 1^{st} term and r be the common ratio.

By the question $t_4 = 8$ and $t_8 = 128/625$

So $ar^3 = 8$ and $ar^7 = 128 / 625$

Therefore $ar^7 / ar^3 = \dfrac{128}{625 \times 8}$ => $r^4 = 16 / 625 = (\pm 2/5)^4$ => $r = 2/5$ and $-2/5$

Now $ar^3 = 8$ => $a \times (2/5)^3 = 8$ => $a = 125$

Thus the G. P is

125, 50, 20, 8, 16/5, ………..

When $r = -2/5$, $a = -125$ and the G.P is –125, 50, –20, 8, –16/5 ,………

Finally, the G.P. is 125, 50, 20, 8, 16/5, ………..

or, –125, 50, –20, 8, –16/5,………

Sum of first n terms of a G P

Let a be the first term and r be the common ratio. So the first n terms are a, ar, ar^2, …... ar^{n-1}.

If S be the sum of n terms,

$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ ....(i)

Now $rS_n = ar + ar^2 + \ldots\ldots + ar^{n-1} + ar^n$ ....(ii)

Subtracting (i) from (ii)

$S_n - rS_n = a - ar^n$

or $S_n(1 - r) = a(1 - r^n)$

or $S_n = a(1 - r^n) / (1 - r)$ when $r < 1$

$S_n = a(r^n - 1) / (r - 1)$ when $r > 1$

If $r = 1$, then $S_n = a + a + a + \ldots\ldots\ldots$ to n terms

$= na$

If the nth term of the G. P be l then $\ell = ar^{n-1}$

Therefore, $S_n = (ar^n - a) / (r - 1) = (a\, r^{n-1}\, r - a) / (r - 1) = \dfrac{\ell r - a}{r - 1}$

So, when the last term of the G. P is known, we use this formula.

Sum of infinite geometric series

$S = a(1 - r^n) / (1 - r)$ when $r < 1$

$= a(1 - 1/R^n) / (1 - 1/R)$ (since $r < 1$, we take $r = 1/R$).

If $n \to \infty$, $1/R^n \to 0$

Thus $$S_\infty = \frac{a}{1-r}, \quad r < 1$$

i.e. Sum of G.P. upto infinity is $\frac{a}{1-r}$, where $r < 1$

Also, $S_\infty = \frac{a}{1-r}$, if $-1<r<1$.

Example 1: Find the sum of 1 + 2 + 4 + 8 + … to 8 terms.,

Solution: Here $a = 1, r = 2/1 = 2, n = 8$

Let $S = 1 + 2 + 4 + 8 + \ldots\ldots$ to 8 terms

$= 1(2^8 - 1)/(2 - 1) = 2^8 - 1 = 255$

Example 2: Find the sum to n terms of $6 + 27 + 128 + 629 + \ldots\ldots.$

Solution: Required Sum $= (5 + 1) + (5^2 + 2) + (5^3 + 3) + (5^4 + 4) + \ldots$ to n terms

$= (5 + 5^2 + 5^3 + \ldots\ldots + 5^n) + (1 + 2 + 3 + .. + n \text{ terms})$

$= \{5(5^n - 1)/(5 - 1)\} + \{n(n + 1)/2\}$

$= \{5(5^n - 1)/4\} + \{n(n + 1)/2\}$

Example 3: Find the sum to n terms of the series

$3 + 33 + 333 + \ldots\ldots.$

Solution: Let S denote the required sum.

i.e. $S = 3 + 33 + 333 + \ldots\ldots\ldots\ldots\ldots$ to n terms

$= 3(1 + 11 + 111 + \ldots\ldots.$ to n terms)

$= \frac{3}{9}(9 + 99 + 999 + \ldots.$ to n terms)

$= \frac{3}{9}\{(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots + (10^n - 1)\}$

$= \frac{3}{9}\{(10 + 10^2 + 10^3 + \ldots. + 10^n) - n\}$

$= \frac{3}{9}\{10(1 + 10 + 10^2 + \ldots + 10^{n-1}) - n\}$

$= \frac{3}{9}[\{10(10^n - 1)/(10 - 1)\} - n]$

$= \frac{3}{81}(10^{n+1} - 10 - 9n)$

$= \frac{1}{27}(10^{n+1} - 9n - 10)$

Example 4: Find the sum of n terms of the series 0.7 + 0.77 + 0.777 + …. to n terms

Solution: Let S denote the required sum.

i.e. $S = 0.7 + 0.77 + 0.777 + \ldots$ to n terms

$= 7(0.1 + 0.11 + 0.111 + \ldots$ to n terms$)$

$= \frac{7}{9}(0.9 + 0.99 + 0.999 + \ldots$ to n terms $)$

$= \frac{7}{9}\{(1 - 1/10) + (1 - 1/10^2) + (1 - 1/10^3) + \ldots + (1 - 1/10^n)\}$

$= \frac{7}{9}\{n - \frac{1}{10}(1 + 1/10 + 1/10^2 + \ldots + 1/10^{n-1})\}$

So $S = \frac{7}{9}\{n - \frac{1}{10}(1 - 1/10^n)/(1 - 1/10)\}$

$= \frac{7}{9}\{n - (1 - 10^{-n})/9\}$

$= \frac{7}{81}\{9n - 1 + 10^{-n}\}$

Example 5: Evaluate $0.21\dot{7}\dot{5}$ using the sum of an infinite geometric series.

Solution: $0.21\dot{7}\dot{5} = 0.2175757575 \ldots\ldots$

$0.21\dot{7}\dot{5} = 0.21 + 0.0075 + 0.000075 + \ldots$

$= 0.21 + 75(1 + 1/10^2 + 1/10^4 + \ldots)/10^4$

$= 0.21 + 75\{1/(1 - 1/10^2\}/10^4$

$= 0.21 + (75/10^4) \times 10^2/99$

$= 21/100 + (3/4) \times (1/99)$

$= 21/100 + 1/132$

$= (693 + 25)/3300 = 718/3300 = 359/1650$

Example 6: Find three numbers in G. P whose sum is 19 and product is 216.

Solution: Let the 3 numbers be a/r, a, ar.

According to the question $a/r \times a \times ar = 216$

or $a^3 = 6^3 \Rightarrow a = 6$

So the numbers are 6/r, 6, 6r

Again $6/r + 6 + 6r = 19$

or $6/r + 6r = 13$

or $6 + 6r^2 = 13r$

or $6r^2 - 13r + 6 = 0$

or $6r^2 - 4r - 9r + 6 = 0$

or $2r(3r - 2) - 3(3r - 2) = 2$

or $(3r - 2)(2r - 3) = 0$ or, $r = 2/3, 3/2$

So the numbers are

$6/(2/3), 6, 6 \times (2/3) = 9, 6, 4$

or $6/(3/2), 6, 6 \times (3/2) = 4, 6, 9$

EXERCISE 6 (B)

Choose the most appropriate option (a), (b), (c) or (d)

1. The 7^{th} term of the series 6, 12, 24,……is
   (a) 384 (b) 834 (c) 438 (d) none of these
2. t_8 of the series 6, 12, 24,…is
   (a) 786 (b) 768 (c) 867 (c) none of these
3. t_{12} of the series –128, 64, –32, ….is
   (a) – 1/16 (b) 16 (c) 1/16 (d) none of these
4. The 4^{th} term of the series 0.04, 0.2, 1, … is
   (a) 0.5 (b) 1/2 (c) 5 (d) none of these
5. The last term of the series 1, 2, 4,…. to 10 terms is
   (a) 512 (b) 256 (c) 1024 (d) none of these
6. The last term of the series 1, –3, 9, –27 up to 7 terms is
   (a) 297 (b) 729 (c) 927 (d) none of these
7. The last term of the series x^2, x, 1, …. to 31 terms is
   (a) x^{28} (b) $1/x$ (c) $1/x^{28}$ (d) none of these
8. The sum of the series –2, 6, –18, …. to 7 terms is
   (a) –1094 (b) 1094 (c) – 1049 (d) none of these
9. The sum of the series 243, 81, 27, …. to 8 terms is
   (a) 36 (b) $\left(36\frac{13}{30}\right)$ (c) $36\frac{1}{9}$ (d) none of these
10. The sum of the series $\frac{1}{\sqrt{3}} + 1 + \frac{3}{\sqrt{3}} + \ldots\ldots$ to 18 terms is
    (a) $9841\frac{(1+\sqrt{3})}{\sqrt{3}}$ (b) 9841 (c) $\frac{9841}{\sqrt{3}}$ (d) none of these

11. The second term of a G P is 24 and the fifth term is 81. The series is
    (a) 16, 36, 24, 54,.. (b) 24, 36, 53,... (c) 16, 24, 36, 54,.. (d) none of these
12. The sum of 3 numbers of a G P is 39 and their product is 729. The numbers are
    (a) 3, 27, 9 (b) 9, 3, 27 (c) 3, 9, 27 (d) none of these
13. In a G. P, the product of the first three terms is 27/8. The middle term is
    (a) 3/2 (b) 2/3 (c) 2/5 (d) none of these
14. If you save 1 paise today, 2 paise the next day 4 paise the succeeding day and so on, then your total savings in two weeks will be
    (a) ₹ 163 (b) ₹ 183 (c) ₹ 163.83 (d) none of these
15. Sum of n terms of the series 4 + 44 + 444 + … is
    (a) $4/9 \{ 10/9 (10^n -1) -n \}$ (b) $10/9 (10^n -1) -n$
    (c) $4/9 (10^n -1) -n$ (d) none of these
16. Sum of n terms of the series 0.1 + 0.11 + 0.111 + … is
    (a) $1/9 \{n - (1- (0.1)^n)\}$ (b) $1/9 \{n - (1-(0.1)^n)/9\}$
    (c) $n- 1 - (0.1)^n/9$ (d) none of these
17. The sum of the first 20 terms of a G. P is 244 times the sum of its first 10 terms. The common ratio is
    (a) $\pm\sqrt{3}$ (b) ±3 (c) $\sqrt{3}$ (d) none of these
18. Sum of the series 1 + 3 + 9 + 27 +….is 364. The number of terms is
    (a) 5 (b) 6 (c) 11 (d) none of these
19. The product of 3 numbers in G P is 729 and the sum of squares is 819. The numbers are
    (a) 9, 3, 27 (b) 27, 3, 9 (c) 3, 9, 27 (d) none of these
20. The sum of the series 1 + 2 + 4 + 8 + .. to n term
    (a) $2^n -1$ (b) 2n – 1 (c) $1/2^n - 1$ (d) none of these
21. The sum of the infinite GP 14, – 2, + 2/7, – 2/49, + … is
    (a) $4\frac{1}{12}$ (b) $12\frac{1}{4}$ (c) 12 (d) none of these
22. The sum of the infinite G. P. 1 - 1/3 + 1/9 - 1/27 +... is
    (a) 0.33 (b) 0.57 (c) 0.75 (d) none of these
23. The number of terms to be taken so that 1 + 2 + 4 + 8 + will be 8191 is
    (a) 10 (b) 13 (c) 12 (d) none of these
24. Four geometric means between 4 and 972 are
    (a) 12, 36, 108, 324 (b) 12, 24, 108, 320 (c) 10, 36, 108, 320 (d) none of these

ILLUSTRATIONS:

(I) A person is employed in a company at ₹ 3000 per month and he would get an increase of ₹ 100 per year. Find the total amount which he receives in 25 years and the monthly salary in the last year.

SOLUTION:

He gets in the 1st year at the Rate of 3000 per month;

In the 2nd year he gets at the rate of ₹ 3100 per month;

In the 3rd year at the rate of ₹ 3200 per month so on.

In the last year the monthly salary will be

₹ {3000 + (25 – 1) × 100} = ₹ 5400

Total amount = ₹ 12 (3000 + 3100 + 3200 +… + 5400) $\left[\text{Use } S_n = \frac{n}{2}(a+l)\right]$

= ₹ 12 × 25/2 (3000 + 5400)

= ₹ 150 × 8400

= ₹ 12,60,000

(II) A person borrows ₹ 8,000 at 2.76% Simple Interest per annum. The principal and the interest are to be paid in the 10 monthly instalments. If each instalment is double the preceding one, find the value of the first and the last instalment.

SOLUTION:

Interest to be paid = 2.76 × 10 × 8000 / 100 × 12 = ₹ 184

Total amount to be paid in 10 monthly instalment is ₹ (8000 + 184) = ₹ 8184

The instalments form a G P with common ratio 2 and so ₹ 8184 = a $(2^{10} – 1) / (2 – 1)$,

a = 1st instalment

Here a = ₹ 8184 / 1023 = ₹ 8

The last instalment = $ar^{10-1} = 8 \times 2^9 = 8 \times 512$ = ₹ 4096

SUMMARY

- **Sequence:** An ordered collection of numbers a_1, a_2, a_3, a_4, ..., a_n, ... is a sequence if according to some definite rule or law, there is a definite value of a_n, called the term or element of the sequence, corresponding to any value of the natural number n.
- An expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ which is the sum of the elements of the sequence $\{a_n\}$ is called a *series*. If the series contains a finite number of elements, it is called a *finite series*, otherwise called *an infinite series.*
- **Arithmetic Progression:** A sequence $a_1, a_2, a_3, \ldots, a_n$ is called an Arithmetic Progression (A.P.) when $a_2 – a_1 = a_3 – a_2 = \ldots = a_n – a_{n-1}$. That means A. P. is a sequence in which each term is obtained by adding a constant d to the preceding term. This constant 'd' is called the *common difference* of the A.P. If 3 numbers a, b, c are in A.P., we say

  b – a = c – b or a + c = 2b; b is called the arithmetic mean between a and c.

  nth term (t_n) = a + (n – 1) d,

Where a = First Term

d = Common difference= $t_n - t_{n-1}$

Sum of n terms of AP=

$$s = \frac{n}{2}\left[2a + (n-1)d\right]$$

- Sum of the first n terms : Sum of 1st n natural or counting numbers

  S = n(n + 1)/2

  Sum of 1st n odd numbers : $S = n^2$

  Sum of the Squares of the first, n natural numbers

$$= \frac{n(n+1)(2n+1)}{6}$$

sum of the cubes of the first n natural numbers is

$$\left\{\frac{n(n+1)}{2}\right\}^2$$

- **Geometric Progression (G.P).** If in a sequence of terms each term is constant multiple of the proceeding term, then the sequence is called a Geometric Progression (G.P). The constant multiplier is called the *common ratio*

$$= \frac{\text{Any term}}{\text{Preceding term}} = \frac{t_n}{t_{n-1}}$$

$$= ar^{n-1}/ar^{n-2} = r$$

- Sum of first n terms of a G P:

$$S_n = a(1 - r^n)/(1 - r) \text{ when } r < 1$$

$$S_n = a(r^n - 1)/(r - 1) \text{ when } r > 1$$

  Sum of infinite geometric series

$$S_\infty = \frac{a}{1-r}, \quad r < 1$$

- A.M. of a & b is = (a + b) /2
- If a, b, c are in G.P we get b/a = c/b => b^2= ac, b is called the geometric mean between a and c

EXERCISE 6 (C)

Choose the most appropriate option (a), (b), (c) or (d).

1. Three numbers are in AP and their sum is 21. If 1, 5, 15 are added to them respectively, they form a G. P. The numbers are
   (a) 5, 7, 9 (b) 9, 5, 7 (c) 7, 5, 9 (d) none of these
2. The sum of $1 + 1/3 + 1/3^2 + 1/3^3 + \ldots + 1/3^{n-1}$ is
   (a) 2/3 (b) 3/2 (c) 4/5 (d) none of these
3. The sum of the infinite series 1 + 2/3 + 4/9 + .. is
   (a) 1/3 (b) 3 (c) 2/3 (d) none of these
4. The sum of the first two terms of a G.P. is 5/3 and the sum to infinity of the series is 3. The common ratio is
   (a) 1/3 (b) 2/3 (c) – 2/3 (d) none of these
5. If p, q and r are in A.P. and x, y, z are in G.P. then $x^{q-r}. y^{r-p}. z^{p-q}$ is equal to
   (a) 0 (b) –1 (c) 1 (d) none of these
6. The sum of three numbers in G.P. is 70. If the two extremes by multiplied each by 4 and the mean by 5, the products are in AP. The numbers are
   (a) 12, 18, 40 (b) 10, 20, 40 (c) 40, 20, 10 (d) none of these
7. The sum of 3 numbers in A.P. is 15. If 1, 4 and 19 be added to them respectively, the results are is G. P. The numbers are
   (a) 26, 5, –16 (b) 2, 5, 8 (c) 5, 8, 2 (d) none of these
8. Given x, y, z are in G.P. and $x^p = y^q = z^\sigma$, then $1/p$, $1/q$, $1/\sigma$ are in
   (a) A.P. (b) G.P. (c) Both A.P. and G.P. (d) none of these
9. If the terms 2x, (x+10) and (3x+2) be in A.P., the value of x is
   (a) 7 (b) 10 (c) 6 (d) none of these
10. If A be the A.M. of two positive unequal quantities x and y and G be their G. M, then
    (a) $A < G$ (b) $A>G$ (c) $A \geq G$ (d) $A \leq G$
11. The A.M. of two positive numbers is 40 and their G. M. is 24. The numbers are
    (a) (72, 8) (b) (70, 10) (c) (60, 20) (d) none of these
12. Three numbers are in A.P. and their sum is 15. If 8, 6, 4 be added to them respectively, the numbers are in G.P. The numbers are
    (a) 2, 6, 7 (b) 4, 6, 5 (c) 3, 5, 7 (d) none of these
13. The sum of four numbers in G. P. is 60 and the A.M. of the first and the last is 18. The numbers are
    (a) 4, 8, 16, 32 (b) 4, 16, 8, 32 (c) 16, 8, 4, 20 (d) none of these
14. A sum of ₹ 6240 is paid off in 30 instalments such that each instalment is ₹ 10 more than the preceeding installment. The value of the 1st instalment is
    (a) ₹ 36 (b) ₹ 30 (c) ₹ 60 (d) none of these
15. The sum of $1.03 + (1.03)^2 + (1.03)^3 + \ldots$ to n terms is
    (a) $103 \{(1.03)^n – 1\}$ (b) $103/3 \{(1.03)^n – 1\}$ (c) $(1.03)^n –1$ (d) none of these

16. If x, y, z are in A.P. and x, y, (z + 1) are in G.P. then
(a) $(x - z)^2 = 4x$ (b) $z^2 = (x - y)$ (c) $z = x - y$ (d) none of these

17. The numbers x, 8, y are in G.P. and the numbers x, y, –8 are in A.P. The value of x and y are
(a) (–8, –8) (b) (16, 4) (c) (8, 8) (d) none of these

18. The nth term of the series 16, 8, 4,…. in $1/2^{17}$. The value of n is
(a) 20 (b) 21 (c) 22 (d) none of these

19. The sum of n terms of a G.P. whose first terms 1 and the common ratio is 1/2, is equal to $1\frac{127}{128}$. The value of n is
(a) 7 (b) 8 (c) 6 (d) none of these

20. t_4 of a G.P. in x, t_{10} = y and t_{16} = z. Then
(a) $x^2 = yz$ (b) $z^2 = xy$ (c) $y^2 = zx$ (d) none of these

21. If x, y, z are in G.P., then
(a) $y^2 = xz$ (b) $y(z^2 + x^2) = x(z^2 + y^2)$ (c) $2y = x+z$ (d) none of these

22. The sum of all odd numbers between 200 and 300 is
(a) 11,600 (b) 12,490 (c) 12,500 (d) 24,750

23. The sum of all natural numbers between 500 and 1000 which are divisible by 13, is
(a) 28,405 (b) 24,805 (c) 28,540 (d) none of these

24. If unity is added to the sum of any number of terms of the A.P. 3, 5, 7, 9,…... the resulting sum is
(a) 'a' perfect cube (b) 'a' perfect square (c) 'a' number (d) none of these

25. The sum of all natural numbers from 100 to 300 which are exactly divisible by 4 or 5 is
(a) 10,200 (b) 15,200 (c) 16,200 (d) none of these

26. The sum of all natural numbers from 100 to 300 which are exactly divisible by 4 and 5 is
(a) 2,200 (b) 2,000 (c) 2,220 (d) none of these

27. A person pays ₹ 975 by monthly instalment each less then the former by ₹ 5. The first instalment is ₹ 100. The time by which the entire amount will be paid is
(a) 10 months (b) 15 months (c) 14 months (d) none of these

28. A person saved ₹ 16,500 in ten years. In each year after the first year he saved ₹ 100 more than he did in the preceding year. The amount of money he saved in the 1st year was
(a) ₹ 1000 (b) ₹ 1500 (c) ₹ 1200 (d) none of these

29. At 10% C.I. p.a., a sum of money accumulate to ₹ 9625 in 5 years. The sum invested initially is
(a) ₹ 5976.37 (b) ₹ 5970 (c) ₹ 5975 (d) ₹ 5370.96

30. The population of a country was 55 crores in 2005 and is growing at 2% p.a C.I. the population is the year 2015 is estimated as
(a) 5705 (b) 6005 (c) 6700 (d) none of these

ANSWERS

Exercise 6 (A)

1.	(b)	**2.**	(a)	**3.**	(a)	**4.**	(a)	**5.**	(a)	**6.**	(b)	**7.**	(c)	**8.**	(d)
9.	(a), (b)	**10**	(c)	**11.**	(a)	**12.**	(c)	**13.**	(b)	**14.**	(a)	**15.**	(b)	**16.**	(c),(d)
17.	(a)	**18.**	(b)	**19.**	(b)	**20.**	(c)	**21.**	(c)	**22.**	(a)	**23.**	(b)	**24.**	(a)
25.	(c)														

Exercise 6 (B)

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(c)	**5.**	(a)	**6.**	(b)	**7.**	(c)	**8.**	(a)
9.	(d)	**10.**	(a)	**11.**	(c)	**12.**	(c)	**13.**	(a)	**14.**	(c)	**15.**	(a)	**16.**	(b)
17.	(a)	**18.**	(b)	**19.**	(c)	**20.**	(a)	**21.**	(b)	**22.**	(c)	**23.**	(b)	**24.**	(a)

Exercise 6 (C)

1.	(a)	**2.**	(d)	**3.**	(b)	**4.**	(b), (c)	**5.**	(c)	**6.**	(b), (c)	**7.**	(a), (b)	**8.**	(a)
9.	(c)	**10.**	(b)	**11.**	(a)	**12.**	(c)	**13.**	(a)	**14.**	(d)	**15.**	(b)	**16.**	(a)
17.	(a), (b)	**18.**	(c)	**19.**	(b)	**20.**	(c)	**21.**	(a)	**22.**	(c)	**23.**	(a)	**24.**	(b)
25.	(c)	**26.**	(a)	**27.**	(b)	**28.**	(c)	**29.**	(a)	**30.**	(d)				

ADDITIONAL QUESTION BANK

1. If a, b, c are in A.P. as well as in G.P. then –

   (a) They are also in H.P. (Harmonic Progression) (b) Their reciprocals are in A.P.

   (c) Both (a) and (b) are true (d) Both (a) and (b) are false

2. If a, b, c be respectively p^{th}, q^{th} and r^{th} terms of an A.P. the value of $a(q-r)+b(r-p)+c(p-q)$ is _________.

   (a) 0 (b) 1 (c) –1 (d) None

3. If the p^{th} term of an A.P. is q and the q^{th} term is p the value of the r^{th} term is_________.

   (a) $p - q - r$ (b) $p + q - r$ (c) $p + q + r$ (d) None

4. If the p^{th} term of an A.P. is q and the q^{th} term is p the value of the $(p + q)^{th}$ term is_______.

   (a) 0 (b) 1 (c) –1 (d) None

5. The sum of first n natural number is _______.

   (a) $(n/2)(n+1)$ (b) $(n/6)(n+1)(2n+1)$ (c) $[(n/2)(n+1)]^2$ (d) None

6. The sum of square of first n natural number is __________.

(a) $(n/2)(n+1)$ (b) $(n/6)(n+1)(2n+1)$ (c) $[(n/2)(n+1)]^2$ (d) None

7. The sum of cubes of first n natural number is __________.

(a) $(n/2)(n+1)$ (b) $(n/6)(n+1)(2n+1)$ (c) $[(n/2)(n+1)]^2$ (d) None

8. The sum of a series in A.P. is 72 the first term is 17 and the common difference –2. the number of terms is __________.

(a) 6 (b) 12 (c) 6 or 12 (d) None

9. Find the sum to n terms of $(1-1/n) + (1-2/n) + (1-3/n) + ...$

(a) $\frac{1}{2}(n-1)$ (b) $\frac{1}{2}(n+1)$ (c) $(n-1)$ (d) $(n+1)$

10. If S_n the sum of first n terms in a series is given by $2n^2 + 3n$ the series is in ______.

(a) A.P. (b) G.P. (c) H.P. (d) None

11. The sum of all natural numbers between 200 and 400 which are divisible by 7 is ______.

(a) 7,730 (b) 8,729 (c) 7,729 (d) 8,730

12. The sum of natural numbers upto 200 excluding those divisible by 5 is ________.

(a) 20,100 (b) 4,100 (c) 16,000 (d) None

13. If a, b, c be the sums of p, q, r terms respectively of an A.P. the value of $(a/p)(q-r)+(b/q)(r-p)+(c/r)(p-q)$ is ______.

(a) 0 (b) 1 (c) –1 (d) None

14. If S_1, S_2, S_3 be the respectively the sum of terms of $n, 2n, 3n$ an A.P. the value of $S_3 \div (S_2 - S_1)$ is given by ______.

(a) 1 (b) 2 (c) 3 (d) None

15. The sum of n terms of two A.P.s are in the ratio of $(7n-5)/(5n+17)$. Then the _______ term of the two series are equal.

(a) 12 (b) 6 (c) 3 (d) None

16. Find three numbers in A.P. whose sum is 6 and the product is –24

(a) –2, 2, 6 (b) –1, 1, 3 (c) 1, 3, 5 (d) 1, 4, 7

17. Find three numbers in A.P. whose sum is 6 and the sum of whose square is 44.

(a) –2, 2, 6 (b) –1, 1, 3 (c) 1, 3, 5 (d) 1, 4, 7

18. Find three numbers in A.P. whose sum is 6 and the sum of their cubes is 232.

(a) –2, 2, 6 (b) –1, 1, 3 (c) 1, 3, 5 (d) 1, 4, 7

19. Divide 12.50 into five parts in A.P. such that the first part and the last part are in the ratio of 2:3

(a) 2, 2.25, 2.5, 2.75, 3 (b) –2, –2.25, –2.5, –2.75, –3

(c) 4, 4.5, 5, 5.5, 6 (d) –4, –4.5, –5, –5.5, –6

20. If *a, b, c* are in A.P. then the value of $(a^3 + 4b^3 + c^3)/[b(a^2 + c^2)]$ is

(a) 1 (b) 2 (c) 3 (d) None

21. If *a, b, c* are in A.P. then the value of $(a^2 + 4ac + c^2)/(ab + bc + ca)$ is

(a) 1 (b) 2 (c) 3 (d) None

22. If *a, b, c* are in A.P. then $(a/bc)(b + c), (b/ca)(c + a), (c/ab)(a + b)$ are in ____________.

(a) A.P. (b) G.P. (c) H.P. (d) None

23. If *a, b, c* are in A.P. then $a^2(b + c), b^2(c + a), c^2(a + b)$ are in ________.

(a) A.P. (b) G.P. (c) H.P. (d) None

24. If $(b + c)^{-1}, (c + a)^{-1}, (a + b)^{-1}$ are in A.P. then a^2, b^2, c^2 are in _________.

(a) A.P. (b) G.P. (c) H.P. (d) None

25. If a^2, b^2, c^2 are in A.P. then $(b + c), (c + a), (a + b)$ are in ________.

(a) A.P. (b) G.P. (c) H.P. (d) None

26. If a^2, b^2, c^2 are in A.P. then $a/(b + c), b/(c + a), c/(a + b)$ are in ____________.

(a) A.P. (b) G.P. (c) H.P. (d) None

27. If $(b + c - a)/a, (c + a - b)/b, (a + b - c)/c$ are in A.P. then a, b, c are in __________.

(a) A.P. (b) G.P. (c) H.P. (d) None

28. If $(b - c)^2, (c - a)^2, (a - b)^2$ are in A.P. then $(b - c), (c - a), (a - b)$ are in _______.

(a) A.P. (b) G.P. (c) H.P. (d) None

29. If $a\ b\ c$ are in A.P. then (b + c), (c + a), (a + b) are in ________.

(a) A.P. (b) G.P. (c) H.P. (d) None

30. Find the number which should be added to the sum of any number of terms of the A.P. 3, 5, 7, 9, 11 …….resulting in a perfect square.

(a) –1 (b) 0 (c) 1 (d) None

31. The sum of n terms of an A.P. is $2n^2 + 3n$. Find the n^{th} term.

(a) 4n + 1 (b) 4n - 1 (c) 2n + 1 (d) 2n - 1

32. The p^{th} term of an A.P. is $1/q$ and the q^{th} term is $1/p$. The sum of the $(pq)^{th}$ term is_______.

(a) $\frac{1}{2}(pq+1)$ (b) $\frac{1}{2}(pq-1)$ (c) pq+1 (d) pq-1

33. The sum of p terms of an A.P. is q and the sum of q terms is p. The sum of $p + q$ terms is ________.

(a) – (p + q) (b) p + q (c) $(p - q)^2$ (d) $p^2 - q^2$

34. If S_1, S_2, S_3 be the sums of n terms of three A.P.s the first term of each being unity and the respective common differences 1, 2, 3 then $(S_1 + S_3) / S_2$ is ______.

(a) 1 (b) 2 (c) –1 (d) None

35. The sum of all natural numbers between 500 and 1000, which are divisible by 13, is _______.

(a) 28,400 (b) 28,405 (c) 28,410 (d) None

36. The sum of all natural numbers from 100 and 300, which are divisible by 4, is ____.

(a) 10,200 (b) 30,000 (c) 8,200 (d) 2,200

37. The sum of all natural numbers from 100 to 300 excluding those, which are divisible by 4, is _______.

(a) 10,200 (b) 30,000 (c) 8,200 (d) 2,200

38. The sum of all natural numbers from 100 to 300, which are divisible by 5, is ______.

(a) 10,200 (b) 30,000 (c) 8,200 (d) 2,200

39. The sum of all natural numbers from 100 to 300, which are divisible by 4 and 5, is ______.

(a) 10,200 (b) 30,000 (c) 8,200 (d) 2,200

40. The sum of all natural numbers from 100 to 300, which are divisible by 4 or 5, is ______.

(a) 10,200 (b) 8,200 (c) 2,200 (d) 16,200

41. If the n terms of two A.P.s are in the ratio (3n+4) : (n+4) the ratio of the fourth term is ______.

(a) 2 (b) 3 (c) 4 (d) None

42. If a, b, c, d are in A.P. then

(a) $a^2 - 3b^2 + 3c^2 - d^2 = 0$ (b) $a^2 + 3b^2 + 3c^2 + d^2 = 0$ (c) $a^2 + 3b^2 + 3c^2 - d^2 = 0$ (d) None

43. If a, b, c, d, e are in A.P. then

(a) $a - b - d + e = 0$ (b) $a - 2c + e = 0$ (c) $b - 2c + d = 0$ (d) all the above

44. The three numbers in A.P. whose sum is 18 and product is 192 are _______.

(a) 4, 6, 8 (b) –4, –6, –8 (c) 8, 6, 4 (d) both (a) & (c)

45. The three numbers in A.P., whose sum is 27 and the sum of their squares is 341, are _______.

(a) 2, 9, 16 (b) 16, 9, 2 (C) both (a) and (b) (d) –2, –9, –16

46. The four numbers in A.P., whose sum is 24 and their product is 945, are _______.

(a) 3, 5, 7, 9 (b) 2, 4, 6, 8 (c) 5, 9, 13, 17 (d) None

47. The four numbers in A.P., whose sum is 20 and the sum of their squares is 120, are _______.

(a) 3, 5, 7, 9 (b) 2, 4, 6, 8 (c) 5, 9, 13, 17 (d) None

48. The four numbers in A.P. with the sum of second and third being 22 and the product of the first and fourth beinf 85 are _______.

(a) 3, 5, 7, 9 (b) 2, 4, 6, 8 (c) 5, 9, 13, 17 (d) None

49. The five numbers in A.P. with their sum 25 and the sum of their squares 135 are _______.

(a) 3, 4, 5, 6, 7 (b) 3, 3.5, 4, 4.5, 5 (c) –3, –4, –5, –6, –7

(d) –3, –3.5, –4, –4.5, –5

50. The five numbers in A.P. with the sum 20 and product of the first and last 15 are _______.

(a) 3, 4, 5, 6, 7 (b) 3, 3.5, 4, 4.5, 5 (c) –3, –4, –5, –6, –7

(d) –3, –3.5, –4, –4.5, –5

51. The sum of n terms of 2, 4, 6, 8….. is

(a) n(n+1) (b) (n/2)(n+1) (c) n(n–1) (d) (n/2)(n-1)

52. The sum of n terms of a+b, 2a, 3a–b, ….. is

(a) n(a–b)+2b (b) n(a+b) (c) both the above (d) None

53. The sum of n terms of $(x + y)^2$, $(x^2 + y^2)$, $(x - y)^2$,...... is
(a) $(x + y)^2 - 2(n - 1)xy$ (b) $n(x+y)^2 - n(n-1)xy$ (c) both the above (d) None

54. The sum of n terms of $(1/n)(n-1)$, $(1/n)(n-2)$, $(1/n)(n-3)$....... is
(a) 0 (b) $(1/2)(n-1)$ (c) $(1/2)(n+1)$ (d) None

55. The sum of n terms of 1.4, 3.7, 5.10 .... Is
(a) $(n/2)(4n^2+5n-1)$ (b) $n(4n^2+5n-1)$ (c) $(n/2)(4n^2-5n-1)$ (d) None

56. The sum of n terms of $1^2, 3^2, 5^2, 7^2$,.......is
(a) $(n/3)(4n^2-1)$ (b) $(n/2)(4n^2-1)$ (c) $(n/3)(4n^2+1)$ (d) None

57. The sum of n terms of 1, (1 + 2), (1 + 2 + 3) .... is
(a) $(n/3)(n+1)(n-2)$ (b) $(n/3)(n+1)(n+2)$ (c) $n(n+1)(n+2)$ (d) None

58. The sum of n terms of the series $1^2/1+(1^2+2^2)/2+(1^2+2^2+3^2)/3+$.......is
(a) $(n/36)(4n^2+15n+17)$ (b) $(n/12)(4n^2+15n+17)$
(c) $(n/12)(4n^2+15n+17)$ (d) None

59. The sum of n terms of the series 2.4.6 + 4.6.8 + 6.8.10 + ... is
(a) $2n(n^3+6n^2+11n+6)$ (b) $2n(n^3-6n^2+11n-6)$
(c) $n(n^3+6n^2+11n+6)$ (d) $n(n^3+6n^2+11n-6)$

60. The sum of n terms of the series $1.3^2+4.4^2+7.5^2+10.6^2+$........ is
(a) $(n/12)(n+1)(9n^2+49n+44)-8n$ (b) $(n/12)(n+1)(9n^2+49n+44)+8n$
(c) $(n/6)(2n+1)(9n^2+49n+44)-8n$ (d) None

61. The sum of n terms of the series 4 + 6 + 9 + 13 .... is
(a) $(n/6)(n^2+3n+20)$ (b) $(n/6)(n+1)(n+2)$ (c) $(n/3)(n+1)(n+2)$ (d) None

62. The sum to n terms of the series 11, 23, 59, 167 ....is
(a) $3^{n+1}+5n-3$ (b) $3^{n+1}+5n+3$ (c) 3^n+5n-3 (d) None

63. The sum of n terms of the series $1/(4.9)+1/(9.14)+1/(14.19)+1/(19.24)+$........is
(a) $(n/4)(5n+4)^{-1}$ (b) $(n/4)(5n+4)$ (c) $(n/4)(5n-4)^{-1}$ (d) None

64. The sum of n terms of the series 1 + 3 + 5 + ... Is
(a) n^2 (b) $2n^2$ (c) $n^2/2$ (d) None

65. The sum of n terms of the series 2 + 6 + 10 + ……. is

(a) $2n^2$ (b) n^2 (c) $n^2/2$ (d) $4n^2$

66. The sum of n terms of the series 1.2 + 2.3 + 3.4 + ……. Is

(a) $(n/3)(n+1)(n+2)$ (b) $(n/2)(n+1)(n+2)$ (c) $(n/3)(n+1)(n-2)$(D) None

67. The sum of n terms of the series 1.2.3 + 2.3.4 + 3.4.5 + …….is

(a) $(n/4)(n+1)(n+2)(n+3)$ (b) $(n/3)(n+1)(n+2)(n+3)$

(c) $(n/2)(n+1)(n+2)(n+3)$ (d) None

68. The sum of n terms of the series $1.2+3.2^2+5.2^3+7.2^4+$....... is

(a) $(n-1)2^{n+2}-2^{n+1}+6$ (b) $(n+1)2^{n+2}-2^{n+1}+6$ (c) $(n-1)2^{n+2}-2^{n+1}-6$ (d) None

69. The sum of n terms of the series $1/(3.8)+1/(8.13)+1/(13.18)+$...... is

(a) $(n/3)(5n+3)^{-1}$ (b) $(n/2)(5n+3)^{-1}$ (c) $(n/2)(5n-3)^{-1}$ (d) None

70. The sum of n terms of the series $1/1+1/(1+2)+1/(1+2+3)+$..... is

(a) $2n(n+1)^{-1}$ (b) $n(n+1)$ (c) $2n(n-1)^{-1}$ (d) None

71. The sum of n terms of the series $2^2+5^2+8^2+$........ is

(a) $(n/2)(6n^2+3n-1)$ (b) $(n/2)(6n^2-3n-1)$

(c) $(n/2)(6n^2+3n+1)$ (d) None

72. The sum of n terms of the series $1^2+3^2+5^2+$........ is

(a) $\frac{n}{3}(4n^2-1)$ (b) $n^2(2n^2+1)$ (c) $n(2n-1)$ (d) $n(2n+1)$

73. The sum of n terms of the series 1.4 + 3.7 + 5.10 + …… is

(a) $\frac{n}{3}(4n^2+5n+5)$ (b) $(n/2)(5n^2+4n-1)$

(c) $(n/2)(4n^2+5n+1)$ (d) None

74. The sum of n terms of the series $2.3^2+5.4^2+8.5^2+$........ is

(a) $(n/12)(9n^3+62n^2+123n+22)$ (b) $(n/12)(9n^3-62n^2+123n-22)$

(c) $(n/6)(9n^3+62n^2+123n+22)$ (d) None

75. The sum of n terms of the series 1 + (1 + 3) + (1 + 3 + 5) + ……. is

(a) $(n/6)(n+1)(2n+1)$ (b) $(n/6)(n+1)(n+2)$ (c) $(n/3)(n+1)(2n+1)$ (d) None

76. The sum of n terms of the series $1^2+(1^2+2^2)+(1^2+2^2+3^2)+........$ is

(a) $(n/12)(n+1)^2(n+2)$ (b) $(n/12)(n-1)^2(n+2)$ (c) $(n/12)(n^2-1)(n+2)$ (d) None

77. The sum of n terms of the series $1+(1+1/3)+(1+1/3+1/3^2)+........$ is

(a) $(3/2)(1-3^{-n})$ (b) $(3/2)[n-(1/2)(1-3^{-n})]$ (c) Both (d) None

78. The sum of n terms of the series $n.1+(n-1).2+(n-2).3+$ .... is

(a) $(n/6)(n+1)(n+2)$ (b) $(n/3)(n+1)(n+2)$ (c) $(n/2)(n+1)(n+2)$ (d) None

79. The sum of n terms of the series 1 + 5 + 12 + 22 + ….. is

(a) $(n^2/2)(n+1)$ (b) $n^2(n+1)$ (c) $(n^2/2)(n-1)$ (d) None

80. The sum of n terms of the series 4 + 14 + 30 + 52 + 80 + …… is

(a) $n(n+1)^2$ (b) $n(n-1)^2$ (c) $n(n^2-1)$ (d) None

81. The sum of n terms of the series 3 + 6 + 11 + 20 + 37 + …….. is

(a) $2^{n+1}+(n/2)(n+1)-2$ (b) $2^{n+1}+(n/2)(n+1)-1$ (c) $2^{n+1}+(n/2)(n-1)-2$ (d) None

82. The n^{th} terms of the series is 1/(4.7) + 1/(7.10) + 1/(10.13) + ……. is

(a) $(1/3)[(3n+1)^{-1}-(3n+4)^{-1}]$ (b) $(1/3)[(3n-1)^{-1}-(3n+4)^{-1}]$

(c) $(1/3)[(3n+1)^{-1}-(3n-4)^{-1}]$ (d) None

83. In question No.(82) the sum of the series upto n is

(a) $(n/4)(3n+4)^{-1}$ (b) $(n/4)(3n-4)^{-1}$ (c) $(n/2)(3n+4)^{-1}$ (d) None

84. The sum of n terms of the series $1^2/1+(1^2+2^2)/(1+2)+(1^2+2^2+3^2)/(1+2+3)+....$ is

(a) $(n/3)(n+2)$ (b) $(n/3)(n+1)$ (c) $(n/3)(n+3)$ (d) None

85. The sum of n terms of the series $1^3/1+(1^3+2^3)/2+(1^3+2^3+3^3)/3+....$ is

(a) $(n/48)(n+1)(n+2)(3n+5)$ (b) $(n/24)(n+1)(n+2)(3n+5)$

(c) $(n/48)(n+1)(n+2)(5n+3)$ (d) None

86. The value of $n^2 + 2n[1+2+3+.....+(n-1)]$ is

(a) n^3 (b) n^2 (c) n (d) None

87. $2^{4n}-1$ is divisible by

(a) 15 (b) 4 (c) 6 (d) 64

88. 3^n-2n-1 is divisible by

(a) 15 (b) 4 (c) 6 (d) 64

89. $n(n-1)(2n-1)$ is divisible by

(a) 15 (b) 4 (c) 6 (d) 64

90. $7^{2n}+16n-1$ is divisible by

(a) 15 (b) 4 (c) 6 (d) 64

91. The sum of n terms of the series whose n^{th} term $3n^2+2n$ is is given by

(a) $(n/2)(n+1)(2n+3)$ (b) $(n/2)(n+1)(3n+2)$

(c) $(n/2)(n+1)(3n-2)$ (d) $(n/2)(n+1)(2n-3)$

92. The sum of n terms of the series whose n^{th} term $n.2^n$ is is given by

(a) $(n-1)2^{n+1}+2$ (b) $(n+1)2^{n+1}+2$ (c) $(n-1)2^n+2$ (d) None

93. The sum of n terms of the series whose n^{th} term $5.3^{n+1}+2n$ is is given by

(a) $(5/2)(3^{n+2}-9)+n(n+1)$ (b) $(2/5)(3^{n+2}-9)+n(n+1)$

(c) $(5/2)(3^{n+2}+9)+n(n+1)$ (d) None

94. If the third term of a G.P. is the square of the first and the fifth term is 64 the series would be ________.

(a) 4 + 8 + 16 + 32 + …. (b) 4 – 8 + 16 – 32 + ……..

(c) both (d) None

95. Three numbers whose sum is 15 are in A.P. but if they are added by 1, 4, 19 respectively they are in G.P. The numbers are _______.

(a) 2, 5, 8 (b) 26, 5, –16 (c) Both (d) None

96. If a, b, c are the p^{th}, q^{th} and r^{th} terms of a G.P. respectively the value of $a^{q-r}.b^{r-p}.c^{p-q}$ is ________

(a) 0 (b) 1 (c) –1 (d) None

97. If a, b, c are in A.P. and x, y, z in G.P. then the value of $x^{b-c}.y^{c-a}.z^{a-b}$ is ________

(a) 0 (b) 1 (c) –1 (d) None

98. If *a, b, c* are in A.P. and *x, y, z* in G.P. then the value of $(x^b.y^c.z^a)\div(x^c.y^a.z^b)$ is ____

(a) 0 (b) 1 (c) –1 (d) None

99. The sum of *n* terms of the series 7 + 77 + 777 + …… is

(a) $(7/9)[(1/9)(10^{n+1}-10)-n]$ (b) $(9/10)[(1/9)(10^{n+1}-10)-n]$

(c) $(10/9)[(1/9)(10^{n+1}-10)-n]$ (d) None

100. The least value of n for which the sum of n terms of the series $1 + 3 + 3^2 +$ …… is greater than 7000 is _______.

(a) 9 (b) 10 (c) 8 (d) 7

101. If '*S*' be the sum, '*P*' the product and '*R*' the sum of the reciprocals of *n* terms in a G.P. then '*P*' is the _______ of S^n and R^{-n}.

(a) Arithmetic Mean (b) Geometric Mean (c) Harmonic Mean (d) None

102. Sum upto ∞ of the series $8+4\sqrt{2}+4+.....$ is

(a) $8(2+\sqrt{2})$ (b) $8(2-\sqrt{2})$ (c) $4(2+\sqrt{2})$ (d) $4(2-\sqrt{2})$

103. Sum upto ∞ of the series $1/2+1/3^2+1/2^3+1/3^4+1/2^5+1/3^6+.....$ is

(a) 19/24 (b) 24/19 (c) 5/24 (d) None

104. If $1+a+a^2+.......\infty=x$ and $1+b+b^2+.......\infty=y$ then $1+ab+a^2b^2+.......\infty$ is given by __________.

(a) $(xy)/(x+y-1)$ (b) $(xy)/(x-y-1)$ (c) $(xy)/(x+y+1)$ (d) None

105. If the sum of three numbers in G.P. is 35 and their product is 1000 the numbers are ____.

(a) 20, 10, 5 (b) 5, 10, 20 (c) both (d) None

106. If the sum of three numbers in G.P. is 21 and the sum of their squares is 189 the numbers are ____.

(a) 3, 6, 12 (b) 12, 6, 3 (c) both (d) None

107. If *a, b, c* are in G.P. then the value of $a(b^2+c^2)-c(a^2+b^2)$ is ____

(a) 0 (b) 1 (c) –1 (d) None

108. If a, b, c, d are in G.P. then the value of $b(ab-cd)-(c+a)(b^2-c^2)$ is ____

(a) 0 (b) 1 (c) –1 (d) None

109. If a, b, c, d are in G.P. then the value of $(ab+bc+cd)^2-(a^2+b^2+c^2)(b^2+c^2+d^2)$ is _________.

(a) 0 (b) 1 (c) –1 (d) None

110. If a, b, c, d are in G.P. then $a+b, b+c, c+d$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None

111. If a, b, c are in G.P. then $a^2+b^2, ab+bc, b^2+c^2$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None

112. If a, b, x, y, z are positive numbers such that a, x, b are in A.P. and a, y, b are in G.P. and $z=(2ab)/(a+b)$ then

(a) x, y, z are in G.P. (b) $x \geq y \geq z$ (c) both (d) None

113. If a, b, c are in G.P. then the value of $(a-b+c)(a+b+c)^2-(a+b+c)(a^2+b^2+c^2)$ is given by

(a) 0 (b) 1 (c) –1 (d) None

114. If a, b, c are in G.P. then the value of $a(b^2+c^2)-c(a^2+b^2)$ is given by

(a) 0 (b) 1 (c) –1 (d) None

115. If a, b, c are in G.P. then the value of $a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})-(a^3+b^3+c^3)$ is given by

(a) 0 (b) 1 (c) –1 (d) None

116. If a, b, c, d are in G.P. then $(a-b)^2, (b-c)^2, (c-d)^2$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None

117. If a, b, c, d are in G.P. then the value of $(b-c)^2+(c-a)^2+(d-b)^2-(a-d)^2$ is given by

(a) 0 (b) 1 (c) –1 (d) None

118. If $(a-b), (b-c), (c-a)$ are in G.P. then the value of $(a+b+c)^2-3(ab+bc+ca)$ is given by

(a) 0 (b) 1 (c) –1 (d) None

119. If $a^{1/x}=b^{1/y}=c^{1/z}$ and a, b, c are in G.P. then x, y, z are in

(a) A.P. (b) G.P. (c) H.P. (d) None

120. If $x = a + a/r + a/r^2 + ... \infty$, $y = b - b/r + b/r^2 - ..... \infty$, and $z = c + c/r^2 + c/r^4 + ... \infty$, then the value of $\frac{xy}{z} - \frac{ab}{c}$ is

(a) 0 (b) 1 (c) –1 (d) None

121. If *a, b, c* are in A.P. *a, x, b* are in G.P. and *b, y, c* are in G.P then x^2, b^2, y^2 are in

(a) A.P. (b) G.P. (c) H.P. (d) None

122. If a, b-a, c-a are in G.P. and a=b/3=c/5 then a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) None

123. If a, b, (c+1) are in G.P. and $a = (b-c)^2$ then a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) None

124. If $S_1, S_2, S_3, ...S_n$ are the sums of infinite G.P.s whose first terms are 1, 2, 3 …..n and whose common ratios are 1/2, 1/3, ……1/(*n*+1) then the value of $S_1+S_2+S_3+ ....S_n$ is

(a) (n/2) (n+3) (b) (n/2) (n+2) (c) (n/2) (n+1) (d) $n^2/2$

125. The G.P. whose 3rd and 6th terms are 1, –1/8 respectively is

(a) 4, –2, 1 ….. (b) 4, 2, 1 ……. (c) 4, –1, 1/4 …….. (d) None

126. In a G.P. if the $(p+q)^{th}$ term is *m* and the $(p-q)^{th}$ term is *n* then the p^{th} term is_________.

(a) $(mn)^{1/2}$ (b) mn (c) (m+n) (d) (m-n)

127. The sum of *n* terms of the series is $1/\sqrt{3}+1+3/\sqrt{3}+....$

(a) $(1/6)(3+\sqrt{3})(3^{n/2}-1)$ (b) $(1/6)(\sqrt{3}+1)(3^{n/2}-1)$

(c) $(1/6)(3+\sqrt{3})(3^{n/2}+1)$ (d) None

128. The sum of *n* terms of the series 5/2 – 1 + 2/5 – …… is

(a) $(1/14)(5^n+2^n)/5^{n-2}$ (b) $(1/14)(5^n-2^n)/5^{n-2}$ (c) both (d) None

129. The sum of *n* terms of the series 0.3 + 0.03 + 0.003 + …….. is

(a) $(1/3)(1-1/10^n)$ (b) $(1/3)(1+1/10^n)$ (c) both (d) None

130. The sum of first eight terms of G.P. is five times the sum of the first four terms. The common ratio is __________.

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) both (d) None

131. If the sum of n terms of a G.P. with first term 1 and common ratio 1/2 is 1+127/128, the value of n is _______.

(a) 8 (b) 5 (c) 3 (d) None

132. If the sum of n terms of a G.P. with last term 128 and common ratio 2 is 255, the value of n is _________.

(a) 8 (b) 5 (c) 3 (d) None

133. How many terms of the G.P. 1, 4, 16 …. are to be taken to have their sum 341?

(a) 8 (b) 5 (c) 3 (d) None

134. The sum of n terms of the series 5 + 55 + 555 + …….. is

(a) $(50/81)\ (10^n-1)-(5/9)n$ (b) $(50/81)\ (10^n+1)-(5/9)n$

(c) $(50/81)\ (10^n+1)+(5/9)n$ (d) None

135. The sum of n terms of the series 0.5 + 0.55 + 0.555 + ………. is

(a) $(5/9)n-(5/81)(1-10^{-n})$ (b) $(5/9)n+(5/81)(1-10^{-n})$

(c) $(5/9)n+(5/81)(1+10^{-n})$ (d) None

136. The sum of n terms of the series $1.03+1.03^2+1.03^3+......$ is

(a) $(103/3)(1.03^n-1)$ (b) $(103/3)(1.03^n+1)$ (c) $(103/3)(1.03^{n+1}-1)$ (d) None

137. The sum upto infinity of the series 1/2 + 1/6 + 1/18 + …… is

(a) 3/4 (b) 1/4 (c) 1/2 (d) None

138. The sum upto infinity of the series 4 + 0.8 + 0.16 + …… is

(a) 5 (b) 10 (c) 8 (d) None

139. The sum upto infinity of the series $\sqrt{2}+1/\sqrt{2}+1/(2\sqrt{2})+.......$ is

(a) $2\sqrt{2}$ (b) 2 (c) 4 (d) None

140. The sum upto infinity of the series 2/3 + 5/9 + 2/27 + 5/81 + ……. is

(a) 11/8 (b) 8/11 (c) 3/11 (d) None

141. The sum upto infinity of the series $(\sqrt{2}+1)+1+(\sqrt{2}-1)+......$ is

(a) $(1/2)(4+3\sqrt{2})$ (b) $(1/2)(4-3\sqrt{2})$ (c) $4+3\sqrt{2}$ (d) None

142. The sum upto infinity of the series $(1+2^{-2})+(2^{-1}+2^{-4})+(2^{-2}+2^{-6})+ ....$ is

(a) 7/3 (b) 3/7 (c) 4/7 (d) None

143. The sum upto infinity of the series $4/7-5/7^2+4/7^3-5/7^4+ ....$ is

(a) 23/48 (b) 25/48 (c) 1/2 (d) None

144. If the sum of infinite terms in a G.P. is 2 and the sum of their squares is 4/3 the series is

(a) 1, 1/2, 1/4 …… (b) 1, –1/2, 1/4 ……. (c) –1, –1/2, –1/4 …. (d) None

145. The infinite G.P. with first term 1/4 and sum 1/3 is

(a) 1/4, 1/16, 1/64 … (b) 1/4, –1/16, 1/64 …(C) 1/4, 1/8, 1/16 …. (d) None

146. If the first term of a G.P. exceeds the second term by 2 and the sum to infinity is 50 the series is __________.

(a) 10, 8, 32/5 … (b) 10, 8, 5/2 … (c) 10, 10/3, 10/9 …. (d) None

147. Three numbers in G.P. with their sum 130 and their product 27,000 are _________.

(a) 10, 30, 90 … (b) 90, 30, 10 … (c) both (d) None

148. Three numbers in G.P. with their sum 13/3 and sum of their squares 91/9 are ____.

(a) 1/3, 1, 3 (b) 3, 1, 1/3 (c) both (d) None

149. Find five numbers in G.P. such that their product is 32 and the product of the last two is 108.

(a) 2/9, 2/3, 2, 6, 18 (b) 18, 6, 2, 2/3, 2/9 (c) both (d) None

150. If the continued product of three numbers in G.P. is 27 and the sum of their products in pairs is 39 the numbers are _________.

(a) 1, 3, 9 (b) 9, 3, 1 (c) both (d) None

151. The numbers x, 8, y are in G.P. and the numbers x, y, –8 are in A.P. The values of x, y are ___________.

(a) 16, 4 (b) 4, 16 (c) both (d) None

ANSWERS

1.	(c)	**31.**	(a)	**61.**	(a)	**91.**	(a)	**121.**	(a)
2.	(a)	**32.**	(a)	**62.**	(a)	**92.**	(a)	**122.**	(a)
3.	(b)	**33.**	(a)	**63.**	(a)	**93.**	(a)	**123.**	(a)
4.	(a)	**34.**	(b)	**64.**	(a)	**94.**	(c)	**124.**	(a)
5.	(a)	**35.**	(b)	**65.**	(a)	**95.**	(c)	**125.**	(a)
6.	(b)	**36.**	(a)	**66.**	(a)	**96.**	(b)	**126.**	(a)
7.	(c)	**37.**	(b)	**67.**	(a)	**97.**	(b)	**127.**	(a)
8.	(c)	**38.**	(c)	**68.**	(d)	**98.**	(b)	**128.**	(c)
9.	(a)	**39.**	(d)	**69.**	(a)	**99.**	(a)	**129.**	(a)
10.	(a)	**40.**	(d)	**70.**	(a)	**100.**	(a)	**130.**	(c)
11.	(b)	**41.**	(a)	**71.**	(a)	**101.**	(b)	**131.**	(a)
12.	(c)	**42.**	(a)	**72.**	(a)	**102.**	(a)	**132.**	(a)
13.	(a)	**43.**	(d)	**73.**	(a)	**103.**	(a)	**133.**	(b)
14.	(c)	**44.**	(d)	**74.**	(a)	**104.**	(a)	**134.**	(a)
15.	(b)	**45.**	(c)	**75.**	(a)	**105.**	(c)	**135.**	(a)
16.	(a)	**46.**	(a)	**76.**	(a)	**106.**	(c)	**136.**	(a)
17.	(a)	**47.**	(b)	**77.**	(b)	**107.**	(a)	**137.**	(a)
18.	(a)	**48.**	(c)	**78.**	(a)	**108.**	(a)	**138.**	(a)
19.	(a)	**49.**	(a)	**79.**	(a)	**109.**	(a)	**139.**	(a)
20.	(c)	**50.**	(b)	**80.**	(a)	**110.**	(b)	**140.**	(a)
21.	(b)	**51.**	(a)	**81.**	(a)	**111.**	(b)	**141.**	(a)
22.	(a)	**52.**	(d)	**82.**	(a)	**112.**	(c)	**142.**	(a)
23.	(a)	**53.**	(b)	**83.**	(a)	**113.**	(a)	**143.**	(a)
24.	(a)	**54.**	(b)	**84.**	(a)	**114.**	(a)	**144.**	(a)
25.	(c)	**55.**	(a)	**85.**	(a)	**115.**	(a)	**145.**	(a)
26.	(a)	**56.**	(a)	**86.**	(a)	**116.**	(b)	**146.**	(a)
27.	(c)	**57.**	(d)	**87.**	(a)	**117.**	(a)	**147.**	(c)
28.	(c)	**58.**	(a)	**88.**	(b)	**118.**	(a)	**148.**	(c)
29.	(a)	**59)**	(a)	**89.**	(c)	**119.**	(a)	**149.**	(a)
30.	(c)	**60.**	(a)	**90.**	(d)	**120.**	(a)	**150.**	(c)
151.	(a)								

NOTES

NOTES

CHAPTER L

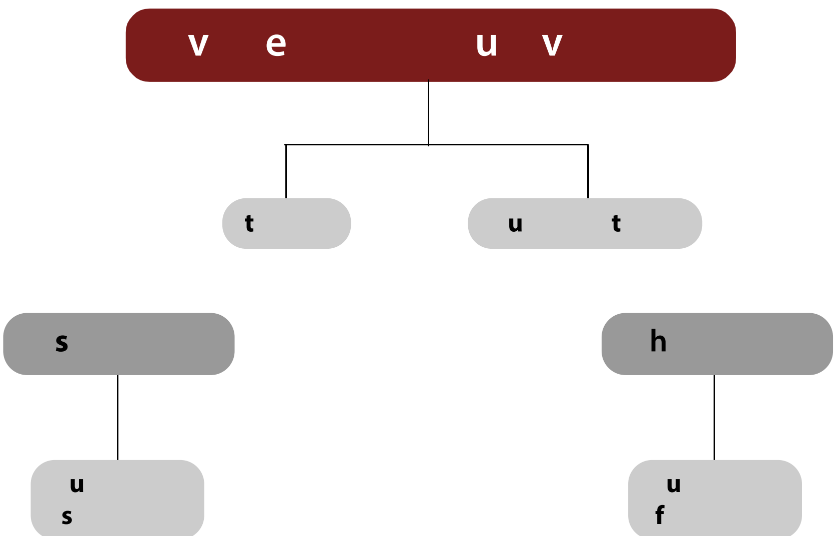
SETS, RELATIONS AND FUNCTIONS

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- Understand the concept of set theory.
- Appreciate the basics of functions and relations.
- Understand the types of functions and relations.
- Solve problems relating to sets, functions and relations.

CHAPTER OVERVIEW



In our mathematical language, everything in this universe, whether living or non-living, is called an object.

If we consider a collection of objects given in such a way that it is possible to tell beyond doubt whether a given object is in the collection under consideration or not, then such a collection of objects is called a *well-defined collection of objects.*

7.1 SETS

A set is defined to be a collection of well-defined distinct objects. This collection may be listed or described. Each object is called an element of the set. We usually denote sets by capital letters and their elements by small letters.

Example: A = {a, e, i, o, u}

B = {2, 4, 6, 8, 10}

C = {pqr, prq, qrp, rqp, qpr, rpq}

D = {1, 3, 5, 7, 9}

E = {1,2}

etc.

This form is called Roster or Braces form. In this form we make a list of the elements of the set and put it within braces { }.

Instead of listing we could describe them as follows :

A = the set of vowels in the alphabet

B = The set of even numbers between 2 and 10 both inclusive.

C = The set of all possible arrangements of the letters p, q and r

D = The set of odd digits between 1 and 9 both inclusive.

E = The set of roots of the equation $x^2 - 3x + 2 = 0$

Set B, D and E can also be described respectively as

B = $\{x : x = 2m$ and m being an integer lying in the interval $0 < m < 6\}$

D = $\{2x - 1 : 0 < x < 5$ and x is an integer$\}$

E = $\{x : x^2 - 3x + 2 = 0\}$

This form is called set-Builder or Algebraic form or Rule Method. This method of writing the set is called Property method. The symbol : or/reads 'such that'. In this method, we list the property or properties satisfied by the elements of the set.

We write, {x:x satisfies properties P}. This means, "the set of all those x such that x satisfies the properties P".

A set may contain either a finite or an infinite number of members or elements. When the number of members is very large or infinite it is obviously impractical or impossible to list them all. In such case.

we may write as :

N = The set of natural numbers = {1, 2, 3…..}

W = The set of whole numbers = {0, 1, 2, 3,…)

etc.

I. The members of a set are usually called elements. In A = {a, e, i, o, u}, a is an element and we write $a \in A$ i.e. a belongs to A. But 3 is not an element of B = {2, 4, 6, 8, 10} and we write $3 \notin B$. i.e. 3 does not belong to B.

II. If every element of a set P is also an element of set Q we say that P is a subset of Q. We write $P \subset Q$. Q is said to be a superset of P. For example {a, b} $\subset$ {a, b, c}, {2, 4, 6, 8, 10} $\subset$ N. If there exists even a single element in A, which is not in B then A is not a subset of B.

III. If P is a subset of Q but P is not equal to Q then P is called a proper subset of Q.

IV. Φ has no proper subset.

Illustration: {3} is a proper subset of {2, 3, 5}. But {1, 2} is not a subset of {2, 3, 5}.

Thus if P = {1, 2} and Q = {1, 2 ,3} then P is a subset of Q but P is not equal to Q. So, P is a proper subset of Q.

To give completeness to the idea of a subset, we include the set itself and the empty set. The empty set is one which contains no element. The empty set is also known as **null or void** set usually denoted by { } or Greek letter Φ, to be read as phi. For example the set of prime numbers between 32 and 36 is a null set. The subsets of {1, 2, 3} include {1, 2, 3}, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3} and { }.

A set containing n elements has 2^n subsets. Thus a set containing 3 elements has 2^3 (=8) subsets. A set containing n elements has $2^n - 1$ proper subsets. Thus a set containing 3 elements has $2^3 - 1 = 7$ subsets. The proper subsets of { 1,2,3} include {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3], { } .

Suppose we have two sets A and B. The intersection of these sets, written as $A \cap B$ contains those elements which are in A and are also in B.

For example A = {2, 3, 6, 10, 15}, B = {3, 6, 15, 18, 21, 24} and C = { 2, 5, 7}, we have $A \cap B$ = { 3, 6, 15}, $A \cap C$ = {2}, $B \cap C = \Phi$, where the intersection of B and C is empty set. So, we say B and C are disjoint sets since they have no common element. Otherwise sets are called overlapping or intersecting sets. The union of two sets, A and B, written as $A \cup B$ contain all these elements which are in either A or B or both.

So $A \cup B$ = {2, 3, 6, 10, 15, 18, 21, 24}

$A \cup C$ = {2, 3, 5, 6, 7, 10, 15}

A set which has at least one element is called non-empty set . Thus the set { 0 } is non-empty set. It has one element say 0.

Singleton Set: A set containing one element is called Singleton Set.

For example {1} is a singleton set, whose only element is 1.

Equal Set: Two sets A & B are said to be equal, written as A = B if every element of A is in B and every element of B is in A.

Illustration: If A = {2, 4, 6} and B = {2, 4, 6} then A = B.

Remarks : (I) The elements of the two sets may be listed in any order.

Thus, {1, 2, 3} = {2, 1, 3} = {3, 2, 1} etc.

(II) The repetition of elements in a set is meaningless.

Example: {x : x is a letter in the word "follow"} = {f, o, l, w}

Example: Show that Φ, {0} and 0 are all different.

Solution: Φ is a set containing no element at all; {0} is a set containing one element, namely 0. And 0 is a number, not a set.

Hence Φ,{0} and 0 are all different.

The set which contains all the elements under consideration in a particular problem is called *the universal set* denoted by S. Suppose that P is a subset of S. Then the complement of P, written as P^c (or P') contains all the elements in S but not in P. This can also be written as S – P or S ~ P. S – P = {x : x $\in$ S, x $\notin$ P}.

For example let S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

P = {0, 2, 4, 6, 8}

Q = {1, 2, 3, 4, 5)

Then P' = {1, 3 ,5 ,7, 9} and Q'= {0 , 6 , 7, 8, 9}

Also $P \cup Q$ = {0, 1, 2, 3, 4, 5, 6, 8}, $(P \cup Q)'$ = {7, 9}

$P \cap Q$ = {2, 4}

$P \cup Q'$ = {0, 2, 4, 6, 7, 8, 9}, $(P \cap Q)'$ = {0, 1, 3, 5, 6, 7, 8, 9}

$P' \cup Q'$ = {0, 1, 3, 5, 6, 7, 8, 9}

$P' \cap Q'$ = {7, 9}

So it can be noted that $(P \cup Q)' = P' \cap Q'$ and $(P \cap Q)' = P' \cup Q'$. These are known as De Morgan's laws.

7.2 VENN DIAGRAMS

We may represent the above operations on sets by means of Euler - Venn diagrams.

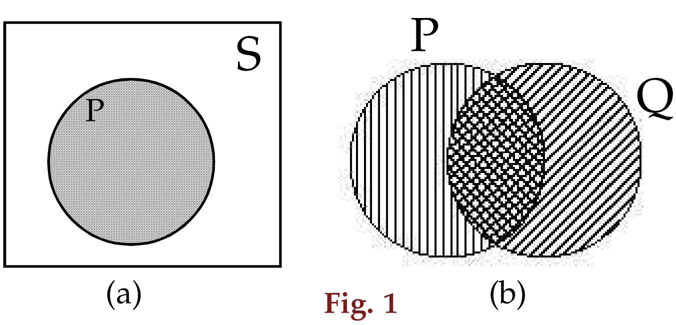


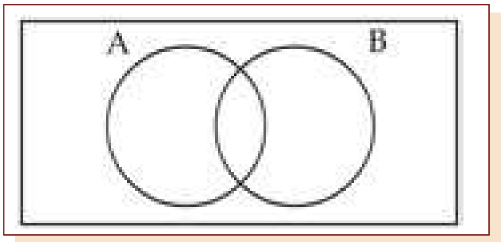
Fig. 1

Thus Fig. 1(a) shows a universal set S represented by a rectangular region and one of its subsets P represented by a circular shaded region.

The un-shaded region inside the rectangle represents P'.

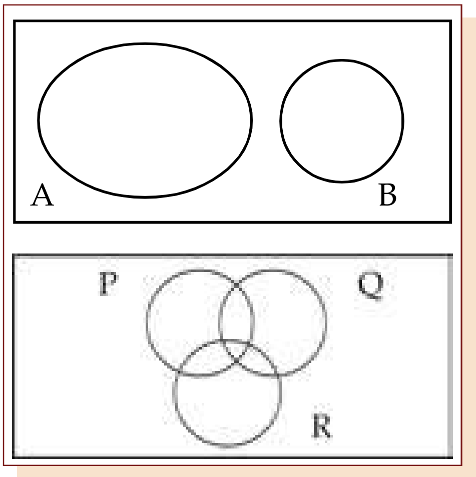
Figure 1(b) shows two sets P and Q represented by two intersecting circular regions. The total shaded area represents PUQ, the cross-hatched "intersection" represents $P \cap Q$.

The number of distinct elements contained in a finite set A is called its **cardinal number**. It is denoted by n(A). For example, the number of elements in the set R = {2, 3, 5, 7} is denoted by n(R). This number is called the cardinal number of the set R.



Thus $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

If A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$ as $n(A \cap B) = 0$



For three sets P, Q and R

$n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(P \cap R) + n(P \cap Q \cap R)$

When P, Q and R are disjoint sets

$= n(P) + n(Q) + n(R)$

Illustration: If A = {2, 3, 5, 7} , then n(A) = 4

Equivalent Set: Two finite sets A & B are said to be equivalent if n (A) = n(B).

Clearly, **equal sets are equivalent but equivalent sets need not be equal.**

Illustration: The sets A = {1, 3, 5} and B = {2, 4, 6} are equivalent but not equal.

Here n (A) = 3 = n (B) so they are equivalent sets. But the elements of A are not in B. Hence they are not equal though they are equivalent.

Power Set : The collection of all possible subsets of a given set A is called the power set of A, to be denoted by P(A).

Illustration: (i) If A = {1, 2, 3} then

P(A) = { {1, 2, 3}, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3}, Φ }

(ii) If A = {1, {2}}, we may write A = {1, B} when B = {2} then

P(A) = {Φ, {1}, {B }, {1, B}} = {Φ, {1}, {{2}}, {1,{2}}}

EXERCISE 7 (A)

Choose the most appropriate option or options (a) (b) (c) or (d).

1. The number of subsets of the set {2, 3, 5} is
   (a) 3 (b) 8 (c) 6 (d) none of these
2. The number of subsets of a set containing n elements is
   (a) 2^n (b) 2^{-n} (c) n (d) none of these
3. The null set is represented by
   (a){Φ } (b) { 0 } (c) Φ (d) none of these
4. A = {2, 3, 5, 7} , B = { 4, 6, 8, 10} then $A \cap B$ can be written as
   (a) { } (b) {Φ} (c) (AUB)' (d) None of these
5. The set {x | 0<x<5} represents the set when x may take integral values only
   (a) {0, 1, 2, 3, 4, 5} (b) {1, 2, 3, 4} (c) {1, 2, 3, 4, 5 } (d) none of these
6. The set {0, 2, 4, 6, 8, 10} can be written as
   (a) {2x | 0<x <5} (b) {x : 0<x<5} (c) {2x : $0 \le x \le 5$} (d) none of these

From Q.7 to Q.10 The data to be used If P = {1, 2, 3, 5, 7}, Q = {1, 3, 6, 10, 15},
Universal Set S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

7. The cardinal number of $P \cap Q$ is
   (a) 3 (b) 2 (c) 0 (d) none of these
8. The cardinal number of $P \cup Q$ is
   (a) 10 (b) 9 (c) 8 (d) none of these
9. n (P') is
   (a) 10 (b) 5 (c) 6 (d) none of these
10. n(Q') is
    (a) 4 (b) 10 (c) 4 (d) none of these
11. The set of cubes of the natural number is
    (a) a finite set (b) an infinite set (c) a null set (d) none of these

12. The set $\{2^x \mid x$ is any positive rational number$\}$ is
    (a) an infinite set (b) a null set (c) a finite set (d) none of these
13. $\{1-(-1)^x\}$ for all integer x is the set
    (a) $\{0\}$ (b) $\{2\}$ (c) $\{0, 2\}$ (d) none of these
14. E is a set of positive even numbers and O is a set of positive odd numbers, then $E \cup O$ is a
    (a) set of whole numbers (b) N (c) a set of rational number (d) none of these
15. If R is the set of positive rational numbers and E is the set of real numbers then
    (a) $R \subseteq E$ (b) $R \subset E$ (c) $E \subset R$ (d) none of these
16. If N is the set of natural numbers and I is the set of positive integers, then
    (a) N = I (b) $N \subset I$ (c) $N \subseteq I$ (d) none of these
17. If I is the set of isosceles triangles and E is the set of equilateral triangles, then
    (a) $I \subset E$ (b) $E \subset I$ (c) E = I (d) none of these
18. If R is the set of isosceles right angled triangles and I is set of isosceles triangles, then
    (a) R = I (b) $R \supset I$ (c) $R \subset I$ (d) none of these
19. $\{n(n+1)/2 : n$ is a positive integer$\}$ is
    (a) a finite set (b) an infinite set (c) is an empty set (d) none of these
20. If A = {1, 2, 3, 5, 7}, and $B = \{x^2 : x \in A\}$
    (a) n(b) = n(A) (b) n(B) > n(A) (c) n(A)= n(B) (d) n(A)<n(B)
21. $A \cup A$ is equal to
    (a) A (b) E (c) ϕ (d) none of these
22. $A \cap A$ is equal to
    (a) ϕ (b) A (c) E (d) none of these
23. $(A \cup B)'$ is equal to
    (a) $(A \cap B)'$ (b) $A \cup B'$ (c) $A' \cap B'$ (d) none of these
24. $(A \cap B)'$ is equal to
    (a) $(A' \cup B)'$ (b) $A' \cup B'$ (c) $A' \cap B'$ (d) none of these
25. $A \cup E$ is equal to (E is a superset of A)
    (a) A (b) E (c) ϕ (d) none of these
26. $A \cap E$ is equal to (E is a superset of A)
    (a) A (b) E (c) ϕ (d) none of these

27. $E \cup E$ is equal to E is a superset of A

(a) E (b) ϕ (c) 2E (d) none of these

28. $A \cap E'$ is equal to E is a superset of A

(a) E (b) ϕ (c) A (d) none of these

29. $A \cap \phi$ is equal to E is a superset of A

(a) A (b) E (c) ϕ (d) none of these

30. AUA' is equal to E is a superset of A

(a) E (b) ϕ (c) A (d) none of these

31. If E = {1, 2, 3, 4, 5, 6, 7, 8, 9}, the subset of E satisfying 5 + x > 10 is

(a) {5, 6, 7, 8, 9} (b) {6, 7, 8, 9} (c) {7, 8, 9} (d) none of these

32. If $A\Delta B = (A–B) \cup (B–A)$ and A = {1, 2, 3, 4}, B = {3,5,7} than $A\Delta B$ is

(a) {1, 2, 4, 5, 7} (b) {3} (c) {1, 2, 3, 4, 5, 7} (d) none of these

[Hint : If A and B are any two sets, then

$A - B = \{ x : x \in A, x \notin B\}$.

[i.e. A - B Contains all elements of A but not in B] .

7.3 PRODUCT OF SETS

Ordered Pair : Two elements a and b, listed in a specific order, form an ordered pair, denoted by (a, b).

Cartesian Product of sets : If A and B are two non-empty sets, then the set of all ordered pairs (a, b) such that a belongs to A and b belongs to B , is called the Cartesian product of A and B, to be denoted by A × B.

Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If $A = \Phi$ or $B = \Phi$, we define $A \times B = \Phi$

Illustration: Let A = {1, 2, 3}, B = {4, 5}

Then A × B = { (1, 4), (1, 5), (2, 4) (2, 5), (3, 4), (3, 5) }

Example: If A × B = { (3, 2) , (3, 4) , (5, 2), (5, 4) } , find A and B.

Solution: Clearly A is the set of all first co-ordinates of A × B, while B is the set of all second co-ordinates of elements of A × B.

Therefore A = {3, 5} and B = {2 , 4}

Example: Let P = {1, 3, 6} and Q {3, 5}

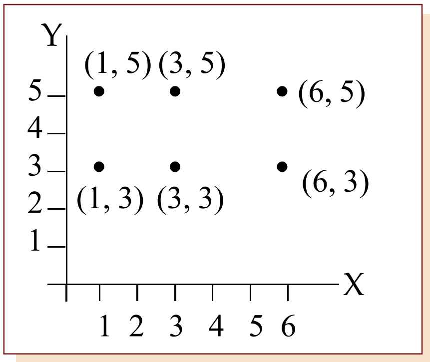
The product set P × Q = {(1, 3), (1, 5), (3, 3), (3, 5), (6, 3), (6, 5)} .

Notice that n(P×Q) = n(P) × n(Q) and ordered pairs (3, 5) and (5, 3) are not equal. and Q × P = {(3, 1), (3, 3), (3, 6), (5, 1), (5, 3), (5, 6)}

So P × Q $\neq$ Q × P; but n (P × Q) = n (Q × P).

Illustration: Here n(P) = 3 and n(Q) = 2, n (P × Q) = 6. Hence n(P×Q) = n(P) × n(Q). and n(P × Q) = n(Q × P) = 6.

We can represent the product set of ordered pairs by plotting points in the XY plane.



If X=Y= the set of all natural numbers, the product set XY represents an infinite equal lattice of points in the first quadrant of the XY plane.

7.4 RELATIONS AND FUNCTIONS

Any subset of the product set X.Y is said to define a **relation** from X to Y and any relation from X to Y in which no two different ordered pairs have the same first element is called a **function**.

Let A and B be two non-empty sets. Then, a rule or a correspondence f which associates to each element x of A, a unique element, denoted by f(x) of B , is called a function or **mapping** from A to B and we write f : A→B

The element f(x) of B is called the image of x, while x is called the pre-image of f (x).

7.5 DOMAIN & RANGE OF A FUNCTION

Let f : A→B, then A is called the domain of f, while B is called the co-domain of f.

The set f(A) = {f (x) : x ∈ A} is called the range of f.

Illustration: Let A = {1, 2, 3, 4} and B ={1, 4, 9, 16, 25}

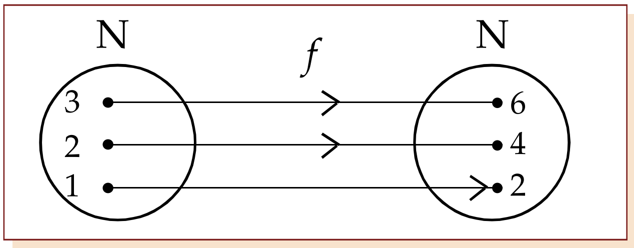
We consider the rule f(x) = x^2 . Then f(1) = 1 ; f(2) =4 ; f (3) = 9 & f (4) = 16.

Then clearly each element in A has a unique image in B.

So, f : A → B : f (x) = x^2 is a function from A to B.

Here domain of f = {1, 2, 3, 4} and range of f = {1, 4, 9, 16}

Example: Let N be the set of all natural numbers. Then, the rule



$f : N \rightarrow N : f(x) = 2x$, for all $x \in N$

is a function from N to N, since twice a natural number is unique.

Now, f (1) = 2; f (2) = 4; f(3) = 6 and so on.

Here domain of f = N = {1, 2 ,3, 4 ,………..} ; range of f = { 2, 4, 6,………..}

This may be represented by the mapping diagram or arrow graph.

7.6 VARIOUS TYPES OF FUNCTIONS

One-One Function : Let $f : A \rightarrow B$. If different elements in A have different images in B, then f is said to be a one-one or an injective function or mapping.

Illustration: (i) Let A = {1, 2, 3} and B = {2, 4, 6}

Let us consider $f : A \rightarrow B : f(x) = 2x$.

Then f(1) = 2; f(2) =4 ; f (3) = 6

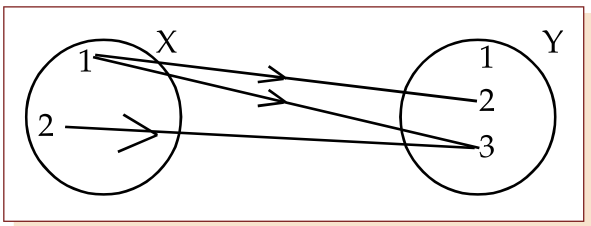
Clearly, f is a function from A to B such that different elements in A have different images in B. Hence f is one-one.

Remark : Let $f : A \rightarrow B$ and let $x_1 , x_2 \in A$.

Then $x_1 = x_2$ implies $f(x_1) = f(x_2)$ is always true.

But **$f(x_1) = f(x_2)$ implies $x_1 = x_2$ is true only when f is one-one.**

(ii) let x={1, 2, 3, 4} and y={1, 2, 3}, then the subset {(1, 2), (1, 3), (2, 3)} defines a relation on X.Y.



Notice that this particular subset contains all the ordered pair in X.Y for which the X element (x) is less than the Y element (y). So in this subset we have X<Y and the relation between the set, is "less than". This relation is not a function as it includes two different ordered pairs (1, 2), (1, 3) have same first element.

X.Y={(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)

(3, 1), (3, 2), (3, 3),(4, 1), (4, 2), (4, 3)}

The subset {(1, 1), (2, 2), (3, 3)} defines the function y= x as different ordered pairs of this subset have different first element.

Onto or Surjective Functions : Let f : A→B. If every element in B has at least one pre-image in A, then f is said to be an onto function.

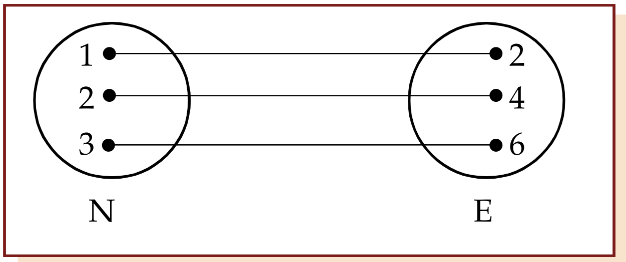
If f is onto, then corresponding to each y ∈ B, we must be able to find at least one element x ∈ A such that y = f (x)

Clearly, f is onto if and only if range of f = B

Illustration: Let N be the set of all natural numbers and E be the set of all even natural numbers. Then, the function

f : N → E : f (x) = 2x , for all x ∈ N

is onto, since each element of E is of the form 2x, where x ∈ N and the same is the f-image of x ∈ N.



Represented on a mapping diagram it is a on to mapping of X onto Y.

Bijection Function : A one-one and onto function is said to be bijective.
A bijective function is also known as a one-to-one correspondence.

Identity Function : Let A be a non-empty set. Then, the function I defined by

I : A → A : I (x) = x for all x ∈ A is called an identity function on A.

It is a one-to-one and onto function with domain A and range A.

Into Functions: Let f : A → B. There exists even a single element in B having no pre-image in A, then f is said to be an into function.

Illustration: Let A = {2, 3, 5, 7} and B = {0, 1, 3, 5, 7} . Let us consider f : A → B;

f (x) = x – 2. Then f(2) = 0; f(3) = 1 ; f (5) =3 & f(7) = 5.

It is clear that f is a function from A to B .

Here there exists an element 7 in B, having no pre-mage in A.

So, f is an into function.

Constant Function: Let f : A → B, defined in such a way that all the elements in A have the same image in B, then f is said to be a constant function.

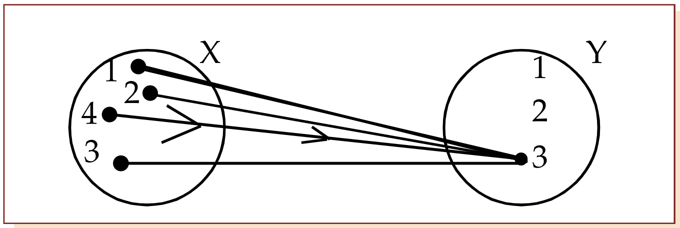
Illustration: Let A = {1, 2, 3} and B = {5, 5, 5} or B={5}. Let f : A → B : f (x) = 5 for all x ∈ A.

Then, all the elements in A have the same image namely 5 in B.

So, f is a constant function.

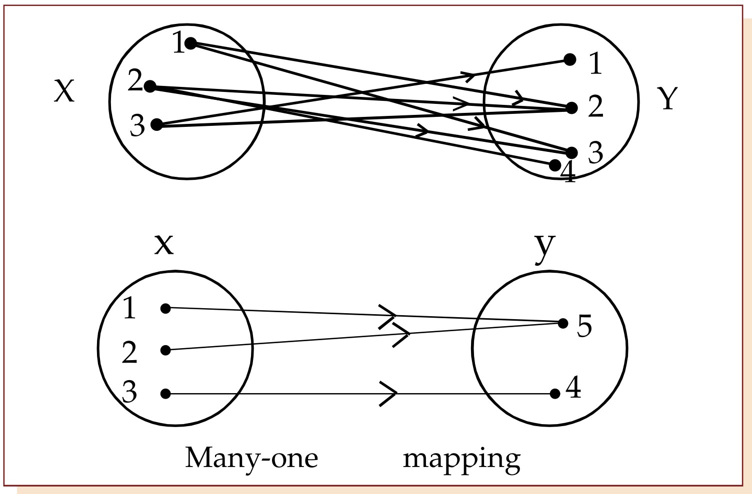
Remark: The range set of a constant function is a singleton set.

Example: Another subset of X.Y is {(1, 3), (2, 3), (3, 3), (4, 3)}



This relation is a function (a constant function). It is represented on a mapping diagram and is a many-one mapping of X into Y.

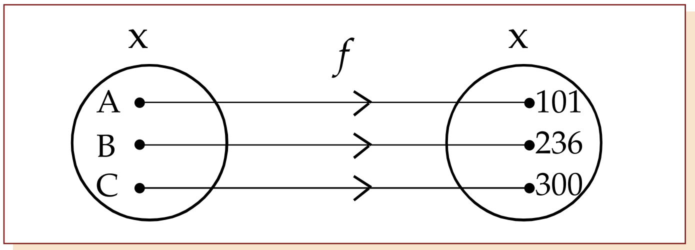
Let us take another subset {(4, 1), (4, 2), (4, 3)} of X.Y. This is a relation but not a function. Here different ordered pairs have same first element so it is not a function.



There are two example of many-one mapping.

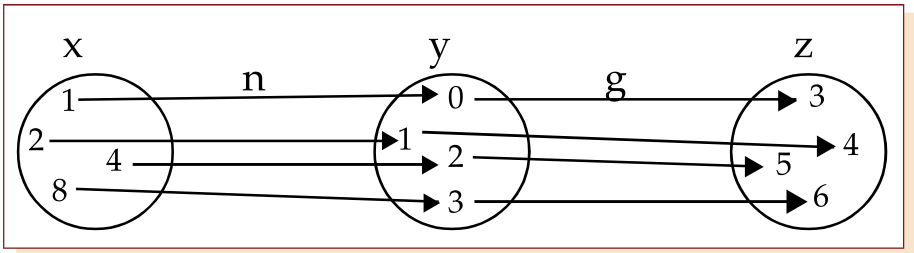
Equal Functions: Two functions f and g are said to be equal, written as f = g if they have the same domain and they satisfy the condition f(x) = g(x), for all x.

A function may simply pair people and the corresponding seat numbers in a theatre. It may simply associate weights of parcels with portal delivery charge or it may be the operation of squaring, adding to doubling, taking the log of etc.



The function f here assigning a locker number to each of the persons A, B and C. Names are associated with or mapped onto, locker numbers under the function f.

We can write $f : X \to Y$ or $f(x) = y$ or $f(B) = 236$



This diagram shows the effect of two functions n and g on the sets X, Y and Z

$n : X \to Y$ and $g : Y \to Z$

If x, y, z are corresponding elements of X, Y and Z, we write n(x) = y and g(y) = z

Thus g n(1) = 0 and g (0) = 3, so that g (n(1)) = g(0) = 3 we can write it as

g (n(1)) or g o n (1) = 3 But g(1) = 4 and n(g(1))= n(4) = 2

So $g(n) \neq n(g)$ [(or, g (o n) $\neq$ n (o g))]

The function g(n) or n(g) is called a composite function. As n(8) = 3, we say that 3 is the image of 8 under the mapping (or function) n.

Inverse Function: Let f be a one-one onto function from A to B. Let y be an arbitrary element of B. Then f being onto, there exists an element x in A such that f (x) = y.

As f is one-one this x is unique.

Thus for each $y \in B$, there exists a unique element $x \in A$ such that f (x) = y.

So, we may define a function, denoted by f^{-1} as:

$f^{-1} : B \to A : f^{-1}(y) = x$ if and only if f (x) = y.

The above function f^{-1} is called the inverse of f.

A function is invertible if and only if f is one-one onto.

Remarks: If f is one-one onto then f^{-1} is also one-one onto.

Example: net A = {1, 3, 5, 7} and B = {3, 4, 9} So, f; $A \to B$ given by {(1, 2), (3, 3), (5, 4) (7, 9)} and it is one-one on to mapping. Now if then is there of f.

If $f : A \to B$ then $f^{-1} : B \to A$.

EXERCISE 7 (B)

Choose the most appropriate option/options (a) (b) (c) or (d).

1. If A = {x, y, z}, B = {p, q, r, s} which of the relations on A to B are functions.
   (a) {(x, p), (x, q), (y, r), (z, s)}, (b) {(x, s), (y, s), (z, s)}
   (c) {(y, p), (y, q), (y, r),(z, s), (d) {(x, p), (y, r), (z, s)}
2. {(x, y) | x+y = 5} where x, y $\in$ R is a
   (a) not a function (b) a composite function (c) one-one mapping (d) none of these
3. {(x, y) | x = 4} where x, y $\in$ R is a
   (a) not a function (b) function (c) one-one mapping (d) none of these
4. {(x, y), y=x^2} where x, y $\in$ R is
   (a) not a function (b) a function (c) inverse mapping (d) none of these
5. {(x, y) | x<y} where x, y $\in$ R is
   (a) not a function (b) a function (c) one-one mapping (d) none of these
6. The domain of {(1, 7), (2, 6)} is
   (a) {1, 6} (b) {7, 6} (c) {1, 2} (d) {6, 7}
7. The range of {(3, 0), (2, 0), (1, 0), (0, 0)} is
   (a) {0, 0} (b) {0} (c) {0, 0, 0, 0} (d) none of these
8. The domain and range of {(x,y) : y = x^2} where x, y $\in$ R is
   (a) (reals, natural numbers) (b) (reals, non-negative reals)
   (c) (reals, reals) (d) none of these
9. Let the domain of x be the set {1}. Which of the following functions gives values equal to 1
   (a) f(x) = x^2, g(x) = x (b) f(x) = x, g(x) = 1–x
   (c) f(x) = x^2 + x + 2, g(x) = $(x+ 1)^2$ (d) none of these
10. If f(x) = 1/1–x, f(–1) is
    (a) 0 (b) ½ (c) 0 (d) none of these
11. If g(x) = (x–1)/x, g(–½) is
    (a) 1 (b) 2 (c) 3/2 (d) 3
12. If f(x) = 1/1–x and g(x) = (x–1)/x, then fog(x) is
    (a) x (b) 1/x (c) –x (d) none of these
13. If f(x) = 1/1–x and g(x) = (x–1)/x, then gof(x) is
    (a) x–1 (b) x (c) 1/x (d) none of these

14. The function $f(x) = 2^x$ is

(a) one-one mapping (b) one-many

(c) many-one (d) none of these

15. The range of the function $f(x) = \log_{10}(1 + x)$ for the domain of real values of x when $0 \leq x \leq 9$ is

(a) [0, 1] (b) [0, 1, 2] (c) {0, 1} (d) none of these

16. The Inverse function f^{-1} of $f(x) = 2x$ is

(a) 1/2x (b) $\frac{x}{2}$ (c) 1/x (d) none of these

17. If $f(x) = x+3$, $g(x) = x^2$, then fog(x) is

(a) $x^2 + 3$ (b) $x^2 + x + 3$ (c) $(x + 3)^2$ (d) none of these

18. If $f(x) = x+3$, $g(x) = x^2$, then f(x).g(x) is

(a) $(x + 3)^2$ (b) $x^2 + 3$ (c) $x^3 + 3x^2$ (d) none of these

19. The Inverse h^{-1} when $h(x) = \log_{10}x$ is

(a) $\log_{10}x$ (b) 10^x (c) $\log_{10}(1/x)$ (d) none of these

20. For the function $h(x) = 10^{1+x}$ the domain of real values of x where $0 \leq x \leq 9$, the range is

(a) $10 \leq h(x) \leq 10^{10}$ (b) $0 \leq h(x) \leq 10^{10}$ (c) $0 < h(x) < 10$

(d) none of these

Different types of relations

Let S = {a, b, c, ….} be any set then the relation R is a subset of the product set S×S

i) If R contains all ordered pairs of the form (a, a) in S×S, then R is called reflexive. In a *reflexive* relation 'a' is related to itself .

For example, 'Is equal to' is a reflexive relation for a = a is true.

ii) If $(a, b) \in R \Rightarrow (b, a) \in R$ for every $a, b \in S$ then R is called symmetric

For example $a=b \Rightarrow b = a$. Hence the relation 'is equal to' is a symmetric relation.

iii) If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \Rightarrow R$ for every a, b, c, $\in$ S then R is called *transistive.*

For example a =b, $b=c \Rightarrow a=c$. Hence the relation 'is equal to' is a transitive relation.

A relation which is reflexive, symmetric and transitive is called an *equivalence relation* or simply an *equivalence*. 'is equal to' is an equivalence relation.

Similarly, the relation " is parallel to " on the set S of all straight lines in a plane is an equivalence relation.

Illustration: The relation "is parallel to" on the set S is

(1) reflexive, since a $||$ a for a $\in$ S

(2) symmetric, since a || b $\Rightarrow$ b || a

(3) transitive, since a || b , b || c $\Rightarrow$ a || c

Hence it is an equivalence relation.

Domain & Range of a relation : If R is a relation from A to B, then the set of all first co-ordinates of elements of R is called the domain of R, while the set of all second co-ordinates of elements of R is called the range of R.

So, Dom (R) = { a : (a, b) $\in$ R } & Range (R) = { b : (a, b) $\in$ R}

Illustration: Let A = {1, 2, 3} and B = {2, 4, 6}

Then A × B = {(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6)}

By definition every subset of A × B is a relation from A to B.

Thus, if we consider the relation

R = { (1, 2), (1, 4), (3, 2), (3, 4)} then Dom (R) = {1, 3} and Range (R)= {2, 4}

From the product set X. Y = {(1, 3), (2, 3), (3, 3), (4, 3), (2, 2), (3, 2), (4, 2), (1, 1), (2, 1), (3, 1), (4, 1)}, the subset {(1, 1), (2, 2), (3, 3)} defines the relation 'Is equal to' , the subset {(1, 3), (2, 3), (1, 2)} defines 'Is less than', the subset {(4, 3), (3, 2), (4, 2), (2, 1), (3, 1), (4, 1)} defines 'Is greater than' and the subset {(4, 3), (3, 2), (4, 2), (2, 1), (3, 1), (4, 1), (1, 1), (2, 2) (3, 3)} defines to greater 'In greater than or equal'.

Illustration: Let A = {1, 2, 3} and b = {2, 4, 6}

Then A × B = {(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6)}

If we consider the relation = {(1, 2), (1, 4), (3, 4)} then Dom (R) = {1, 3} and Range = {2, 4}. Here the relation "Is less than".

Identity Relation: The relation I = {(a, a) : a $\in$A} is called the identity relation on A.

Illustration: Let A = {1, 2, 3} then I = {(1, 1), (2, 2), (3, 3)}

Inverse Relation: If R be a relation on A, then the relation R^{-1} on A, defined by

R^{-1} = {(b, a) : (a, b)$\in$ R} is called an inverse relation on A.

Clearly , Dom (R^{-1}) = Range (R) & Range (R^{-1}) = Dom (R).

Illustration: Let A = {1, 2, 3} and R = {(1, 2), (2, 2), (3, 1), (3, 2)}

Then R being a subset of a × a, it is a relation on A. Dom (R) = {1, 2, 3} and Range (R) = {2, 1}

Now, R^{-1} = {(2, 1), (2, 2), (1, 3), (2, 3)}. Here, Dom (R^{-1}) = {2, 1} = Range (R) and

Range (R^{-1}) = {1, 2, 3} = Dom (R).

Illustration: Let A = {1, 2, 3}, then

(i) R1 = {(1, 1), (2, 2), (3, 3), (1, 2)}

Is reflexive and transitive but not symmetric, since (1, 2)$\in$ R_1 but (2, 1) does not belongs to R_1.

(ii) R2 = {(1, 1), (2, 2), (1, 2), (2, 1)}

is symmetric and transitive but not reflexive, since (3, 3) does not belong to R_2.

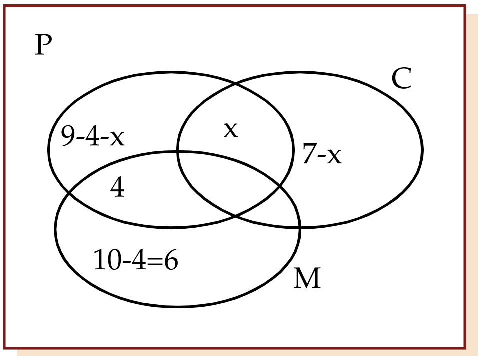
(iii) R3 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)}

is reflexive and symmetric but not transitive , since (1, 2) $\in$ R3 & (2, 3) $\in$ R3 but

(1, 3) does not belong to R3.

Problems and solution using Venn Diagram

1. Out of a group of 20 teachers in a school, 10 teach Mathematics, 9 teach Physics and 7 teach Chemistry. 4 teach Mathematics and Physics but none teach both Mathematics and Chemistry. How many teach Chemistry and Physics? How many teach only Physics ?



Let x be the no. of teachers who teach both Physics & Chemistry.

$9 - 4 - x + 6 + 7 - x + 4 + x = 20$

or $22 - x = 20$

or $x = 2$

Hence, 2 teachers teach both Physics and Chemistry and $9 - 4 - 2 = 3$ teachers teach only Physics.

2. A survey shows that 74% of the Indians like grapes, whereas 68% like bananas.

What percentage of the Indians like both grapes and bananas?

Solution: Let P & Q denote the sets of Indians who like grapes and bananas respectively. Then n(P) = 74, n(Q) = 68 and n($P \cup Q$) = 100.

We know that n($P \cap Q$) = n(P) + n(Q) – n($P \cup Q$) = 74 + 68 – 100 = 42.

Hence, 42% of the Indians like both grapes and bananas.

3. In a class of 60 students, 40 students like Maths, 36 like Science, and 24 like both the subjects. Find the number of students who like

(i) Maths only (ii) Science only (iii) either Maths or Science

(iv) neither Maths nor Science

Solution: Let M = students who like Maths and S = students who like Science

Then n(M) = 40, n(S) = 36 and n $(M \cap S)$ = 24

Hence,

(i) $n(M) - n(M \cap S) = 40 - 24 = 16$ = number of students like Maths only.

(ii) $n(S) - n(M \cap S) = 36 - 24 = 12$ = number of students like Science only.

(iii) $n(M \cup S) = n(M) + n(S) - n(M \cap S) = 40 + 36 - 24 = 52$ = number of students who like either Maths or Science.

(iv) $n(M \cup S)^c = 60 - n(M \cup S) = 60 - 52 = 8$ = number of students who like neither Maths nor Science.

SUMMARY

- A set is defined to be a collection of well-defined distinct objects. This collection may be listed or described. Each object is called an element of the set. We usually denote sets by capital letters and their elements by small letters.
- **Singleton Set:** A set containing one element is called Singleton.
- **Equal Set:** Two sets A & B are said to be equal, written as A = B if every element of A is in B and every element of B is in A.
- **Universal Set:** The set which contains all the elements under consideration in a particular problem is called *the universal set* denoted by S. Suppose that P is a subset of S. Then the complement of P, written as P^c (or P') contains all the elements in S but not in P. This can also be written as S – P or S ~ P. $S - P = \{x : x \in S, x \notin P\}$.
- If A and B are two sets then

  $n(AUB) = n(A) + n(B) - n(A \cap B)$
- If A and B are disjoint sets, then $n(AUB) = n(A) + n(B)$ as $n(A \cap B) = 0$
- For three sets P, Q and R

  $n(PUQUR) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(P \cap R) + n(P \cap Q \cap R)$

  When P, Q and R are disjoint sets

  $= n(P) + n(Q) + n(R)$
- **Equivalent Set:** Two finite sets A & B are said to be equivalent if n (A) = n(B).
- **Power Set:** The collection of all possible subsets of a given set A is called the power set of A, to be denoted by P(A).

  1. A set containing n elements has 2^n subsets.
  2. A set containing n elements has 2^n-1 proper subsets

- **Ordered Pair:** Two elements a and b, listed in a specific order, form an ordered pair, denoted by (a, b).
- **Cartesian Product of sets:** If A and B are two non-empty sets, then the set of all ordered pairs (a, b) such that a belongs to A and b belongs to B , is called the Cartesian product of A and B, to be denoted by A × B.

  Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

  If $A = \Phi$ or $B = \Phi$, we define $A \times B = \Phi$
- **Relation and Function:** Any subset of the product set X.Y is said to define a **relation** from X to Y and any relation from X to Y in which no two different ordered pairs have the same first element is called a **function**.

  Let A and B be two non-empty sets. Then, a rule or a correspondence f which associates to each element x of A, a unique element, denoted by f(x) of B , is called a function or **mapping** from A to B and we write $f : A \to B$

  The element f(x) of B is called the image of x, while x is called the pre-image of f (x).

  Let $f : A \to B$, then A is called the domain of f, while B is called the co-domain of f.

  The set $f(A) = \{ f(x) : x \in A \}$ is called the range of f.
- **One-one Function:** Let $f : A \to B$. If different elements in A have different images in B, then f is said to be a one-one or an injective function or mapping.
- **Onto or Surjective Functions:** Let $f : A \to B$. If every element in B has at least one pre-image in A, then f is said to be an onto function.

  If f is onto, then corresponding to each $y \in B$, we must be able to find at least one element $x \in A$ such that y = f (x)

  Clearly, f is onto if and only if range of f = B
- **Bijection Function:** A one-one and onto function is said to be bijective.

  A bijective function is also known as a one-to-one correspondence.
- **Identity Function:** Let A be a non-empty set . Then, the function I defined by

  $I : A \to A : I(x) = x$ for all $x \in A$ is called an identity function on A.
- It is a one-to-one onto function with domain A and range A.

  Into Functions: Let $f : A \to B$. There exists even a single element in B having no pre-image in A, then f is said to be an into function.
- **Constant Function:** Let $f : A \to B$, defined in such a way that all the elements in A have the same image in B, then f is said to be a constant function.
- **Equal Functions:** Two functions f and g are said to be equal, written as f = g if they have the same domain and they satisfy the condition f(x) = g(x), for all x.
- **Inverse Function:** Let f be a one-one onto function from A to B. Let y be an arbitrary element of B. Then f being onto, there exists an element x in A such that f (x) = y.

A function is invertible if and only if f is one-one onto.

- **Different types of relations:**

Let S = {a, b, c, ….} be any set then the relation R is a subset of the product set S×S

i) If R contains all ordered pairs of the form (a, a) in S×S, then R is called reflexive. In a *reflexive* relation 'a' is related to itself.

For example, 'Is equal to' is a reflexive relation for a = a is true.

ii) If $(a, b) \in R \Rightarrow (b, a) \in R$ for every $a, b \in S$ then R is called symmetric

For Example $a = b \Rightarrow b = a$. Hence the relation 'is equal to' is a symmetric relation.

iii) If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \Rightarrow R$ for every $a, b, c, \in S$ then R is called *transistive.*

For Example a = b, $b = c \Rightarrow a = c$. Hence the relation 'is equal to' is a transitive relation.

A relation which is reflexive, symmetric and transitive is called an *equivalence relation* or simply an *equivalence*. 'is equal to' is an equivalence relation.

Similarly, the relation "is parallel to" on the set S of all straight lines in a plane is an equivalence relation.

- **Domain & Range of a relation:** If R is a relation from A to B, then the set of all first co-ordinates of elements of R is called the domain of R, while the set of all second co-ordinates of elements of R is called the range of R.

So, Dom (R) = $\{a : (a, b) \in R\}$ & Range (R) = $\{b : (a, b) \in R\}$

EXERCISE 7 (C)

Choose the most appropriate option/options (a) (b) (c) or (d).

1. "Is smaller than" over the set of eggs in a box is

(a) Transitive (T) (b) Symmetric (S) (c) Reflexive (R) (d) Equivalence (E)

2. "Is equal to" over the set of all rational numbers is

(a) (T) (b) (S) (c) (R) (d) E

[By using using R = Reflexive; T = Transitive, S = Symmetric and E = Equivalence from Q.No. 2 to 8]

3. "has the same father as" …… over the set of children

(a) R (b) S (c) T (d) E

4. "is perpendicular to" over the set of straight lines in a given plane is

(a) R (b) S (c) T (d) E

5. "is the reciprocal of" …….. over the set of non-zero real numbers is

(a) S (b) R (c) T (d) none of these

6. $\{(x, y)/x \in x, y \in y, y = x\}$ is

(a) R (b) S (c) T (d) E

7. {(x,y) / x + y = 2x where x and y are positive integers}, is

(a) R (b) S (c) T (d) E

8. "Is the square of" over n set of real numbers is

(a) R (b) S (c) T (d) none of these

9. If A has 32 elements, B has 42 elements and $A \cup B$ has 62 elements, the number of elements in $A \cap B$ is

(a) 12 (b) 74 (c) 10 (d) none of these

10. In a group of 20 children, 8 drink tea but not coffee and 13 like tea. The number of children drinking coffee but not tea is

(a) 6 (b) 7 (c) 1 (d) none of these

11. The number of subsets of the sets {6, 8, 11} is

(a) 9 (b) 6 (c) 8 (d) none of these

12. The sets V = {x / x+2=0}, R={x / x^2+2x=0} and S = {x : $x^2 + x - 2 = 0$} are equal to one another if x is equal to

(a) –2 (b) 2 (c) ½ (d) none of these

13. If the universal set E = {x | x is a positive integer <25}, A = {2, 6, 8, 14, 22}, B = {4, 8, 10, 14} then

(a) $(A \cap B)' = A' \cup B'$ (b) $(A \cap B)' = A' \cap B'$ (c) $(A' \cap B)' = \phi$ (d) none of these

14. If the set P has 3 elements, Q four and R two then the set P × Q × R contains

(a) 9 elements (b) 20 elements (c) 24 elements (d) none of these

15. Given A = {2, 3}, B = {4, 5}, C = {5, 6} then $A \times (B \cap C)$ is

(a) {(2, 5), (3, 5)} (b) {(5, 2), (5, 3)} (c) {(2, 3), (5, 5)} (d) none of these

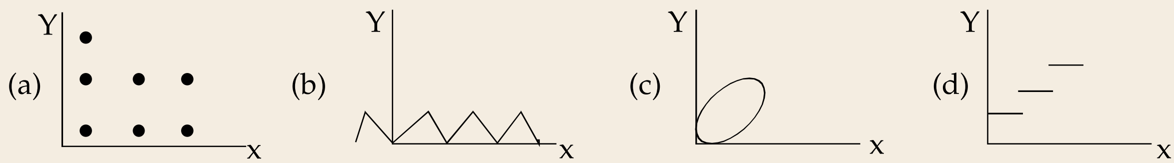
16. A town has a total population of 50,000. Out of it 28,000 read the newspaper X and 23,000 read Y while 4,000 read both the papers. The number of persons not reading X and Y both is

(a) 2,000 (b) 3,000 (c) 2,500 (d) none of these

17. If A = { 1, 2, 3, 5, 7} and B = {1, 3, 6, 10, 15}. Cardinal number of A-B is

(a) 3 (b) 4 (c) 6 (d) none of these

18. Which of the diagram is graph of a function



19. At a certain conference of 100 people there are 29 Indian women and 23 Indian men. Out of these Indian people 4 are doctors and 24 are either men or doctors. There are no foreign doctors. The number of women doctors attending the conference is

    (a) 2 (b) 4 (c) 1 (d) none of these

20. Let A = {a, b}. Set of subsets of A is called power set of A denoted by P(A). Now n(P(A) is

    (a) 2 (b) 4 (c) 3 (d) none of these

21. Out of 2000 employees in an office 48% preferred Coffee (c), 54% liked (T), 64% used to smoke (S). Out of the total 28% used C and T, 32% used T and S and 30% preferred C and S, only 6% did none of these. The number having all the three is

    (a) 360 (b) 300 (c) 380 (d) none of these

22. Referred to the data of Q. 21 the number of employees having T and S but not C is

    (a) 200 (b) 280 (c) 300 (d) none of these

23. Referred to the data of Q. 21 the number of employees preferring only coffee is

    (a) 100 (b) 260 (c) 160 (d) none of these

24. If f(x) = x + 3, g(x) = x^2, then g of (x) is

    (a) $(x + 3)^2$ (b) $x^2 + 3$ (c) $x^2 (x + 3)$, (d) none of these

25. If f(x) = 1/1–x, then $f^{-1}(x)$ is

    (a) 1–x (b) (x–1)/x (c) x/(x–1) (d) none of these

ANSWERS

Exercise 7(A)

1. b	**2.** a	**3.** c	**4.** a	**5.** b	**6.** c	**7.** b	**8.** c
9. a	**10.** b	**11.** b	**12.** a	**13.** c	**14.** b	**15.** b	**16.** a
17. b	**18.** c	**19.** b	**20.** c	**21.** a	**22.** b	**23.** c	**24.** b
25. b	**26.** a	**27.** a	**28.** b	**29.** c	**30.** a	**31.** b	**32.** a

Exercise 7(B)

1. b, d	**2.** c	**3.** a	**4.** b	**5.** a	**6.** c	**7.** b	**8.** b
9. a	**10.** b	**11.** d	**12.** a	**13.** b	**14.** a	**15.** a	**16.** b
17. a	**18.** c	**19.** b	**20.** a				

Exercise 7(C)

1. a	**2.** d	**3.** d	**4.** b	**5.** a	**6.** d	**7.** d	**8.** d
9. a	**10.** b	**11.** c	**12.** a	**13.** a	**14.** c	**15.** a	**16.** b
17. a	**18.** b	**19.** c	**20.** b	**21.** a	**22.** b	**23.** c	**24.** a
25. b							

7.7 CONCEPT OF LIMIT

I) We consider a function f(x) = 2 x. If x is a number approaching to the number 2 then f(x) is a number approaching to the value 2x2= 4

The following table shows f(x) for different values of x approaching 2

x	f(x)
1.90	3.8
1.99	3.98
1.999	3.998
1.9999	3.9998
2	4

Here x approaches 2 from values of x<2 and for x being very close to 2 f(x) is very close 4. This situation is defined as left-hand limit of f(x) as x approaches 2 and is written as lim f(x) = 4, $x \to 2-$

x	f(x)
2.0001	4.0002
2.001	4.002
2.01	1.02
2.0	4

Here x approaches 2 from values of x greater than 2 and for x being very close 2f(x) is very close to 4. This situation is defined as right-hand limit of f(x) as x approaches 2 and is written as lim f(x)= 4 as $x \to {}_{2+}$

So we write

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-+} f(x) = 4$$

Thus $\lim_{x \to a} f(x)$ is said to exist when both left-hand and right-hand limits exists and they are equal. We write as

$$\lim_{x \to a-} f(x) = \lim_{x \to a-+} f(x) = \lim_{x \to a} f(x)$$

Thus $\lim_{x \to a} f(x)$ is said to exist when both left-hand and right-hand limits exists and they are equal. We write as

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a} f(x)$$

Thus, if $\lim_{h \to o} f(a + h) = \lim_{h \to o} f(a - h)$, $\quad (h > o)$

then $\lim_{x \to a} f(x)$ exists

We now consider a function defined by

$$fx = \begin{cases} 2x-2 & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ 2x+2 & \text{for } x > 0 \end{cases}$$

We calculate limit of f(x) as x tend to zero. At x =0 , f(x) =1 (given). If x tends to zero from left-hand side for the value of x<0, f(x) is approaching (2x0) –2 + –2 which is defined as left-hand limit of f(x) as $x \to 0$. We can write this as

$$\lim_{x \to 0-} = -2$$

Similarly if x approaches zero from right-hand side for values of x>0, f(x) is approaching 2 x 0 + 2 = 2. We can write this as $\lim_{x \to 0+} f(x) = -2$.

In this case both left-hand and right-hand exist, but they are not equal. So we may conclude that $\lim_{x \to 0} f(x)$ does not exist.

7.8. USEFUL RULES (THEOREMS) ON LIMITS

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$

where l and m are finite quantities

i) $\lim_{x \to a} f\{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m$

That is limit of the sum of the difference of two functions is equal to the sum of their limits

ii) $\lim_{x \to a} \{f(x) - g(x)\} = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l - m$

That is limit of the difference of two functions is equal to difference of their limits.

iii) $\lim_{x \to a} \{f(x) . g(x)\} = \lim_{x \to a} f(x) . \lim_{x \to a} g(x) = lm$

That is limit of the product of two functions is equal to the product of their limits

iv) $\lim_{x \to a} \{f(x) / g(x)\} = \left\{\lim_{x \to a} f(x)\right\} / \left\{\lim_{x \to a} g(x)\right\} = l / m$ if $m \neq 0$

That is limit of the quotient of two functions is equal to the quotient of their limits.

v) $\lim_{x \to a} c = c$ where c is constant

That is limit of a constant is the constant

vi) $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

vii) $\lim_{x \to a} F\{f(x)\} = F\{\lim_{x \to} F(x)\} = F(l)$

viii) $\lim\limits_{x \to 0+} \frac{1}{x} = \lim\limits_{h \to 0} \frac{1}{h} \to +\infty$ (h>0)

$\lim\limits_{x \to 0+} \frac{1}{x} = \lim\limits_{h \to 0} \frac{1}{-h} \to -\infty$ (h>0)

∞ is a very-every large number called infinity

Thus $\lim\limits_{x \to 0+} \frac{1}{x}$ does not exist.

Example 1: Evaluate (i) $\lim\limits_{x \to 2}(3x+9)$; (ii) $\lim\limits_{x \to 5} \frac{1}{x-1}$ (iii) $\lim\limits_{x \to a} \frac{1}{x-a}$

Solution:

(i) $\lim\limits_{x \to 2} (3x+9) = 3.2 + 9 = (6+9) = 15$

(ii) $\lim\limits_{x \to 5} \frac{1}{x-1} = \frac{1}{5-1} = \frac{1}{4}$

(iii) $\lim\limits_{x \to a} \frac{1}{x-a}$ does not exist, since $\lim\limits_{x \to a+} \frac{1}{x-a} \to +\infty$ and $\lim\limits_{x \to a-} \frac{1}{x-a} \to -\infty$

Example-2 Evaluate $\lim\limits_{x \to 2} \frac{x^2 - 5x + 6}{x-2}$

Solution: At x = 2 the function becomes undefined as 2-2 =0 and division by zero is not mathematically defined.

So $\lim\limits_{x \to 2} \left\{x^2 - 5x + 6 / (x-2)\right\} = \lim\limits_{x \to 2} \{(x-3)/(x-2)\} = \lim\limits_{x \to 2}(x- = 3)$ ($\because x - 2 \neq 0$)

= 2-3 = -1

Example-2 Evaluate $\lim\limits_{x \to 2} \frac{x^2 + 2x - 1}{\sqrt{x^2+2}}$

$$\lim_{x \to 2} \frac{x^2 + 2x - 1}{\sqrt{x^2+2}} = \frac{\lim\limits_{x \to 2}(x^2 + 2x - 1)}{\lim\limits_{x \to 2} \sqrt{x^2+2}} = \frac{\lim\limits_{x \to 2}(x^2 + \lim\limits_{x \to 2} 2x - 1)}{\sqrt{\lim\limits_{x \to 2} x^2 + 2}}$$

$$\frac{(2)^2 + 2\text{x } 2 - 1}{\sqrt{(2)^2 + 2}} = \frac{7}{\sqrt{6}}$$

SOME IMPORTANT LIMITS

We now state some important limits

a) $\lim_{x \to 0} \frac{(e^x -1)}{x} = 1$

b) $\lim_{x \to 0} \frac{a^x -1}{x} = \log_e \quad a(a > 0)$

c) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

d) $\lim_{x \to x} \left(1+\frac{1}{x}\right)^x = e \quad \text{or} \quad \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}}}{x} = e$

e) $\lim_{x\to 0} \frac{x^n - a^n}{x} = na^{n-1}$

f) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(A) The number e called exponential number is given by e = 2.718281828____________=2.7183. This number e is one of the useful constants in mathematics.

(B) In calculus all logarithms are taken with respect to base 'that is $\log x = \log_e x$.

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate : $\lim_{x \to 3} \frac{x^2 - 6x+9}{x-3}$, where $f(x) = \frac{x^2 - 6x+9}{x-3}$. Aso find f (3)

Solution: At x = 3 the function is undefined as division by zero is meaningless. While taking the limit as $x \to 3$, x cannot be exactly equal to 3 i.e. x-3 $\neq 0$ and consequently division by x-3 is permissible.

Now $\lim_{x\to 3} \frac{x^2 - 6x+9}{x-3} = \lim_{x \to 3} \frac{(x-3)^2}{x-3} = \lim_{x-3} (x-3) = 3-3 = 0 \qquad f(3)\frac{0}{0}$ is undefined

The reader may compute the left-hand and the right-hand limits as an exercise.

Example 2: A function is defined as follows:

$$fx = \begin{cases} -3x \text{ when } x < 0 \\ 2x \text{ when } x > 0 \end{cases}$$

Test the existence of $\lim_{x\to 0} f(x)$.

Solution: For x approaching 0 from the left $x < 0$.

Left-hand limit = $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-3x) = 0$

When x approaches 0 from the right $x > 0$

Right-hand limit = $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 2x = 0$

Since L.H. limit = R.H. Limit, the limit exists. Thus, $\lim_{x \to 0} f(x) = 0$.

Example 3: $\lim_{x \to 3} \frac{x^2 + 4x + 3}{x^2 + 6x + 9}$.

Solution: $\frac{x^2 + 4x + 3}{x^2 + 6x + 9} = \frac{x^2 + 3x + x + 3}{(x+3)^2} = \frac{x(x+3) + 1(x+3)}{(x+3)^2} = \frac{(x+3)(x+1)}{(x+3)^2} = \frac{x+1}{x+3}$

$\therefore \lim_{x \to 3} \frac{x^2 + 4x + 3}{x^2 + 6x + 9} = \lim_{x \to 3} \frac{x+1}{x+3} = \frac{4}{3} = \frac{2}{3}$.

Example 4: Find the following limits:

(i) $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$ (ii) $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ if $h > 0$.

Solution:

(i) $\frac{\sqrt{x} - 3}{x - 9} = \frac{\sqrt{x} - 3}{\left(\sqrt{x} + 3\right)\left(\sqrt{x} - 3\right)} = \frac{1}{\sqrt{x} + 3}$. $\therefore \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}$.

(ii) $\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$ $\therefore \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$

$= \frac{1}{\lim_{h \to 0} \sqrt{x+h} + \lim_{h \to 0} \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Example 5: Find $\lim_{x \to 0} \frac{3x + |x|}{7x - 5|x|}$.

Solution: Right-hand limit $= \lim_{x \to 0+} \frac{3x + |x|}{7x - 5|x|} = \lim_{x \to 0+} \frac{3x + x}{7x - 5x} = \lim_{x \to 0+} 2 = 2$

Left-hand limit $= \lim_{x \to 0-} \frac{3x + |x|}{7x - 5|x|} = \lim_{x \to 0} \frac{3x - (x)}{7x - 5(-x)} = \lim_{x \to 0-} \frac{1}{6} = \frac{1}{6}$.

Since Right-hand limit ≠ Left-hand limit the limit does not exist.

Example 6: Evaluate $=\lim\limits_{x\to 0}\frac{e^{x}-e^{-x}}{x}$

Solution: $=\lim\limits_{x\to 0}\frac{e^{x}-e^{-x}}{x}==\lim\limits_{x\to 0}\frac{(e^{x}-1)-(e^{-x}-1)}{x}=\lim\limits_{x\to 0}\frac{e^{x}-1}{x}-=\lim\limits_{x\to 0}\frac{e^{x}-1}{x}=1-1=0$

Example 7: Find $\lim\limits_{x\to á}\left(1+\frac{9}{x}\right)^{x}$. (Form 1)

Solution: It may be noted that $\frac{x}{9}$ approaches α as x approaches ∞. i.e. $\lim\limits_{x\to\infty}\frac{x}{9}\to\infty$

$$\lim_{x\to\infty}\left(1+\frac{9}{x}\right)^{x}=\lim_{x/9\to\alpha}\left\{\left(1+\frac{1}{\frac{x}{9}}\right)^{x/9}\right\}^{9}$$

Substitution x / 9 = z the above expression takes the form $\lim\limits_{z\to\alpha}\left\{\left(1+\frac{1}{z}\right)^{z}\right\}^{9}$

$$=\left\{\lim_{z\to\infty}\left(1+\frac{1}{z}\right)^{z}\right\}^{9}=e^{9}.$$

Example 8: Evaluate: $\lim\limits_{z\to\infty}\frac{2x+1}{x^{3}+1}$. $\left[\text{Form}\frac{\infty}{\infty}\right]$

Solution: As x approaches ∞, $2x+1$ and $x^{3}+1$ both approach ∞ and therefore the given function takes the form $\frac{\infty}{\infty}$ which is determinate. Therefore, instead of evaluating directly let us try for suitable algebraic transformation so that the indeterminate for is avoided.

$$\lim_{x\to\infty}\frac{x^{\frac{2}{2}+\frac{1}{3}}}{1+x^{\frac{1}{3}}}=\frac{\lim\limits_{x\to\infty}\left(x^{\frac{2}{2}+\frac{1}{3}}\right)}{\lim\limits_{x\to\infty}\left(1+x^{\frac{1}{3}}\right)}=\frac{\lim\limits_{x\to\infty}x^{\frac{2}{2}+}\lim\limits_{x\to\infty}x^{\frac{1}{3}}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}x^{\frac{1}{3}}}=\frac{0+0}{1+0}=\frac{0}{1}=0$$

Example 9: Find $\lim\limits_{x\to\infty}\frac{1^{2}+2^{2}+3^{2}+.............+x^{2}}{x^{3}}$

$$\lim_{x\to\infty}\frac{[x(x+)(2x+1)]}{6x^{3}}=\frac{1}{6}\lim_{x\to\infty}\left\{\left(1+\frac{1}{x}\right)\left(2+\frac{1}{x}\right)\right\}$$

$$=\frac{1}{6}\text{ x }1\text{ x }2=\frac{1}{3}.$$

7.9 CONTINUITY:

By the term continuous we mean some thing which goes on without interruption and without abrupt changes. Here in mathematics the term continuous carries the same meaning. Thus, we define continuity of a function in the following way.

A function f(x) is said to be continuous at x= a if and only if

(i) f(x) is defined x= a

(ii) $\lim_{x\mapsto a-} f(x) = \lim_{x\mapsto a+} f(x)$

(iii) $\lim_{x\mapsto a} f(x) = f(a)$

In the second condition both left-hand and right-hand limits are equal. In the third condition limiting value of the function must be equal to its function value at x= a

Useful Information:

(i) The sum, difference and product of two continuous functions is a continuous function. This property holds good for any finite number of functions.

(ii) The quotient of two continuous functions is continuous function provided the denominator is not equal to zero.

Example:1 f(x) = [

$= \frac{3}{2} - x$ when $\frac{1}{2}$<x<1

$= \frac{1}{2}$ when x=$\frac{1}{2}$

Discuss the continuity of f(x) at x= $\frac{1}{2}$

$$\lim_{x\mapsto \frac{1}{2}-} f(x) = \lim_{x\to \frac{1}{2}+}\left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\lim_{x\mapsto \frac{1}{2}+} f(x) = \lim_{x\to \frac{1}{2}+}\left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = 1$$

Solution: Since LHL $\neq$ RHL

$\lim_{x\mapsto \frac{1}{2}} f(x)$ does not exist

Moreover f(1/2)

Hence f(x) is not continuous at x=1/2 , ie f(x) is discontinuous at x= 1/2

Example 2: Find points of discontinuity of the function f(x) = $\frac{x^2+2x+5}{x^2-3x+2}$

Solution: f(x) = $\frac{x^2+2x+5}{x^2-3x+2}$

For x=1 and x=2 the denominator becomes zero and the function f(x) is undefined at x=1 and x= 2. Hence the points of discontinuity are at x=1 and x= 2

Example 3: A function g(x) is defined as follows

g(x) = x when 0<x<1

=2-x when x≥1

is g(x) is continuous at x= 1

Solution:

$\lim_{x\mapsto 1-} g(x) = \lim_{x\mapsto 1+} g(x)$

$\lim_{x\mapsto 1+} g(x) = \lim_{x\mapsto 1+} (2-x) = 2-1 = 1$

$\therefore \lim_{x\mapsto 1-} g(x) = \lim_{x\mapsto 1-} g(x) = 1$

Also g(1) =2-1=1

and $\lim_{x\mapsto 1} g(x) = g(1) = 1$

Hence f(x) is coninuous at x=1

Example 4: The function f(x) = $\frac{x^2-9}{x-3}$ is undefined at x=3 . What value must be assigned to f (3), if f(x) is to be continuous at x=3

Solution: When x approaches 3, x≠3 ie x-3≠0

$\lim_{x\mapsto 3} f(x) = \lim_{x\mapsto 3} \frac{(x-3)(x+3)}{(x-3)}$

$= \lim_{x\mapsto 3} (x+3) = 3+3 = 6$

Therefore if f(x) is to be continuous at x=3 =f(3) = $\lim_{x\mapsto 3} f(x) = 6$

Example 5: Is the function f(x) = $|x|$ continuous at x= 0

Solution: We know that $|x|$ =x when x>0

= 0 when x= 0

= -x when x<0

$\lim_{x\to 0-} f(x) = \lim_{x\mapsto 0-} (-x) = 0$ and $\lim_{x\to 0+} f(x) = \lim_{x\mapsto 0+} x = 0$

Hence $\lim_{x\mapsto 0} f(x) = 0 = f(0)$

So f(x) is continuous at x = 0

ADDITIONAL QUESTION BANK

1. Following set notations represent: $A \subset B$; $x \notin A$; $A \supset B$; $\{0\}$; $A \not\subset B$

   (a) A is a proper subset of B; x is not an element of A; A contains B; singleton with an only element zero; A is not contained in B.

   (b) A is a proper subset of B; x is an element of A; A contains B; singleton with an only element zero; A is contained in B.

   (c) A is a proper subset of B; x is not an element of A; A does not contains B; contains elements other than zero; A is not contained in B.

   (d) None

2. Represent the following sets in set notation: – Set of all alphabets in English language, set of all odd integers less than 25, set of all odd integers, set of positive integers x satisfying the equation $x^2+5x+7=0$:

   (a) A = {x:x is an alphabet in English}, I = {x:x is an odd integer>25}, I = {2, 4, 6, 8 ….}
   I = $\{x: x^2+5x+7=0\}$

   (b) A={x:x is an alphabet in English}, I = {x:x is an odd integer<25}, I = {1, 3, 5, 7 ….}
   I = $\{x: x^2+5x+7=0\}$

   (c) A = {x:x is an alphabet in English}, I = {x:x is an odd integer £ 25}, I = {1, 3, 5, 7 ….}
   I = $\{x: x^2+5x+7=0\}$

   (d) None

3. Rewrite the following sets in a set builder form: - A={a, e, i, o, u} B={1, 2, 3, 4 ….} C is a set of integers between –15 and 15.

   (a) A={x:x is a consonant}, B={x:x is an irrational number}, C={x:–15<x<15$\wedge$ x is a fraction}

   (b) A={x:x is a vowel}, B={x:x is a natural number}, C={x: –15³x³15$\wedge$ x is a whole number}

   (c) A={x:x is a vowel}, B={x:x is a natural number}, C={x: –15 <x<15$\wedge$ x is a whole number}

   (d) None

4. If V={0, 1, 2, …9}, X={0, 2, 4, 6, 8}, Y={3, 5, 7} and Z={3 7} then
   $Y \cup Z, (V \cup Y) \cap X, (X \cup Z) \cup V$ are respectively: –

   (a) {3, 5, 7}, {0, 2, 4, 6, 8}, {0, 1, 2, …9} (b) {2, 4, 6}, {0, 2, 4, 6, 8}, {0, 1, 2, …9}

   (c) {2, 4, 6}, {0, 1, 2, …9}, {0, 2, 4, 6, 8} (d) None

5. In question No. (4) $(X \cup Y) \cap Z$ and $(\phi \cup V) \cap \phi$ are respectively: –

   (a) {0, 2, 4, 6, 8}, ϕ (b) {3, 7}, ϕ (c) {3, 5, 7}, ϕ (d) None

6. If V={x:x + 2 = 0} R={x:x²+x–2=0, x <0} and S={x:x²+4x+4=0} then V, R, S are equal for the value of x equal to ______.

(a) 0 (b) –1 (c) –2 (d) None

7. What is the relationship between the following sets? A={x:x is a letter in the word *flower*} B={x:x is a letter in the word *flow*} C={x:x is a letter in the word *wolf*} D={x:x is a letter in the word *follow*}

(a) B=C=D and all these are subsets of the set A

(b) B=C≠D (c) B≠C≠D (d) None

8. Comment on the correctness or otherwise of the following statements: – (i) {a, b, c}={c, b, a} (ii) {a, c, a, d, c, d} ⊂ {a, c, d} (iii) {b} ∈ {{b}} (iv) {b} ⊂ {{b}} and ϕ ⊂ {{b}}.

(a) Only (iv) is incorrect (b) (i) (ii) are incorrect

(c) (ii) (iii) are incorrect (d) All are incorrect

9. If A={a, b, c}, B={a, b}, C = {a, b, d}, D={c, d} and E={d} state which of the following statements are correct: – (i) B ⊂ A (ii) D ≠ C (iii) C ⊃ E (iv) D ⊂ E (v) D ⊂ B (vi) D = A (vii) B ⊄ C (viii) E ⊂ A (ix) E ⊄ B (x) a ∈ A (xi) a ⊂ A (xii) {a} ∈ A (xiii) {a} ⊂ A

(a) (i) (ii) (iii) (ix) (x) (xiii) only are correct (b) (ii) (iii) (iv) (x) (xii) (xiii) only are correct

(c) (i) (ii) (iv) (ix) (xi) (xiii) only are correct(d) None

10. Let A = {0}, B = {0 1}, C = ϕ, D = {ϕ}, E = {x | x is a human being 300 years old}, F = {x | x ∈ A and x ∈ B} state which of the following statements are true: – (i) A ⊂ B (ii) B = F (iii) C ⊂ D (iv) C = E (v) A = F (vi) F = 1 and (vii) E = C = D

(a) (i) (iii) (iv) and (v) only are true (b) (i) (ii) (iii) and (iv) are true

(c) (i) (ii) (iii) and (vi) only are true (d) None

11. If A = {0, 1} state which of the following statements are true: – (i) {1} ⊂ A (ii) {1} ∈ A (iii) ϕ ∈ A (iv) 0 ∈ A (v) 1 ⊂ A (vi) {0} ∈ A (vii) ϕ ⊂ A

(a) (i) (iv) and (vii) only are true (b) (i) (iv) and (vi) only are true

(c) (ii) (iii) and (vi) only are true (d) None

12. State whether the following sets are finite, infinite or empty: – (i) X = {1, 2, 3, …..500} (ii) Y = {y: $y = a^2$; a is an integer} (iii) A = {x:x is a positive integer multiple of 2} (iv) B = {x:x is an integer which is a perfect root of 26<x<35}

(a) finite infinite infinite empty (b) infinite infinite finite empty

(c) infinite finite infinite empty (d) None

13. If A = {1, 2, 3, 4} B = {2, 3, 7, 9} and C = {1, 4, 7, 9} then

(a) A ∩ B ≠ ϕ B ∩ C ≠ ϕ A ∩ C ≠ ϕ but A ∩ B ∩ C = ϕ

(b) A ∩ B = ϕ B ∩ C = ϕ A ∩ C = ϕ A ∩ B ∩ C = ϕ

(c) A ∩ B ≠ ϕ B ∩ C ≠ ϕ A ∩ C ≠ ϕ A ∩ B ∩ C ≠ ϕ

(d) None

14. If the universal set is X = {x:x ∈ N, 1≤ x ≤ 12} and A = {1, 9, 10}, B = {3, 4, 6, 11, 12} and C = {2, 5, 6} are subsets of X then set A ∪ (B ∩ C) is _______.

    (a) {3, 4, 6, 12} (b) {1, 6, 9, 10} (c) {2, 5, 6, 11} (d) None

15. As per question No.(14) the set (A ∪ B) ∩ (A ∪ C) is _________.

    (a) {3, 4, 6, 12} (b) {1, 6, 9, 10} (c) {2, 5, 6, 11} (d) None

16. A sample of income group of 1172 families was surveyed and noticed that for income groups < ₹6000/-, 6000/- to ₹10999/-, ₹11000/-, to ₹15999/-, ₹16000 and above No. TV set is available to 70, 50, 20, 50 families one set is available to 152, 308, 114, 46 families and two or more sets are available to 10, 174, 84, 94 families.

    If A = {x|x is a family owning two or more sets}, B = {x|x is a family with one set, }C = {x|x is a family with income less than ₹ 6000/-}, D = {x|x| is a family with income ₹ 6000/- to ₹ 10999/-}, E = {x|x is a family with income ₹ 11000/- to ₹ 15999/-}, find the number of families in each of the following sets (i) C ∩ B

    (ii) A ∪ E

    (a) 152, 580 (b) 152, 20 (c) 152, 50 (d) 152, 496

17. As per question No.(16) find the number of families in each of the following sets: –

    (i) (A ∪ B)′ ∩ E (ii) (C ∪ D ∪ E) ∩ (A ∪ B)′

    (a) 20, 50 (b) 152, 20 (c) 152, 50 (d) 20, 140

18. As per question No.(16) express the following sets in set notation: –

    (i) {x|x is a family with one set and income of less than ₹ 11000/-}

    (ii) {x|x is a family with no set and income over ₹ 16000/-}

    (a) (C ∪ D) ∩ B (b) (A ∪ B)′ ∩ (C ∪ D ∪ E)′

    (c) Both (d) None

19. As per question No.(16) express the following sets in set notation: –

    (i) {x|x is a family with two or more sets or income of ₹ 11000/- to ₹ 15999/-}

    (ii) {x|x is a family with no set}

    (a) (A ∪ E) (b) (A ∪ B)′ (c) Both (d) None

20. If A = {a, b, c, d} list the element of power set P (a)

    (a) {ϕ {a} {b}({c} {d} {a, b} {a, c} {a, d} {b, c} {b, d} {c, d}

    (b) {a, b, c} {a, b, d} {a, c, d} {b, c, d}

    (c) {a, b, c, d}

    (d) All the above elements are in P (a)

21. If four members a, b, c, d of a decision making body are in a meeting to pass a resolution where rule of majority prevails list the wining coalitions. Given that a, b, c, d own 50%, 20%, 15%, 15% shares each.

    (a) {a, b} {a, c} {a, d} {a, b, c} {a, b, d} {a, b, c, d} (b) {b, c, d}

    (c) {b, c} {b, d} {c, d} {a, c, d} {b, c, d} {a} {b} {c} {d} ϕ (d) None

22. As per question No.(21) with same order of options (a) (b) (c) and (d) list the blocking conditions.

23. As per question No.(21) with same order of options (a) (b) (c) and (d) list the losing conditions.

24. If A ={a, b, c, d, e, f} B = {a, e, i, o, u} and C = {m, n, o, p, q, r, s, t, u} then $A \cup B$ is

    (a) {a, b, c, d, e, f, i, o, u} (b) {a, b, c, i, o, u}

    (c) {d, e, f, i, o, u} (d) None

25. As per question No.(24) $A \cup C$ is

    (a) {a, b, c, d, e, f, m, n, o, p, q, r, s, t, u} (b) {a, b, c, s, t, u}

    (c) {d, e, f, p, q, r} (d) None

26. As per question No.(24) $B \cup C$ is

    (a) {a, e, i, o, u, m, n, p, q, r, s, t} (b) {a, e, i, r, s, t}

    (c) {i, o, u, p, q, r} (d) None

27. As per question No.(24) A – B is

    (a) {b, c, d, f} (b) {a, e, i, o} (c) {m, n, p, q} (d) None

28. As per question No.(24) $A \cap B$ is

    (a) {a, e} (b) {i, o} (c) {o, u} (d) None

29. As per question No.(24) $B \cap C$ is

    (a) {a, e} (b) {i, o} (c) {o, u} (d) None

30. As per question No.(24) $A \cup (B – C)$ is

    (a) {a, b, c, d, e, f, i} (b) {a, b, c, d, e, f, o} (c) {a, b, c, d, e, f, u} (d) None

31. As per question No.(24) $A \cup B \cup C$ is

    (a) {a, b, c, d, e, f, i, o, u, m, n, p, q, r, s, t} (b) {a, b, c, r, s, t}

    (c) { d, e, f, n, p, q} (d) None

32. As per question No.(24) $A \cap B \cap C$ is

    (a) ϕ (b) {a, e} (c) { m, n} (d) None

33. If A = {3, 4, 5, 6} B = {3, 7, 9, 5} and C = {6, 8, 10, 12, 7} then A′ is (given that the universal set U = {3, 4, ….., 11, 12, 13}

    (a) {7, 8, ….12, 13} (b) {4, 6, 8, 10, ….13}

    (c) { 3, 4, 5, 9, 11, 13} (d) None

34. As per question No.(33) with the same order of options (a), (b), (c) and (d) the set B′ is

35. As per question No.(33) with the same order of options (a), (b), (c) and (d) the set C′ is

36. As per question No.(33) the set (A′)′ is

(a) {3, 4, 5, 6} (b) {3, 7, 9, 5} (c) {8, 10, 11, 12, 13} (d) None

37. As per question No.(33) the set (B′)′ is

(a) {3, 4, 5, 6} (b) {3, 7, 9, 5} (c) {8, 10, 11, 12, 13} (d) None

38. As per question No.(33) the set $(A \cup B)'$ is

(a) {3, 4, 5, 6} (b) {3, 7, 9, 5} (c) {8, 10, 11, 12, 13} (d) None

39. As per question No.(33) the set $(A \cap B)'$ is

(a) {8, 10, 11, 12, 13} (b) {4, 6, 7, ….13} (c) {3, 4, 5, 7, 8,….13} (d) None

40. As per question No.(33) the set $A' \cup C'$ is

(a) {8, 10, 11, 12, 13} (b) {4, 6, 7, ….13} (c) {3, 4, 5, 7, 8,….13} (d) None

41. If A = {1, 2, …9}, B = {2, 4, 6, 8} C = {1, 3, 5, 7, 9}, D = {3, 4, 5} and E = {3, 5} what is set S if it is also given that $S \subset D$ and $S \not\subset B$

(a) {3, 5} (b) {2, 4} (c) {7, 9} (d) None

42 As per question No.(41) what is set S if it is also given that $S \subset B$ and $S \not\subset C$

(a) {3, 5} (b) {2, 4} (c) {7, 9} (d) None

43. If U = {1, 2, …9} be the universal set A = {1, 2, 3, 4} and B = {2, 4, 6, 8} then the set $A \cup B$ is

(a) {1, 2, 3, 4, 6, 8} (b) {2, 4} (c) {5, 6, 7, 8, 9} (d) {5, 7, 9}

44. As per question No.(43) with the same order of options (a) (b) (c) and (d) the set $A \cap B$ is

45. As per question No.(43) with the same order of options (a) (b) (c) and (d) the set A′ is

46. As per question No.(43) with the same order of options (a) (b) (c) and (d) the set $(A \cup B)'$ is

47. As per question No.(43) the set $(A \cap B)'$ is

(a) {1, 2, 3, 4, 6, 8} (b) {2, 4} (c) {5, 6, 7, 8, 9} (d) {1, 3, 5, 6, 7, 8, 9}

48. Let P = (1, 2, x), Q = (a x y), R = (x, y, z) then $P \times Q$ is

(a) {(1, a) (1, x) (1, y); (2, a) (2, x) (2, y); (x, a) (x, x) (x, y)}

(b) {(1, x); (1, y); (1, z); (2, x); (2, y); (2, z); (x, x) (x, y) (x, z)}

(c) {(a, x) (a, y) (a, z); (x, x) (x, y) (x, z); (y, x) (y, y) (y, z)}

(d) {(1, x) (1, y) (2, x) (2, y) (x, x) (x, y)}

49. As per question No.(48) with the same order of options (a) (b) (c) and (d) then the set $P \times R$ is

50. As per question No.(48) with the same order of options (a) (b) (c) and (d) then the set $Q \times R$ is

51. As per question No.(48) with the same order of options (a) (b) (c) and (d) then the set $(P \times Q) \cap (P \times R)$ is

52. As per question No.(48) the set $(R \times Q) \cap (R \times P)$ is

(a) {(a, x), (a, y), (a, z), (x, x), (x, y), (x, z), (y, x), (y, y), (y, z)}

(b) {(1, x), (1, y), (2, x), (2, y), (x, x), (x, y)}

(c) {(x, x), (y, x), (z, x)}

(d) {(1, a), (1, x), (1, y), (2, a), (2, x), (2, y), (x, a), (x, x), (x, y), (x, 1), (x, 2), (y, 1), (y, 2), (y, x), (z, 1), (z, 2), (z, x)}

53. As per question No. (48) with the same order of options (a) (b) (c) and (d) as in question No.(52) the set $(P \times Q) \cup (R \times P)$ is

54. If P has three elements Q four and R two how many elements does the Cartesian product set $P \times Q \times R$ will have

(a) 24 (b) 9 (c) 29 (d) None

55. Identify the elements of P if set Q = {1, 2, 3} and P × Q = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (6, 1), (6, 2), (6, 3)}

(a) {3, 4, 5} (b) {4, 5, 6} (c) {5, 6, 7} (d) None

56. If A = {2, 3}, B = {4, 5}, C = {5, 6} then $A \times (B \cup C)$ is

(a) {(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}

(b) {(2, 5), (3, 5)}

(c) {(2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (4, 6), (5, 5), (5, 6)}

(d) None

57. As per question No.(56) with the same order of options (a) (b) (c) and (d) the set $A \times (B \cap C)$ is

58. As per question No.(56) with the same order of options (a) (b) (c) and (d) the set $(A \times B) \cup (B \times C)$ is

59. If A has 32 elements B has 42 elements and $A \cup B$ has 62 elements find the number of elements in $A \cap B$

(a) 74 (b) 62 (c) 12 (d) None

60. Out of a total population of 50,000 only 28,000 read Telegraph and 23,000 read Times of India while 4,000 read the both. How many do not read any paper?

(a) 3,000 (b) 2,000 (c) 4,000 (d) None

61. Out 2000 staff 48% preferred coffee 54% tea and 64% cocoa. Of the total 28% used coffee and tea 32% tea and cocoa and 30% coffee and cocoa. Only 6% did none of these. Find the number having all the three.

(a) 360 (b) 280 (c) 160 (d) None

62. As per question No.(61) with the same order of options (a), (b), (c) and (d) find the number having tea and cocoa but not coffee.

63. As per question No.(61) with the same order of options (a), (b), (c) and (d) find the number having only coffee.

64. Complaints about works canteen had been about Mess (M) Food (F) and Service (S). Total complaints 173 were received as follows: –

    $n(M) = 110, n(F) = 55, n(S) = 67, n(M \cap F \cap S') = 20, n(M \cap S \cap F') = 11$

    and $n(F \cap S \cap M') = 16$. Determine the complaints about all the three.

    (a) 6 (b) 53 (c) 35 (d) None

65. As per question No. (64) with the same order of options (a), (b), (c) and (d) determine the complaints about two or more than two.

66. Out of total 150 students 45 passed in Accounts 50 in Maths. 30 in Costing 30 in both Accounts and Maths. 32 in both Maths and Costing 35 in both Accounts and Costing. 25 students passed in all the three subjects. Find the number who passed at least in any one of the subjects.

    (a) 63 (b) 53 (c) 73 (d) None

67. After qualifying out of 400 professionals, 112 joined industry, 120 started practice and 160 joined as paid assistants. There were 32, who were in both practice and service 40 in both practice and assistantship and 20 in both industry and assistantship. There were 12 who did all the three. Find how many could not get any of these.

    (a) 88 (b) 244 (c) 122 (d) None

68. As per question No. (67) with the same order of options (a) (b) (c) and (d) find how many of them did only one of these.

69. A marketing research team interviews 50 people about their drinking habits of tea coffee or milk or A B C respectively. Following data is obtained but the Manager is not sure whether these are consistent.

Category	No.	Category	No.
ABC	3	A	42
AB	7	B	17
BC	13	C	27
AC	18		

    (a) Inconsistent since $42 + 17 + 27 - 7 - 13 - 18 + 3 \neq 50$
    (b) Consistent
    (c) Cannot determine due to data insufficiency
    (d) None

70. On a survey of 100 boys it was found that 50 used white shirt 40 red and 30 blue. 20 were habituated in using both white and red shirts 15 both red and blue shirts and 10 blue and white shirts. Find the number of boys using all the colours.

(a) 20 (b) 25 (c) 30 (d) None

71. As per question No.(70) if 10 boys did not use any of the white red or blue colours and 20 boys used all the colours offer your comments.

(a) Inconsistent since no. of boys used all the three colours can't more then no. of boys used two colours

(b) Consistent

(c) Cannot determine due to data insufficiency

(d) None

72. Out of 60 students 25 failed in paper (1), 24 in paper (2), 32 in paper (3), 9 in paper (1) alone, 6 in paper (2) alone, 5 in papers (2) and (3) only and 3 in papers (1) and (2) only. Find how many failed in all the three papers.

(a) 10 (b) 60 (c) 50 (d) None

73. As per question No. (72) how many passed in all the three papers?

(a) 10 (b) 60 (c) 50 (d) None

74. Asked if you will cast your vote for a party the following feedback is obtained: –

	Yes	No	Don't know
Adult Male	10	20	5
Adult Female	20	15	5
Youth over 18 years	10	5	10

If A = set of Adult C = Common set of Women and Youth Y = set of positive opinion N = set of negative opinion then n(A′) is

(a) 25 (b) 40 (c) 20 (d) None

75. As per question No. (74) with the same order of options (a), (b), (c) and (d) the set $n(A \cap C)$ is

76. As per question No. (74) with the same order of options (a), (b), (c) and (d) the set $n(Y \cup N)'$ is

77. As per question No. (74) with the same order of options (a), (b), (c) and (d) the set $n[A \cap (Y \cap N)']$ is

78. In a market survey you have obtained the following data which you like to examine regarding its correctness:

Did not use the brand	April	May	June	April & May	May & June	April & June	April May June
Percentage answering 'Yes'	59	62	62	35	33	31	22

(a) Inconsistent since 59 + 62 + 62 – 35 – 33 – 31 + 22 ≠ 100

(b) Consistent

(c) cannot determine due to data insufficiency

(d) None

79. In his report an Inspector of an assembly line showed in respect of 100 units the following which you are require to examine.

Defect	Strength (S)	Flexibility (F)	Radius (R)	S and F	S and R	F and R	S F R
No. of pieces	35	40	18	7	11	12	3

(a) No. of pieces with radius defect alone was –2 which was impossible

(b) Report may be accepted

(c) Cannot be determined due to data insufficiency (d) None

80. A survey of 1000 customers revealed the following in respect of their buying habits of different grades:

A grade only	A and C grades	C grade	A grade but not B grade	A grade	C and B grades	None
180	80	480	230	360	80	140

How many buy B grade?

(a) 280 (b) 400 (c) 50 (d) None

81. As per question No. (80) with the same order of options (a) (b) (c) and (d) how many buy C grade if and only if they do not buy B grade?

82. As per question No. (80) with the same order of options (a) (b) (c) and (d) how many buy C and B grades but not the A grade?

83. Consider the following data: –

	Skilled & Direct Worker	Unskilled & Direct Worker	Skilled & Indirect Worker	Unskilled & Indirect Worker
Short Term	6	8	10	20
Medium Term	7	10	16	9
Long Term	3	2	8	0

If S M L T I denote short medium long terms skilled and indirect workers respectively find the number of workers in set M.

(a) 42 (b) 8 (c) 10 (d) 43

84. Consider the problem No. (83) and find the number of workers in set $L \cap I$.

(a) 42 (b) 8 (c) 10 (d) 43

85. Consider the problem No. (83) and find the number of workers in set $S \cap T \cap I$.

(a) 42 (b) 8 (c) 10 (d) 43

86. Consider the problem No. (83) and find the number of workers in set $(M \cup L) \cap (T \cup I)$.

(a) 42 (b) 8 (c) 10 (d) 43

87. Consider the problem No. (83) and find the number of workers in set $S' \cup (S' \cap I)'$.

(a) 42 (b) 44 (c) 43 (d) 99

88. Consider the problem No. (83). Find out which set of the pair has more workers as its members. Pair is $(S \cup M)'$ or L: –

(a) $(S \cup M)' > L$ (b) $(S \cup M)' < L$ (c) $(S \cup M)' = L$ (d) None

89. Consider the problem No. (88). Find out which set of the pair has more workers as its members. Pair is $(I \cap T)'$ or $S - (I \cap S')$: –

(a) $(I \cap T)' > [S - (I \cap S')]$ (b) $(I \cap T)' < [S - (I \cap S')]$

(c) $(I \cap T)' = [S - (I \cap S')]$ (d) None

90. Out of 1000 students 658 failed in the aggregate, 166 in the aggregate and in group-I 434 in aggregate and in group-II, 372 in group-I, 590 in group-II and 126 in both the groups. Find out how many failed in all the three.

(a) 106 (b) 224 (c) 206 (d) 464

91. As per question No.(90) how many failed in the aggregate but not in group-II?

(a) 106 (b) 224 (c) 206 (d) 464

92. As per question No.(90) how many failed in group-I but not in the aggregate?

(a) 106 (b) 224 (c) 206 (d) 464

93. As per question No.(90) how many failed in group-II but not in group-I?

(a) 106 (b) 224 (c) 206 (d) 464

94. As per question No.(90) how many failed in aggregate or group-II but not in group-I?

(a) 206 (b) 464 (c) 628 (d) 164

95. As per question No.(90) how many failed in aggregate but not in group-I and group-II?

(a) 206 (b) 464 (c) 628 (d) 164

ANSWERS

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(a)	**5.**	(b)	**6.**	(c)
7.	(a)	**8.**	(a)	**9.**	(a)	**10.**	(a)	**11.**	(a)	**12.**	(a)
13.	(a)	**14.**	(b)	**15.**	(b)	**16.**	(d)	**17.**	(d)	**18.**	(c)
19.	(c)	**20.**	(d)	**21.**	(a)	**22.**	(b)	**23.**	(c)	**24.**	(a)
25.	(a)	**26.**	(a)	**27.**	(a)	**28.**	(a)	**29.**	(c)	**30.**	(a)
31.	(a)	**32.**	(a)	**33.**	(a)	**34.**	(b)	**35.**	(c)	**36.**	(a)
37.	(b)	**38.**	(c)	**39.**	(b)	**40.**	(c)	**41.**	(a)	**42.**	(b)
43.	(a)	**44.**	(b)	**45.**	(c)	**46.**	(d)	**47.**	(d)	**48.**	(a)
49.	(b)	**50.**	(c)	**51.**	(d)	**52.**	(c)	**53.**	(d)	**54.**	(a)
55.	(b)	**56.**	(a)	**57.**	(b)	**58.**	(c)	**59.**	(c)	**60.**	(a)
61.	(a)	**62.**	(b)	**63.**	(c)	**64.**	(a)	**65.**	(b)	**66.**	(b)
67.	(a)	**68.**	(b)	**69.**	(a)	**70.**	(b)	**71.**	(a)	**72.**	(a)
73.	(a)	**74.**	(a)	**75.**	(b)	**76.**	(c)	**77.**	(c)	**78.**	(a)
79.	(a)	**80.**	(a)	**81.**	(b)	**82.**	(c)	**83.**	(a)	**84.**	(b)
85.	(c)	**86.**	(d)	**87.**	(d)	**88.**	(c)	**89.**	(a)	**90.**	(a)
91.	(b)	**92.**	(c)	**93.**	(d)	**94.**	(c)	**95.**	(d)		

NOTES

CHAPTER N

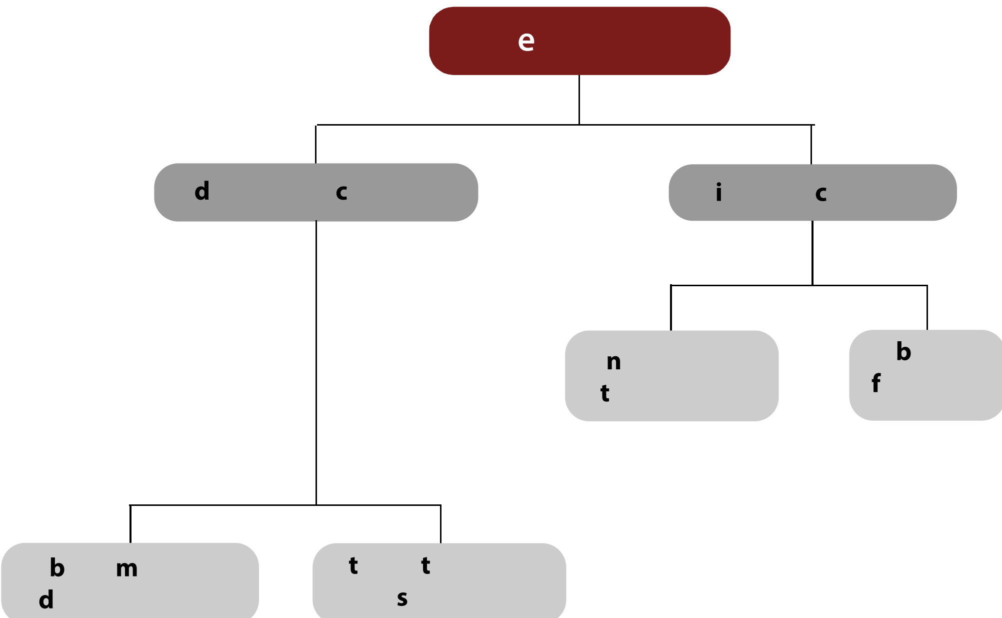
BASIC CONCEPTS OF DIFFERENTIAL AND INTEGRAL CALCULUS

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- Understand the basics of differentiation and integration.
- Know how to compute derivative of a function by the first principle, derivative of a function by the application of formulae and higher order differentiation.
- Appreciate various techniques of integration.
- Understand the concept of definite of integrals of functions and its application.

CHAPTER OVERVIEW



INTRODUCTION TO DIFFERENTIAL AND INTEGRAL CALCULUS (EXCLUDING TRIGONOMETRIC FUNCTIONS)

(A) DIFFERENTIAL CALCULUS

8.A.1 INTRODUCTION

Differentiation is one of the most important fundamental operations in calculus. Its theory primarily depends on the idea of limit and continuity of function.

To express the rate of change in any function we introduce concept of derivative which involves a very small change in the dependent variable with reference to a very small change in independent variable.

Thus differentiation is the process of finding the derivative of a continuous function. It is defined as the limiting value of the ratio of the change (increment) in the function corresponding to a small change (increment) in the independent variable (argument) as the later tends to zero.

8.A.2 DERIVATIVE OR DIFFERENTIAL COEFFICIENT

Let y = f(x) be a function. If h (or Δx) be the small increment in x and the corresponding increment in y or f(x) be Δy = f(x+h) – f(x) then the derivative of f(x) is defined as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ i.e.

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

This is denoted as $f'(x)$ or dy/dx or $\frac{d}{dx}f(x)$. The derivative of f(x) is also known as differential coefficient of f(x) with respect to x. This process of differentiation is called the first principle (or definition or abinitio) (Ab-initio).

Note: In the light of above discussion a function f (x) is said to differentiable at x = c if $\lim_{h \to c} \frac{f(x)-f(c)}{x-c}$ exist which is called the differential coefficient of f(x) at x = c and is denoted by $f'(c)$ or $\left[\frac{dy}{dx}\right]_{x=c}$.

We will now study this with an example.

Consider the function $f(x) = x^2$.

By definition

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0}\frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0}\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0}(2x + \Delta x) = 2x + 0 = 2x$$

Thus, derivative of f(x) exists for all values of x and equals 2x at any point x.

Examples of differentiations from the 1st principle

i) f(x) = c, c being a constant.

Since c is constant we may write f(x+h) = c.

So f(x+h) – f(x) = 0

Hence $f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h \to 0}\frac{0}{h} = 0$

So $\frac{d(c)}{dx} = 0$

ii) Let $f(x) = x^n$; then $f(x+h) = (x+h)^n$

let x+h =t or h= t – x and as h→0, t→x

Now $f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{t \to x}(t^n - x^n)/(t - x) = nx^{n-1}$$

Hence $\frac{d}{dx}(x^n) = nx^{n-1}$

iii) $f(x) = e^x \therefore f(x+h) = e^{x+h}$

So $f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0}\frac{e^{x+h} - e^x}{h} = \lim_{h \to 0}\frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0}\frac{e^h - 1}{h} = e^x.1$$

Hence $\frac{d}{dx}(e^x) = e^x$

iv) Let $f(x) = a^x$ then $f(x+h) = a^{x+h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \left[\frac{a^x(a^h - 1)}{h}\right]$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= a^x \log_e a$$

Thus $\frac{d}{dx}(a^x) = a^x \log_e a$

v) Let $f(x) = \sqrt{x}$. Then $f(x + h) = \sqrt{x+h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Thus $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

vi) $f(x) = \log x \therefore f(x + h) = \log(x + h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log(x+h) - \log x}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\log\left(1 + \frac{h}{x}\right)\right\}$$

Let $\frac{h}{x} = t$ i.e. h=tx and as $h \to 0,\ t \to 0$

$$\therefore f'(x) = \lim_{t\to 0} \frac{1}{tx}\log(1+t) = \frac{1}{x} \lim_{t\to 0} \frac{1}{t}\log(1+t) = \frac{1}{x}\times 1 = \frac{1}{x},\ \text{since} \lim_{t\to 0} \frac{\log(1+t)}{t} = 1$$

Thus $\frac{d}{dx}(\log x) = \frac{1}{x}$

8.A.3 SOME STANDARD RESULTS (FORMULAS)

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$ (2) $\frac{d}{dx}(e^x) = e^x$ (3) $\frac{d}{dx}(a^x) = a^x \log_e a$

(4) $\frac{d}{dx}(\text{constant}) = 0$ (5) $\frac{d}{dx}(e^{ax}) = ae^{ax}$ (5) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Note: $\frac{d}{dx}\{c\, f(x)\} = cf'(x)$ c being constant.

In brief we may write below the above functions and their derivatives:

Table: Few functions and their derivatives

Function	derivative of the function
$f(x)$	$f'(x)$
x^n	$n\,x^{n-1}$
e^{ax}	ae^{ax}
$\log x$	$1/x$
a^x	$a^x \log_e a$
c (a constant)	0

We also tabulate the basic laws of differentiation.

Table: Basic Laws for differentiation

Function	Derivative of the function
(i) h(x) = c.f(x) where c is any real constant (Scalar multiple of a function)	$\frac{d}{dx}\{h(x)\} = c.\frac{d}{dx}\{f(x)\}$
(ii) h(x) = f(x) ± g(x) (Sum/Difference of function)	$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}\{g(x)\}$
(iii) h(x) = f(x). g(x) (Product of functions)	$\frac{d}{dx}\{h(x)\} = f(x)\frac{d}{dx}\{g(x)\}+g(x)\frac{d}{dx}\{f(x)\}$
(iv) $h(x) = \frac{f(x)}{g(x)}$ (Quotient of function)	$\frac{d}{dx}\{h(x)\} = \frac{g(x)\frac{d}{dx}\{f(x)\}-f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
(v) h(x) = f{g(x)}	$\frac{d}{dx}\{h(x)\} = \frac{d}{dz}f(z).\frac{dz}{dx}$, where z = g(x)

It should be noted here even though in (ii), (iii), (iv) and (v) we have considered two functions f and g, it can be extended to more than two functions by taking two by two.

Example: Differentiate each of the following functions with respect to x:

(a) $3x^2 + 5x - 2$ (b) $a^x + x^a + a^a$ (c) $\frac{1}{3}x^3 - 5x^2 + 6x - 2\log x + 3$

(d) $e^x \log x$ (e) $2^x x^5$ (f) $\frac{x^2}{e^x}$

(g) $e^x / \log x$ (h) $2^x . \log x$ (i) $\frac{2x}{3x^3+7}$

Solution: (a) Let $y = f(x) = 3x^2 + 5x - 2$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x)^2 + 5\frac{d}{dx}(x) - \frac{d}{dx}(2)$$

$= 3 \times 2x + 5.1 - 0 = 6x + 5$

(b) Let $h(x) = a^x + x^a + a^a$

$$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}(a^x + x^a + a^a) = \frac{d}{dx}(a^x) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^a),\ a^a \text{ is a constant}$$

$= a^x \log a + ax^{a-1} + 0 = a^x \log a + ax^{a-1}$.

(c) Let $f(x) = \frac{1}{3}x^3 - 5x^2 + 6x - 2\log x + 3 \quad \therefore \frac{d}{dx}\{f(x)\} = \frac{d}{dx}\left(\frac{1}{3}x^3 - 5x^2 + 6x - 2\log x + 3\right)$

$= \frac{1}{3}.3x^2 - 5.2x + 6.1 - 2.\frac{1}{x} + 0 \quad = x^2 - 10x + 6 - \frac{2}{x}$.

(d) Let $y = e^x \log x$

$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(e^x)$ (Product rule)

$= \frac{e^x}{x} + e^x \log x = \frac{e^x}{x}(1 + x \log x)$

So $\frac{dy}{dx} = \frac{e^x}{x}(1 + x \log x)$

(e) $y = 2^x x^5$

$\frac{dy}{dx} = x^5 \frac{d}{dx}(2^x) + 2^x \frac{d}{dx}(x^5)$ (Product Rule)

$= x^5 2^x \log_e 2 + 2^x . 5. x^4$

(f) let $y = \frac{x^2}{e^x}$

$\frac{dy}{dx} = \frac{e^x \frac{8}{dx}(x^2) - x^2 \frac{d}{dx}(e^x)}{(e^x)^2}$ (Quotient Rule)

$= \frac{2xe^x - x^2 e^x}{(e^x)^2} = \frac{x(2-x)}{e^x}$

(g) Let $y = e^x / \log x$

so $\frac{dy}{dx} = \frac{(\log x)\frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\log x)}{(\log x)^2}$ (Quotient Rule)

$= \frac{e^x \log x - e^x/x}{(\log x)^2} = \frac{e^x \ x \log x - e^x}{x(\log x)^2}$

So $\frac{dy}{dx} = \frac{e^x(x \log x - 1)}{x(\log x)^2}$

(h) Let h(x) = 2^x. log x

The given function h(x) is appearing here as product of two functions

f (x) = 2^x and g(x) = log x.

$$=\frac{d}{dx}\{h(x)\} = \frac{d}{dx}(2^x.\log x) = 2^x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(2^x).$$

$$2^x \times \frac{1}{x} + \log x.(2^x \log 2) = \frac{2^x}{x} + 2^x \log 2 \log x$$

(i) Let h(x) = $\frac{2x}{3x^3+7}$ [Given function appears as the quotient of two functions]

f(x) = 2x and g(x) = $3x^3 + 7$

$$\frac{d}{dx}\{h(x)\} = \frac{(3x^3+7)\frac{d}{dx}(2x) - 2x\frac{d}{dx}(3x^3+7)}{(3x^3+7)^2} = \frac{(3x^3+7).2 - 2x.(9x^2+0)}{(3x^3+7)^2}$$

$$= \frac{2\left\{(3x^3+7) - 9x^3\right\}}{(3x^3+7)^2} = \frac{2(7-6x^3)}{(3x^3+7)^2}.$$

8.A.4 DERIVATIVE OF A FUNCTION OF FUNCTION

If y = f [h(x)] then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f'(u) \times h'(x)$ where u = h(x)

Example: Differentiate log (1 + x^2) wrt. x

Solution: Let y = log (1 + x^2) = log t when t = 1 + x^2

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{t} \times (0+2x) = \frac{2x}{t} = \frac{2x}{(1+x^2)}$$

This is an example of derivative of function of a function and the rule is called Chain Rule.

8.A.5 IMPLICIT FUNCTIONS

A function in the form f(x, y) = 0 is known as implicit function. For example $x^2y^2 + 3xy + y = 0$ where y cannot be directly defined as a function of x is called an implicit function of x.

In case of implicit functions if y be a differentiable function of x, no attempt is required to express y as an explicit function of x for finding out $\frac{dy}{dx}$. In such case differentiation of both sides with respect of x and substitution of $\frac{dy}{dx} = y_1$ gives the result. Thereafter y_1 may be obtained by solving the resulting equation.

Example: Find $\frac{dy}{dx}$ for $x^2y^2 + 3xy + y = 0$

Solution: $x^2y^2 + 3xy + y = 0$

Differentiating with respect to x we see

$$x^2 \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^2) + 3x \frac{d(y)}{dx} y + 3y \frac{d}{dx}(x) + \frac{dy}{dx} = 0$$

$$\text{or } 2yx^2 \frac{dy}{dx} + 2xy^2 + 3x \frac{dy}{dx} + 3y \frac{d(x)}{dx} + \frac{dy}{dx} = 0, \frac{d}{dx}(x) = 1, \frac{d(y^2)}{dx} = 2y\frac{dy}{dx} \text{ (chain rule)}$$

$$\text{or } (2yx^2 + 3x + 1)\frac{dy}{dx} + 2xy^2 + 3y = 0$$

$$\text{or } \frac{dy}{dx} = -\frac{(2xy^2 + 3y)}{(2x^2y + 3x + 1)}$$

This is the procedure for differentiation of Implicit Function.

8.A.6 PARAMETRIC EQUATION

When both the variables x and y are expressed in terms of a parameter (a third variable), the involved equations are called parametric equations.

For the parametric equations $x = f(t)$ and $y = h(t)$ the differential coefficient $\frac{dy}{dx}$

is obtained by using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} . = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Example: Find $\frac{dy}{dx}$ if $x = at^3, y = a / t^3$

Solution: $\frac{dx}{dt} = 3at^2$; $\frac{dy}{dt} = -3a / t^4$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-3a}{t^4} \times \frac{1}{3at^2} = \frac{-1}{t^6}$$

This is the procedure for differentiation of parametric functions.

8.A.7 LOGARITHMIC DIFFERENTIATION

The process of finding out derivative by taking logarithm in the first instance is called logarithmic differentiation. The procedure is convenient to adopt when the function to be differentiated involves a function in its power or when the function is the product of number of functions.

Example: Differentiate x^x w.r.t. x

Solution: let $y = x^x$

Taking logarithm,

$\log y = x \log x$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \log x + \frac{x}{x} = 1 + \log x$$

or $\frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$

This procedure is called logarithmic differentiation.

8.A.8 SOME MORE EXAMPLES

(1) If $y = \sqrt{\frac{1-x}{1+x}}$ show that $(1 - x^2)\frac{dy}{dx} + y = 0.$

Solution: Taking logarithm, we may write $\log y = \frac{1}{2}\{\log(1 - x) - \log(1 + x)\}$

Differentiating throughout we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\{\log(1 - x) - \log(1 + x)\} = \frac{1}{2}\left(\frac{-1}{1-x} - \frac{1}{1+x}\right) = -\frac{1}{1-x^2}.$$

By cross–multiplication $(1 - x^2)\frac{dy}{dx} = -y$

Transposing $(1 - x^2)\frac{dy}{dx} + y = 0.$

(2) Differentiate the following w.r.t. x:

(a) $\log(x + \sqrt{x^2 + a^2})$

(b) $\log\left(\sqrt{x-a} + \sqrt{x-b}\right).$

Solution: (a) $y = \log(x + \sqrt{x^2 + a^2})$

$$\frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+a^2})}\left(1 + \frac{1}{2\sqrt{x^2+a^2}}(2x)\right)$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})} + \frac{x}{(x+\sqrt{x^2+a^2})\sqrt{x^2+a^2}}$$

$$= \frac{(x+\sqrt{x^2+a^2})}{(x+\sqrt{x^2+a^2})\ \sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}$$

(b) Let $y = \log(\sqrt{x-a}+\sqrt{x-b})$

$$\text{or } \frac{dy}{dx} = \frac{1}{\sqrt{x-a}+\sqrt{x-b}}\left(\frac{1}{2\sqrt{x-a}}+\frac{1}{2\sqrt{x-b}}\right) = \frac{(\sqrt{x-b}+\sqrt{x-a})}{(\sqrt{x-a}+\sqrt{x-b})\ 2\sqrt{x-a}\sqrt{x-b}}.$$

$$= \frac{1}{2\sqrt{x-a}\sqrt{x-b}}$$

(3) If $x^m y^n = (x+y)^{m+n}$ prove that $\frac{dy}{dx} = \frac{y}{x}$

Solution: $x^m y^n = (x+y)^{m+n}$

Taking log on both sides

$\log x^m y^n = (m+n) \log (x + y)$

or $m \log x + n \log y = (m+n) \log (x+y)$

$$\text{so } \frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\text{or } \left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{(x+y)}-\frac{m}{x}$$

$$\text{or } \frac{(nx+ny-my-ny)}{y(x+y)}\frac{dy}{dx} = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\text{or } \frac{(nx-my)}{y}\frac{dy}{dx} = \frac{nx-my}{x}$$

or $\frac{dy}{dx} = \frac{y}{x}$ proved.

(4) If $x^y = e^{x-y}$ prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution: $x^y = e^{x-y}$

So $y \log x = (x - y) \log e$

or $y \log x = (x - y)$ ....(a)

Differentiating w.r.t. x we get

$$\frac{y}{x} + \log x \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

or $(1 + \log x) \dfrac{dy}{dx} = 1 - \dfrac{y}{x}$

or $\dfrac{dy}{dx} = \dfrac{(x-y)}{x(1+\log x)}$, substituting x–y = log x, from (a) we have

or $\dfrac{dy}{dx} = \dfrac{y(\log x)}{x(1+\log x)}$ ……………….. (b)

From (a) $y(1 + \log x) = x$

or $\dfrac{y}{x} = \dfrac{1}{(1+\log x)}$

From (b) $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$

8.A.9 BASIC IDEA ABOUT HIGHER ORDER DIFFERENTIATION

Let $y = f(x) = x^4 + 5x^3 + 2x^2 + 9$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = 4x^3 + 15x^2 + 4x = f'(x)$$

Since f(x) is a function of x it can be differentiated again.

Thus $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = f''(x) = \dfrac{d}{dx}(4x^3 + 15x^2 + 4x) = 12x^2 + 30x + 4$

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is written as $\dfrac{d^2y}{dx^2}$ (read as d square y by dx square) and is called the second derivative of y with respect to x while $\dfrac{dy}{dx}$ is called the first derivative. Again the second derivative here being a function of x can be differentiated again and $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$

$= f'''(x) = 24x + 30.$

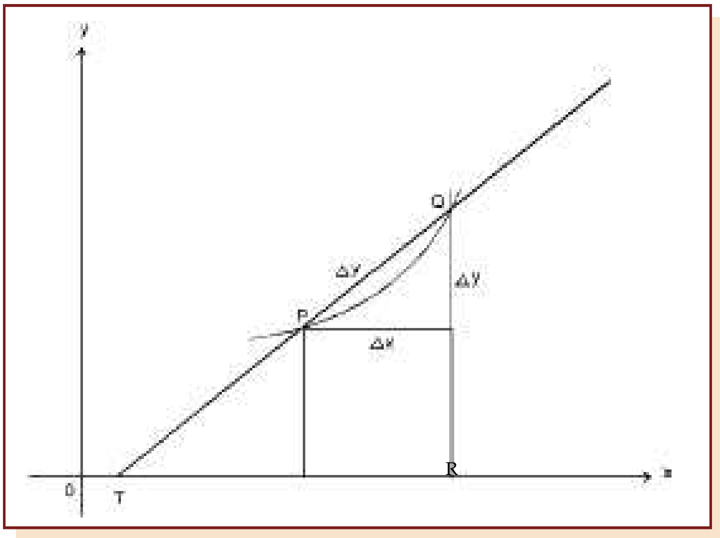
Example: If $y = ae^{mx} + be^{-mx}$ prove that $\dfrac{d^2y}{dx^2} = m^2y$.

Solution: $\dfrac{dy}{dx} = \dfrac{d}{dx}(ae^{mx} + be^{-mx}) = ame^{mx} - bme^{-mx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(ame^{mx} - bme^{-mx})$$

$= am^2e^{mx} + bm^2e^{-mx} = m^2(ae^{mx} + be^{-mx}) = m^2y.$

8.A.10 GEOMETRIC INTERPRETATION OF THE DERIVATIVE



Let f(x) represent the curve in the fig. We take two adjacent pairs P and Q on the curve Let f(x) represent the curve in the fig. We take two adjacent points P and Q on the curve whose coordinates are (x, y) and ($x + \Delta x$, $y+\Delta y$) respectively. The slope of the chord TPQ is given by

$\Delta y/\Delta x$ when $\Delta x \to 0$, $Q \to P$. TPQ becomes the tangent at P and $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

The derivative of f(x) at a point x represents the slope (or sometime called the gradient of the curve) of the tangent to the curve y = f(x) at the point x. If $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ exists for a particular point say x =a and f(a) is finite we say the function is differentiable at x = a and continuous at that point.

Example: Find the gradient of the curve $y = 3x^2 - 5x + 4$ at the point (1, 2).

Solution: $y = 3x^2 - 5x + 4 \quad \therefore \frac{dy}{dx} = 6x - 5$

so $[dy/dx]_{x=1\ y=2} = 6.1 - 5 = 6 - 5 = 1$

Thus the gradient of the curve at (1, 2) is 1.

SUMMARY

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \log_e a$

- $\frac{d}{dx}$ (constant) = 0
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$

Note: $\frac{d}{dx}\{c\, f(x)\} = cf'(x)$ c being constant.

Function	Derivative of the function
(i) h(x) = c.f(x) where c is any real constant (Scalar multiple of a function)	$\frac{d}{dx}\{h(x)\} = c.\frac{d}{dx}\{f(x)\}$
(ii) h(x) = f(x) ± g(x) (Sum/Difference of function)	$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}\{g(x)\}$
(iii) h(x) = f(x). g(x) (Product of functions)	$\frac{d}{dx}\{h(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$
(iv) $h(x) = \frac{f(x)}{g(x)}$ (Quotient of function)	$\frac{d}{dx}\{h(x)\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
(v) h(x) = f{g(x)}	$\frac{d}{dx}\{h(x)\} = \frac{d}{dz}f(z).\frac{dz}{dx}$, where z = g(x)

Applications of Differential Calculus:

In this chapter we have discussed the concept of differentiation. Differentiation helps us to find out the average rate of change in the dependent variable with respect to change in the independent variable. It makes differentiation to have applications. Various scientific formulae and results involves rate of change in price, change in demand with respect change in output, change in revenue obtained with respect to change in price, change in demand with respect change in income, etc.

1) Ate of Change in Quantities: Let there two variables x and y such that y is a function of x. Differential coefficient $\frac{dy}{dx}$ represents the rate of change of y with respect to x.
2) In other words, the expression, "the rate of Change of a function" means the derivative of the function.
3) We write f(x) in place of y and f′(x) in place of $\frac{dy}{dx}$.

Cost Function: Total cost consists of two parts (i) Variable Cost (ii) Fixed Cost.

If C(X) denotes the cost producing x units of a product then C(x) = V(x) + F(x) , where V(x) denotes the variable cost and F(x) is the fixed cost. Variable cost depends upon the number of units produced (i.e value of x) whereas fixed cost is independent of the level of output x. For example,

Average cost (AC or $\bar{C}$) = $\frac{\text{Total Cost}}{\text{OutPut}} = \frac{C(X)}{X}$

Average variable cost (AVC) = $\frac{\text{Variable Cost}}{\text{OutPut}} = \frac{V(x)}{x}$

Average Fixed Cost (AFC) = $\frac{\text{Fixed Cost}}{\text{OutPut}} = \frac{F(x)}{x}$

Marginal Cost: If C(x) the total cost producing x units then the increase in cost in producing one more unit is called marginal cost at an output level of x units and is given as $\frac{dC}{dx}$.

Marginal Cost (MC) = Rate of change in cost C per unit change in OutPut at an OutPut level of x units = $\frac{dC}{dx}$.

To increase profits of a company may decide to increase its production. The question that concerns the management is how will the cost be affected by an increase in production. Economists use the marginal cost to answer the question.

Example 1: The total cost function of a firm is $C(x) = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ where C is the total cost and x is outpout.

A tax at the rate of ₹ 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by, where ₹ p is the price per unit of output, find the profit maximising output and price for maximum profit.

Solution:

After the imposition of tax of ₹ 2 per unit, the total new cost is

$$C(x) = \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x$$

Also, $R(x) = px = (2530 - 5x)x = 2530x - 5x^2$

$\therefore\ P(x) = R(x) - C(x)$

$$= (2530x - 5x^2) - (\frac{1}{3}x^3 - 5x^2 + 30x + 10) \quad = -\frac{1}{3}x^3 + 2500x - 10$$

For maximum total profit, and .

$P'(x) = 0$ gives $-x^2 + 2500 = 0 \therefore x = \pm 50$

Since output cannot be negative, we consider x = 50.

For x = 50, $P''(x) = -2x = -2 \times 50 = -100 < 0$

Thus, the profit is maximum at x = 50.

Putting x = 50 in the demand function, the corresponding price is $p = 2530 - 5 \times 50 =$ ₹ 2280.

Example 2: The cost function of a company is given by:

$$C(x) = 100x - 8x^2 + \frac{x^3}{3},$$

where x denotes the output. Find the level of output at which:

(i) marginal cost is minimum

(ii) average cost is minimum

Solution:

$$M(x) = \text{Marginal Cost} = C(x) = \frac{d}{dx}\left(100x - 8x^2 + \frac{x^3}{3}\right) = 100 - 16x + x^2$$

$$A(x) = \text{Average Cost} = \frac{C(x)}{x} = 100 - 8x + \frac{x^2}{3}.$$

(i) M(x) is maximum or minimum when $M¢(x) = -16 + 2x = 0$ or, x = 8.

$M^2(8) = M''(x)]_{x=8} = [2]_{x=8} = 2 > 0$

Hence, marginal cost is minimum at x = 8.

(ii) A(x) is maximum or minimum when $A'(x) = -8 + \frac{2x}{3} = 0$ or, x = 12.

$A''(12)\ A''(x)]_{x=12} = \left.\frac{2}{3}\right]_{x=12} = \frac{2}{3} > 0$

Hence, average cost is minimum at x = 12.

$$A(x) = \text{Average Cost} = 100 - 8x + \frac{x^2}{3} = 100 - 8(12) + \frac{144}{3}$$

$$= 100 - 96 + 48 = 52$$

1) **Revenue Function**: Revenue, R(x), gives the total money obtained (Total turnover) by selling x units of a product. If x units are sold at 'P per unit, then R(x) = P.X

Marginal Revenue: It is the rate of change I revenue per unit change in output. If R is the revenue and x is the output, then $MR = \frac{dR}{dx}$.

Profit function: Profit P(x), the difference of between total revenue R(x) and total Cost C (x).

P(X)= R(x) – C(x)

Marginal Profit: It is rate of change in profit per unit change in output. i.e $\frac{dP}{dx}$

Example 3: A computer software company wishes to start the production of floppy disks. It was observed that the company had to spend ₹ 2 lakhs for the technical informations. The cost of setting up the machine is ₹ 88,000 and the cost of producing each unit is ₹ 30, while each floppy could be sold at ₹ 45. Find:

(i) the total cost function for producing x floppies; and

(ii) the break-even point.

Solution :

a) Given, fixed cost = ₹ 2,00,000 + ₹ 88,000 = ₹ 2,88,000.

(i) If C (x) be the total cost function for producing floppies, then C(x) = 30x + 2,88,000

(ii) The Revenue function R(x), for sales of x floppies is given by R(x) = 45x.

For break-even point, R(x) = C(x)

i.e., 45x = 30x + 2,88,000

i.e., 15x = 2,88,0000 $\therefore$ x = 19,200, the break-even point

Example 4: A company decided to set up a small production plant for manufacturing electronic clocks. The total cost for initial set up (fixed cost) is ₹ 9 lakhs. The additional cost for producing each clock is ₹ 300. Each clock is sold at ₹ 750. During the first month, 1,500 clocks are produced and sold.

(i) What profit or loss the company incurs during the first month, when all the 1,500 clocks are sold ?

(ii) Determine the break-even point.

(b) Total cost of producing 20 items of a commodity is ₹ 205, while total cost of producing 10 items is ₹ 135. Assuming that the cost function is a linear function, find the cost function and marginal cost function.

Solution:

(a) The total cost function for manufacturing x Clocks is given by C(x) = Fixed cost + Variable cost to produce x Clocks = 9,00,000 + 300x.

The revenue function from the sale of x clocks in given by R(x) = 750 × x = 750x.

(i) Profit function,

$P(x) = R(x) - C(x)$

$= 750x - (9,00,000 + 300x) = 450x - 9,00,000$

$\therefore$ Profit, when all 1500 clocks are sold = P(1500) = 450 × 1500 – 9,00,000 = – ₹ 2,25,000

Thus, there is a loss of ' 2,25,000 when only 1500 clocks are sold.

(ii) At the break-even point, R(x) = C(x)

or, $9,00,000 + 300x = 750x$

or, $450x = 9,00,000$ $\therefore$ $x = 2,000$

Hence, 2000 clocks have to be sold to achieve the break-even point.

(b) Let cost function be

$C(x) = ax + b$,…………(i)

x being number of items and a, b being constants.

Given, C(x) = 205 for x = 20 and C(x) = 135 for x = 10.

Putting these values in (i),

$205 = 20a+b$ …………(ii)

$135 = 10a+b$ …………(iii)

(ii) – (iii) gives,

$70 = 10a$ or, $a = 7$

From (iii), $b = 135 - 10a = 135 - 70 = 65$

$\therefore$ Required cost function is given by $C(x) = 7x + 65$

Marginal cost function, $C'(x) = \frac{d}{dx}(7x+65) = 7$

Marginal Propensity to Consume (MPC): The consumption function C = F(Y) expresses the relationship between the total consumption and total Income (Y), then the marginal propensity to consume is defined as the rate of Change consumption per unit change in Income i.e., $\frac{dC}{dY}$. By consumption we mean expenditure incurred in on Consumption.

Marginal Propensity to save (MPS): Saving, S is the difference between income, I and consumption, c, i.e ., $\frac{dS}{dY}$.

EXERCISE 8(A)

Choose the most appropriate option (a) (b) (c) or (d).

1. The gradient of the curve $y = 2x^3 - 3x^2 - 12x + 8$ at $x = 0$ is

a) –12 b) 12 c) 0 d) none of these

2. The gradient of the curve $y = 2x^3 - 5x^2 - 3x$ at $x = 0$ is

a) 3 b) –3 c) 1/3 d) none of these

3. The derivative of $y = \sqrt{x+1}$ is

a) $1/\sqrt{x+1}$ b) $-1/\sqrt{x+1}$ c) $1/2\sqrt{x+1}$ d) none of these

4. If $f(x) = e^{ax^2+bx+c}$ the $f'(x)$ is

a) e^{ax^2+bx+c} b) $e^{ax^2+bx+c}(2ax+b)$ c) $2ax+b$ d) none of these

5. If $f(x) = \frac{x^2+1}{x^2-1}$ then $f'(x)$ is

a) $-4x/(x^2-1)^2$ b) $4x/(x^2-1)^2$ c) $x/(x^2-1)^2$ d) none of these

6. If $y = x(x-1)(x-2)$ then $\frac{dy}{dx}$ is

a) $3x^2 - 6x + 2$ b) $-6x + 2$ c) $3x^2 + 2$ d) none of these

7. If $xy = 1$ then $y^2 + dy/dx$ is equal to

a) 1 b) 0 c) –1 d) none of these

8. The derivative of the function $\sqrt{x+\sqrt{x}}$ is

a) $\frac{1}{2\sqrt{x+\sqrt{x}}}$ b) $1+\frac{1}{2\sqrt{x}}$ c) $\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)$ d) none of these

9. Given $e^{-xy} - 4xy = 0$, $\frac{dy}{dx}$ can be proved to be

a) $-y/x$ b) y/x c) x/y d) none of these

10. If $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, $\frac{dy}{dx}$ can be expressed as

a) $\frac{x}{y}$ b) $\frac{x}{\sqrt{x^2-a^2}}$ c) $\frac{1}{\sqrt{\frac{x^2}{a^2}-1}}$ d) none of these

11. If $\log (x / y) = x + y$, $\frac{dy}{dx}$ may be found to be

a) $\frac{y(1-x)}{x(1+y)}$ b) $\frac{y}{x}$ c) $\frac{1-x}{1+y}$ d) none of these

12. If $f(x, y) = x^3 + y^3 - 3axy = 0$, $\frac{dy}{dx}$ can be found out as

a) $\frac{ay-x^2}{y^2+ax}$ b) $\frac{ay-x^2}{y^2-ax}$ c) $\frac{ay+x^2}{y^2+ax}$ d) none of these

13. Given $x = at^2$, $y = 2at$; $\frac{dy}{dx}$ is calculated as

a) t b) $-1/t$ c) $1/t$ d) none of these

14. Given $x = 2t + 5$, $y = t^2 - 2$; $\frac{dy}{dx}$ is calculated as

a) t b) $-1/t$ c) $1/t$ d) none of these

15. If $y = \frac{1}{\sqrt{x}}$ then $\frac{dy}{dx}$ is equal to

a) $\frac{1}{2x\sqrt{x}}$ b) $\frac{-1}{x\sqrt{x}}$ c) $-\frac{1}{2x\sqrt{x}}$ d) none of these

16. If $x = 3t^2 - 1$, $y = t^3 - t$, then $\frac{dy}{dx}$ is equal to

a) $\frac{3t^2-1}{6t}$ b) $3t^2-1$ c) $\frac{3t -1}{6t}$ d) none of these

17. For the curve $x^2 + y^2 + 2gx + 2hy = 0$, the value of $\frac{dy}{dx}$ at (0, 0) is

a) $-g/h$ b) g/h c) h/g d) none of these

18. If $y = \frac{e^{3x}-e^{2x}}{e^{3x}+e^{2x}}$, then $\frac{dy}{dx}$ is equal to

a) $2e^{5x}$ b) $1/(e^{5x} + e^{2x})^2$ c) $e^{5x}/(e^{5x} + e^{2x})$ d) none of these

19. Given $x = t + t^{-1}$ and $y = t - t^{-1}$ the value of $\frac{dy}{dx}$ at $t = 2$ is

a) 3/5 b) $-3/5$ c) 5/3 d) none of these

20. If $x^3 - 2x^2 y^2 + 5x + y - 5 = 0$ then $\frac{dy}{dx}$ at x = 1, y = 1 is equal to

a) 4/3 b) – 4/3 c) 3/4 d) none of these

21. The derivative of $x^2 \log x$ is

a) 1+2log x b) x(1 + 2 log x) c) 2 log x d) none of these

22. The derivative of $\frac{3-5x}{3+5x}$ is

a) $30/(3+5x)^2$ b) $1/(3+5x)^2$ c) $-30/(3+5x)^2$ d) none of these

23. Let $y = \sqrt{2x} + 3^{2x}$ then $\frac{dy}{dx}$ is equal to

a) $(1/\sqrt{2x}) + 2.3^{2x} \log_e 3$ b) $1/\sqrt{2x}$

c) $2.3^{2x} \log_e 3$ d) none of these

24. The derivative of e^{3x^2-6x+2} is

a) $30(1-5x)^5$ b) $(1-5x)^5$ c) $6(x-1)e^{3x^2-6x+2}$ d) none of these

25. If $y = \frac{e^x+1}{e^x-1}$ then $\frac{dy}{dx}$ is equal to

a) $\frac{-2e^x}{(e^x-1)^2}$ b) $\frac{2e^x}{(e^x-1)^2}$ c) $\frac{-2}{(e^x-1)^2}$ d) none of these

26. If $x = at^2$, $y = 2at$ then $\left[\frac{dy}{dx}\right]_{t=2}$ is equal to

a) 1/2 b) –2 c) –1/2 d) none of these

27. Let $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ then f′(2) is equal to

a) 3/4 b) 1/2 c) 0 d) none of these

28. If $f(x) = x^2 - 6x + 8$ then f′(5) – f′(8) is equal to

a) f′(2) b) 3f′(2) c) 2f′(2) d) none of these

29. If $f(x) = x^k$ and f′(1) = 10 the value of k is

a) 10 b) –10 c) 1/10 d) none of these

30. If $y = e^x + e^{-x}$ then $\frac{dy}{dx} - \sqrt{y^2 - 4}$ is equal to

a) 1 b) –1 c) 0 d) none of these

31. The derivative of $(x^2-1)/x$ is

a) $1 + 1/x^2$ b) $1 - 1/x^2$ c) $1/x^2$ d) none of these

32. The differential coefficients of $(x^2 +1)/x$ is

a) $1 + 1/x^2$ b) $1 - 1/x^2$ c) $1/x^2$ d) none of these

33. If $y = e^{\sqrt{2x}}$ then $\frac{dy}{dx}$ is equal to ____________.

a) $\frac{e^{\sqrt{2x}}}{\sqrt{2x}}$ b) $e^{\sqrt{2x}}$ c) $\frac{e^{\sqrt{2x}}}{\sqrt{2x}}$ d) none of these

34. If $x = (1 - t^2)/(1 + t^2)$ $y = 2t/(1 + t^2)$ then dy/dx at $t = 1$ is ____________.

a) 1/2 b) 1 c) 0 d) none of these

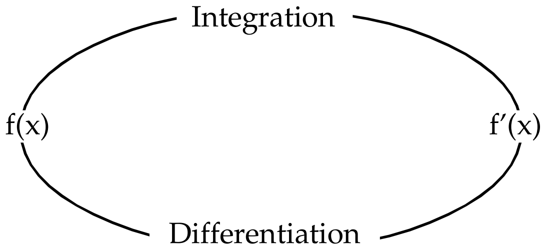
35. $f(x) = x^2/e^x$ then $f'(1)$ is equal to ____________.

a) $-1/e$ b) $1/e$ c) e d) none of these

(B) INTEGRAL CALCULUS

8.B.1 INTEGRATION

Integration is the reverse process of differentiation.



We know

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=\frac{(n+1)x^n}{(n+1)}$$

or $$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n \qquad \text{.....................(1)}$$

Integration is the inverse operation of differentiation and is denoted by the symbol $\int$.

Hence, from equation (1), it follows that

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}$$

i.e. Integral of x^n with respect to variable x is equal to $\frac{x^{n+1}}{n+1}$

Thus if we differentiate $\frac{(x^{n+1})}{n+1}$ we can get back x^n

Again if we differentiate $\frac{(x^{n+1})}{n+1}$ + c and c being a constant, we get back the same x^n .

i.e. $$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}+c\right]=x^n$$

Hence $\int x^n\,dx = \frac{(x^{n+1})}{n+1}$ + c and this c is called the constant of integration.

Integral calculus was primarily invented to determine the area bounded by the curves dividing the entire area into infinite number of infinitesimal small areas and taking the sum of all these small areas.

8.B.2 BASIC FORMULAS

i) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$ (If n=-1, $\frac{x^{n+1}}{n+1} = \frac{1}{0}$ which is not defined)

ii) $\int dx = x + c$, since $\int 1dx = \int x^{\circ} dx = \frac{x1}{1} = x + c$

iii) $\int e^x\, dx = e^x + c$, since $\frac{d}{dx} e^x = e^x$

iv) $\int e^{ax}\, dx = \frac{e^{ax}}{a} + c$, since $\frac{d}{dx}\left(\frac{e^{ax}}{a}\right) = e^{ax}$

v) $\int \frac{dx}{x} = \log x + c$, since $\frac{d}{dx} \log x = \frac{1}{x}$

vi) $\int a^x\, dx = a^x / \log_e a + c$, since $\frac{d}{dx}\left(\frac{a^x}{\log_e^a}\right) = a^x$

Note: *In the answer for all integral sums we add +c (constant of integration) since the differentiation of constant is always zero.*

Elementary Rules:

$\int c\, f(x)\, dx = c \int f(x)\, dx$ where c is constant.

$\int \{ f(x)\, dx \pm g(x)\}\, dx = \int f(x)dx \pm \int g(x)dx$

Examples : Find (a) $\int \sqrt{x}\, dx$, (b) $\int \frac{1}{\sqrt{x}} dx$, (c) $\int e^{-3x} dx$ (d) $\int 3^x dx$ (e) $\int x\sqrt{x}\ dx$.

Solution: (a) $\int \sqrt{x}\ dx = x^{1/2+1} / (1/2 + 1) = \frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3} + c$

(b) $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = 2\sqrt{x} + c$ where c is arbitrary constant.

(c) $\int e^{-3x}\ dx = \frac{e^{-3x}}{-3} + c = -\frac{1}{3} e^{-3x} + c$

(d) $\int 3^x\ dx = \frac{3^x}{\log_e 3} + c.$

(e) $\int x\sqrt{x}\ dx. = \int x^{\frac{3}{2}}dx = \dfrac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}dx = \dfrac{2}{5}x^{5/2} + c.$

Examples : Evaluate the following integral:

i) $\int (x + 1/x)^2\, dx = \int x^2\, dx + 2\int dx + \int dx / x^2$

$= \dfrac{x^3}{3} + 2x + \dfrac{x^{-2+1}}{-2+1}$

$= \dfrac{x^3}{3} + 2x - \dfrac{1}{x} + c$

ii) $\int \sqrt{x}\ (x^3+2x-3)\, dx = \int x^{7/2}\, dx + 2\int x^{3/2}\, dx - 3\int x^{1/2}\, dx$

$= \dfrac{x^{7/2+1}}{7/2+1} + \dfrac{2\,x^{3/2+1}}{3/2+1} - \dfrac{3\,x^{1/2+1}}{1/2+1}$

$= \dfrac{2x^{9/2}}{9} + \dfrac{4x^{5/2}}{5} \quad -2\,x^{3/2} + c$

iii) $\int e^{3x} + e^{-4x}\, dx \qquad = \int e^{2x}\, dx \quad + \quad \int e^{-4x}\, dx$

$= \dfrac{e^{2x}}{2} + \dfrac{e^{-4x}}{-4} = \dfrac{e^{2x}}{2} - \dfrac{1}{4e^{4x}} + c$

iv) $\int \dfrac{x^2}{x+1}dx = \int \dfrac{x^2-1+1}{x+1}dx$

$= \int \dfrac{(x^2-1)}{x+1}dx + \int \dfrac{dx}{x+1}$

$= \int (x-1)dx + \log(x+1) = \dfrac{x^2}{2} - x + \log(x+1) + c$

v) $\int \dfrac{x^3 + 5x^2 - 3}{(x+2)}dx$

By simple division $\int \dfrac{x^3 + 5x^2 - 3}{(x+2)}dx$

$= = \int \left\{x^2 + 3x - 6 + \dfrac{9}{(x+2)}\right\}dx$

$$= \frac{x^3}{3} + \frac{3x^2}{2} - 6x + 9\log(x+2) + c$$

8.B.3 METHOD OF SUBSTITUTION (CHANGE OF VARIABLE)

It is sometime possible by a change of independent variable to transform a function into another which can be readily integrated.

We can show the following rules.

To put z = f (x) and also adjust dz = f ′ (x) dx

Example: $\int F\{h(x)\}\ h'(x)\,dx$, take $e^z = h(x)$ and to adjust $dz = h'(x)\,dx$

then integrate $\int F(z)\,dz$ using normal rule.

Example: $\int (2x+3)^7\,dx$

We put (2x + 3) = t ⇒ so 2 dx = dt or dx = dt / 2

Therefore $\int (2x+3)^7\,dx = ½\int t^7\,dt = \frac{t^8}{2x8} = \frac{t^8}{16} = \frac{(2x+3)^8}{16} + c$

This method is known as Method of Substitution

Example: $\int \frac{x^3}{(x^2+1)^3}\,dx$ We put $(x^2+1) = t$

so 2x dx = dt or x dx = dt / 2

$$= \int \frac{x^2.x}{t^3}\,dx$$

$$= \frac{1}{2}\int \frac{t-1}{t^3}\,dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2} - \frac{1}{2}\int \frac{dt}{t^3}$$

$$= \frac{1}{2} \times \frac{t^{-2+1}}{(-2+1)} - \frac{1}{2} \times \frac{t^{-3+1}}{(-3+1)}$$

$$= -\frac{1}{2}\ \frac{1}{t} + \frac{1}{4}\ \frac{1}{t^2}$$

$$= \frac{1}{4}\frac{1}{t^2} - \frac{1}{2}\frac{1}{t}$$

$$= \frac{1}{4}\cdot\frac{1}{(x^2+1)^2} - \frac{1}{2}\cdot\frac{1}{(x^2+1)} + c$$

IMPORTANT STANDARD FORMULAE

a) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\frac{x-a}{x+a} + c$

b) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x} + c$

c) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

d) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) + c$

e) $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + c$

f) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + c$

g) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$

h) $\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c$

Examples: (a) $\int \frac{e^x}{e^{2x} - 4}dx = \int \frac{dz}{z^2 - 2^2}$ where $z = e^x$ $dz = e^x dx$

$$= \frac{1}{4}\log\left(\frac{e^x - 2}{e^x + 2}\right) + c$$

(b) $\int \frac{1}{x + \sqrt{x^2 - 1}}dx = \int \frac{x - \sqrt{x^2 - 1}}{(x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1})}dx = \int (x - \sqrt{x^2 - 1})\,dx$

$$= \frac{x^2}{2} - \frac{x}{2}\sqrt{x^2 - 1} + \frac{1}{2}\log(x + \sqrt{x^2 - 1}) + c$$

(c) $\int e^x(x^3 + 3x^2)dx = \int e^x\{f(x) + f'(x)\}dx$, where $f(x) = x^3$

[by (e) above)] $= e^x x^3 + c$

8.B.4 INTEGRATION BY PARTS

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{d(u)}{dx}\int v\,dx\right]dx$$

where u and v are two different functions of x

Evaluate:

i) $\int xe^x\, dx$

Integrating by parts we see

$$\int xe^x\, dx = x\int e^x\, dx - \int\left\{\frac{d}{dx}(x)\int e^x\, dx\right\}dx$$

$$= x e^x - \int 1.e^x dx = xe^x - e^x + c$$

ii) $\int x \log x\, dx$

Integrating by parts,

$$= \log x \int x\, dx - \int\left\{\frac{d}{dx}(\log x)\int x dx\right\}dx$$

$$= \frac{x^2}{2}\log x - \int\left[\frac{1}{x}.\frac{x^2}{2}\right]dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

iii) $\int x^2 e^{ax}\, dx$

$$= x^2\int e^{ax}\, dx - \int\left\{\frac{d}{dx}(x^2)\int e^{ax}\, dx\right\}dx$$

$$= \frac{x^2}{a}e^{ax} - \int 2x.\frac{e^{ax}}{a}\, dx$$

$$= \frac{x^2}{a}e^{ax} - \frac{2}{a}\int x.e^{ax} dx$$

$$= \frac{x^2}{a}e^{ax} - \frac{2}{a}\times\int e^{ax}\, dx - \int\left[\frac{d}{dx}(x)\int e^{ax}\, dx\right]dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[\frac{xe^{ax}}{a} - \int 1.\frac{e^{ax}}{a}dx\right]$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^3} e^{ax} + c$$

8.B.5 METHOD OF PARTIAL FRACTION

Type I :

Example: $\int \frac{(3x+2)\,dx}{(x-2)(x-3)}$

Solution: let $\frac{(3x+2)}{(x-2)(x-3)}$

$$= \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

[Here degree of the numerator must be lower than that of the denominator; the denominator contains non–repeated linear factor]

so $3x + 2 = A(x-3) + B(x-2)$

We put $x = 2$ and get

$3.2 + 2 = A(2-3) + B(2-2) \Rightarrow A = -8$

we put $x = 3$ and get

$3.3 + 2 = A(3-3) + B(3-2) \qquad \Rightarrow B = 11$

$$\int \frac{(3x+2)dx}{(x-2)^2(x-3)} \quad -8\int \frac{dx}{x-2} + 11\int \frac{dx}{x-3}$$

$$= -\log(x-2) + 11\log(x-3) + c$$

Type II:

Example: $\int \frac{(3x+2)\,dx}{(x-2)^2(x-3)}$

Solution: let $\frac{(3x+2)}{(x-2)^2(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)}$

or $3x + 2 = A(x-2)(x-3) + B(x-3) + C(x-2)^2$

Comparing coefficients of x^2, x and the constant terms of both sides, we find

$A + C = 0$ ....(i)

$-5A + B - 4C = 3$ ......(ii)

$6A - 3B + 4C = 2$ ....(iii)

By (ii) + (iii) A – 2B = 5 ………(iv)

(i) – (iv) 2B + C = –5 …….(v)

From (iv) A = 5 + 2B

From (v) C = –5 – 2B

From (ii) –5 (5 + 2B) + B – 4 (– 5 – 2B) = 3

or – 25 – 10B + B + 20 + 8B = 3

or – B – 5 = 3

or B = – 8, A = 5 – 16 = – 11, from (iv) C = – A = 11

Therefore $\int \frac{(3x+2)\,dx}{(x-2)^2(x-3)}$

$$= -11\int\frac{dx}{(x-2)} - 8\int\frac{dx}{(x-2)^2} + 11\int\frac{dx}{(x-3)}$$

$$= -11 \log (x-2) + \frac{8}{(x-2)} + 11 \log (x-3)$$

$$= 11 \log \frac{(x-3)}{(x-2)} + \frac{8}{(x-2)} + c$$

Type III:

Example: $\int \frac{(3x^2 - 2x + 5)}{(x-1)^2 (x^2+5)} dx$

Solution: Let $\frac{3x^2-2x+5}{(x-1)^2 (x^2+5)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+5)}$

so $3x^2 - 2x + 5 = A(x^2 + 5) + (Bx + C)(x-1)$

Equating the coefficients of x^2, x and the constant terms from both sides we get

A + B = 3 …………(i)

C – B = –2 …………(ii)

5A – C = 5 ………….(iii)

by (i) + (ii) A + C = 1 ……… (iv)

by (iii) + (iv) 6A = 6 ……… (v)

or A = 1

therefore B = 3 – 1 = 2 and C = 0

Thus $\int \frac{(3x^2 - 2x + 5)}{(x-1)^2 (x^2+5)} dx$

$$= \int \frac{dx}{x-1} + \int \frac{2x}{x^2+5} dx$$

= log (x–1) + log (x^2 + 5)

= log (x^2 + 5) (x–1) + c

Example: $\int \frac{dx}{x(x^3+1)}$

Solution: $\int \frac{dx}{x(x^3+1)}$

$$= \int \frac{x^2 dx}{x^3(x^3+1)}$$ we put x^3 = z, $3x^2$ dx = dz

$$= \frac{1}{3}\int \frac{dz}{z(z+1)}$$

$$= \frac{1}{3}\int \left(\frac{1}{z} - \frac{1}{z+1}\right) dz$$

$$= \frac{1}{3}[\log z - \log (z+1)]$$

$$= \frac{1}{3}\log\left(\frac{x^3}{x^3+1}\right)$$

Example : Find the equation of the curve where slope at (x, y) is 9x and which passes through the origin.

Solution: $\frac{dy}{dx}$ = 9x

$\therefore \int dy =$ or y = $9x^2$ /2 +c

Since it passes through the origin, c = 0; thus required eqn . is $9x^2$ = 2y.

8.B.6 DEFINITE INTEGRATION

Suppose F(x) dx = *f* (x)

As x changes from a to b the value of the integral changes from *f* (a) to *f* (b). This is as

$$\int_a^b F(x)\,dx = f(b) - f(a)$$

'b' is called the upper limit and 'a' the lower limit of integration. We shall first deal with indefinite integral and then take up definite integral.

Example: $\int_0^2 x^5\,dx$

Solution: $\int_0^2 x^5\,dx = \frac{x^6}{6}$

$$\int_0^2 x^5\,dx = \left(\frac{x^6}{6}\right)_0^2$$

$$= \frac{1}{6}(2^6 - 0) \qquad = 64/6 = 32/3$$

Note: In definite integration the constant (c) **should not be added**

Example: $\int_1^2 (x^2 - 5x + 2)dx$

Solution: $\int_1^2 (x^2 - 5x + 2)dx = \frac{x^3}{3} - \frac{5x^2}{2} + 2x$. Now, $\int_1^2 (x^2 - 5x + 2)dx = \left[\frac{x^3}{3} - \frac{5x^2}{2} + 2x\right]_1^2$

$$= \left[\frac{2^3}{3} - \frac{5x2^2}{2} + 2x2\right] - \left[\frac{1}{3} - \frac{5}{2} + 2\right] = -19/6$$

8.B.7 IMPORTANT PROPERTIES

Important Properties of definite Integral

(I) $\int_a^b f(x)dx = \int_a^b f(t)\,dt$

(II) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(III) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx,\ a < c < b$

(IV) $\int_0^a f(x)\,dx = \int_o^a f(a-x)\,dx$

(V) When f(x) =f (a+x) then $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$

(VI) $\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx$ if f(–x) = f(x)

= 0 if f(–x) = –f(x)

Example: $\int_0^2 \frac{x^2dx}{x^2+(2-x)^2}$

Solution: Let $I = \int_0^2 \frac{x^2dx}{x^2+(2-x)^2}$

$= \int_0^2 \frac{(2-x)^2\ dx}{(2-x)^2\ +x^2}$ [by prop. IV]

$\therefore\ 21 = \int_0^2 \frac{x^2dx}{x^2+(2-x)^2} + \int_0^2 \frac{(2-x)^2}{(2-x)^2+x^2}\,dx$

$\int_0^2 \frac{x^2+(2-x)^2}{x^2+(2-x)^2}\,dx$

$= \int_0^2 dx = [x]_0^2 = 2–0 = 2$

or I = 2/2 = 1

Example: Evaluate $\int_{-2}^{2} \frac{x^4\ dx}{a^{10}\ -\ x^{10}}$ $(a > 2)$

Solution: $\frac{x^4\ dx}{a^{10}\ -\ x^{10}} = \frac{x^4\ dx}{(a^5)^2 - (x^5)^2}$

let $x^5 = t$ so that $5x^4\,dx = dt$

Now $\int \frac{x^4dx}{(a^5)^2-(x^5)^2}$

$= \frac{1}{5}\int \frac{5x^4dx}{(a^5)^2-(x^5)^2}$

$= \frac{1}{5}\int \frac{dt}{(a^5)^2-t^2}$

$$= \frac{1}{10a^5} \log \frac{a^5 + x^5}{a^5 - x^5} \qquad \text{(by standard formula b)}$$

Therefore, $\int_{-2}^{2} \frac{x^4 dx}{a^{10} - x^{10}}$

$$= 2\int_{0}^{2} \frac{x^4 dx}{a^{10} - x^{10}} \qquad \text{(by prop. VI)}$$

$$= 2 \times \frac{1}{10a^5} \log\left[\frac{a^5 + x^5}{a^5 - x^5}\right]_0^2$$

$$= \frac{1}{5a^5} \log \frac{a^5 + 32}{a^5 - 32}$$

SUMMARY

- $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1 \quad \left(\text{If } n=-1, \frac{x^{n+1}}{n+1} = \frac{1}{0} \text{ which is not defined}\right)$
- $\int dx = x$, since $\int 1dx = \int x^\circ dx = \frac{x1}{1} = x + c$
- $\int e^x dx = e^x + c$, since $\frac{d}{dx} e^x = e^x$
- $\int e^{ax} dx = \frac{e^{ax}}{a} + c$, since $\frac{d}{dx}\left(\frac{e^{ax}}{a}\right) = e^{ax}$
- $\int \frac{dx}{x} = \log x + c$, since $\frac{d}{dx} \log x = \frac{1}{x}$
- $\int a^x dx = a^x / \log_e a + c$, since $\frac{d}{dx}\left(\frac{a^x}{\log_e^a}\right) = a^x$

$\int c\, f(x)\, dx = c \int f(x)\, dx$ where c is constant.

$\int \{ f(x)\, dx \pm g(x)\}\, dx = \int f(x)dx \pm \int g(x)dx$

- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$

- $\int \frac{dx}{\sqrt{x^2+a^2}} = log\left|x+\sqrt{x^2+a^2}\right| + c$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right) + c$
- $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + c$
- $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + c$
- $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + c$
- $\int \frac{f'(x)}{f(x)}dx = \log f(x) + c$

Integration by parts

Important Properties of definite Integral

- $\int_a^b f(x)dx = \int_a^b f(t)\,dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
- When f(x) =f (a+x) = $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$
- $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

EXERCISE 8(B) [K = CONSTANT]

Choose the most appropriate option (a) (b) (c) or (d).

1. Evaluate $= \int 5x^2\ dx$:

   (a) $5 / 3x^3 + k$ (b) $\frac{5x^3}{3} + k$ (c) $5x^3$ (d) none of these

2. Integration of $3 - 2x - x^4$ will become

   (a) $-x^2 - x^5 / 5$ (b) $3x - x^2 - \frac{x^5}{5} + k$ (c) $3x - x^2 + \frac{x^5}{5} + k$ (d) none of these

3. Given $f(x) = 4x^3 + 3x^2 - 2x + 5$ and $\int f(x)\,dx$ is

(a) $x^4 + x^3 - x^2 + 5x$ (b) $x^4 + x^3 - x^2 + 5x + k$

(c) $12x^2 + 6x - 2x^2$ (d) none of these

4. Evaluate $\int (x^2 - 1)\,dx$

(a) $x^5/5 - 2/3\, x^3 + x + k$ (b) $\frac{x^3}{3} - x + k$

(c) $2x$ (d) none of these

5. $\int (1-3x)(1+x)\,dx$ is equal to

(a) $x - x^2 - x^3$ (b) $x^3 - x^2 + x$ (c) $x - x^2 - x^3 + k$ (d) none of these

6. $\int [\sqrt{x} - 1/\sqrt{x}]\,dx$ is equal to

(a) $\frac{2}{3}x^{3/2} - 2\,x^{½}$ +k (b) $\frac{2}{3}\sqrt{x} - 2\sqrt{x} + k$ (c) $\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}} + k$ (d) none of these

7. The integral of $px^3 + qx^2 + rk + w/x$ is equal to

(a) $px^2 + qx + r + k$ (b) $px^3/3 + qx^2/2 + rx$

(c) $3px + 2q - w/x^2$ (d) none of these

8. Use method of substitution to integrate the function $f(x) = (4x + 5)^6$ and the answer is

(a) $1/28\,(4x + 5)^7 + k$ (b) $(4x + 5)^7/7 + k$ (c) $(4x + 5)^7/7$ (d) none of these

9. Use method of substitution to evaluate $\int x(x^2 + 4)^5\,dx$ and the answer is

(a) $(x^2 + 4)^6 + k$ (b) $1/12\,(x^2 + 4)^6 + k$

(c) $(x^2 + 4)^6/ + k$ (d) none of these

10. Integrate $(x + a)^n$ and the result will be

(a) $\frac{(x+a)^{n+1}}{n+1} + k$ (b) $\frac{(x+a)^{n+1}}{n+1}$

(c) $(x + a)^{n+1}$ (d) none of these

11. $\int 8x^2/(x^3 + 2)^3\,dx$ is equal to

(a) $-4/3(x^3 + 2)^2 + k$ (b) $-\frac{4}{3(x^3+2)^2} + k$

(c) $\frac{4}{3(x^3+2)^2}+k$ (d) none of these

12. Using method of partial fraction the integration of f(x) when $f(x) = \frac{1}{x^2 - a^2}$ and the answer is

(a) $\log x - \frac{a}{x+a} + k$ (b) log (x – a) – log (x + a) + k

(c) $\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+k$ (d) none of these

13. Use integration by parts to evaluate $\int x^2 e^{3x} dx$

(a) $x^2 e^{3x}/3 - 2x e^{3x}/9 + 2/27 e^{3x} + k$ (b) $x^2 e^{3x} - 2x e^{3x} + 2e^{3x} + k$

(c) $e^{3x}/3 - x e^{3x}/9 + 2e^{3x} + k$ (d) none of these

14. $\int$ logx dx is equal to

(a) x logx + k (b) x logx – x^2 + k (c) x logx + k (d) none of these

15. $\int xe^x dx$ is

(a) $(x - 1)e^x + k$ (b) $(x - 1) e^x$ (c) $x e^x + k$ (d) none of these

16. Evaluate $\int_0^1 (2x^2 - x^3)$ dx and the value is

(a) 4/3 + k (b) 5/12 (c) – 4/3 (d) none of these

17. Evaluate $\int_2^4 (3x-2)^2$ dx and the value is

(a) 104 (b) 100 (c) 10 (d) none of these.

18. Evaluate $\int_0^1 xe^x dx$ dx and the value is

(a) –1 (b) 10 (c) 10/9 (d) +1

19. $\int x^x (1 + \log x)\, dx$ is equal to

(a) $x^x \log x + k$ (b) $e^{x^2} + k$ (c) $\frac{x^2}{2} + k$ (d) $x^x + c$

20. If $f(x) = \sqrt{1+x^2}$ then $\int f(x)dx$ is

(a) $\frac{2}{3}\ x\ (1+x^2)^{3/2} + k$

(b) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log(x+\sqrt{x^2+1}) + k$

(c) $\frac{2}{3}\ x\ (1+x^2)^{3/2} + k$

(d) none of these

21. $\int \frac{\sqrt{2}\left(x^2+1\right)}{\sqrt{x^2+2}}\,dx$ is equal to

(a) $\frac{x}{\sqrt{2}}\left(\sqrt{x^2+2}\right)+k$ (b) $\sqrt{x^2+2}\ + k$ (c) $1/(x^2+2)^{3/2} + k$ (d) none of these

22. $\int (e^x + e^{-x})^2 (e^x - e^{-x})\ dx$ is

(a) $\frac{1}{3}(e^x + e^{-x})^3 + k$

(b) $\frac{1}{2}(e^x - e^{-x})^2 + k$

(c) $e^x + k$

(d) none of these

23. $\int_0^a [\ f(x) + f(-x)\,]\,dx$ is equal to

(a) $\int_0^a 2\ f(x)\,dx$ (b) $\int_{-a}^a f(x)\,dx$ (c) 0 (d) $\int_{-a}^a -\ f(-x)\,dx$

24. $\int xe^x/(x+1)^2\ dx$ is equal to

(a) $e^x/(x+1) + k$ (b) $e^x/x + k$ (c) $e^x + k$ (d) none of these

25. $\int (x^4 + 3/x)\ dx$ is equal to

(a) $x^5/5 + 3\log|x|$

(b) $1/5\ x^5 + 3\log|x| + k$

(c) $1/5\ x^5 + k$

(d) none of these

26. Evaluate $\int_1^4 (2x+5)\ dx$ and the value is

(a) 3 (b) 10 (c) 30 (d) none of these

27. $\int_1^2 \frac{2x}{1+x^2}dx$ is equal to

(a) $\log_e(5/2)$

(b) $\log_e 5 - \log_e 2$

(c) $\log_e(2/5)$

(d) none of these

28. $\int_0^4 \sqrt{3x+4}\ dx$ is equal to

(a) 9/112 (b) 112/9 (c) 11/9 (d) none of these

29. $\int_0^2 \frac{x+2}{x+1} dx$ is

(a) $2 + \log_e 2$ e^2 (b) $2 + \log_e 3$ (c) $\log_e 3$ (d) none of these

30. The value of $\int_2^3 f(5-x)\,dx - \int_2^3 f(x)\,dx$ is

(a) 1 (b) 0 (c) –1 (d) none of these

31. $\int (x-1)e^x / x^2 dx$ is equal to

(a) $e^x/x + k$ (b) $e^{-x}/x + k$ (c) $-e^x/x + k$ (d) none of these

32. $\int_0^2 3x^2\ dx$ is

(a) 7 (b) –8 (c) 8 (d) none of these

33. Using integration by parts $\int x^3 \log x dx$

(a) $x^4/16 + k$ (b) $x^4/16\ (4 \log x - 1) + k$

(c) $4 \log x - 1 + k$ (d) none of these

34. Evaluate $\int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right) dx$ and the value is

(a) $\log_e |e^x + e^{-x}|$ (b) $\log e\ |e^x + e^{-x}| + k$

(c) $\log_e |e^x - e^{-x}| + k$ (d) none of these

35. If $f'(x) = x - 1$, the equation of a curve $y = f(x)$ passing through the point (1, 0) is given by

(a) $y = x^2 - 2x + 1$ (b) $y = x^2/2 - x + 1$

(c) $y = x^2/2 - x + 1/2$ (d) none of these

ANSWERS

Exercise 8(A)

1. (a)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (c)
9. (a)	**10.** (a)&(b)	**11.** (a)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (a)
17. (a)	**18.** (d)	**19.** (c)	**20.** (a)	**21.** (b)	**22.** (c)	**23.** (a)	**24.** (c)
25. (a)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (a)	**30.** (c)	**31.** (a)	**32.** (b)
33. (a)	**34.** (c)	**35.** (b)					

Exercise 8(B)

1. (b)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (a)	**7.** (d)	**8.** (a)
9. (b)	**10.** (a)	**11.** (b)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (a)	**16.** (b)
17. (a)	**18.** (d)	**19.** (d)	**20.** (b)	**21.** (a)	**22.** (a)	**23.** (b)	**24.** (a)
25. (b)	**26.** (c)	**27.** (a)	**28.** (b)	**29.** (b)	**30.** (b)	**31.** (a)	**32.** (c)
33. (b)	**34.** (b)	**35.** (c)					

8. C APPLICATION OF INTEGRATION TO COMMERCE AND ECONOMICS

We know that marginal function is obtained by differentiating the total function. Now, when Marginal function is given and initial values are given, then total function can be obtained with the help of integration.

8.C.1 Determination of cost function

If C denotes the total cost and $MC=\frac{dC}{dx}$ is the marginal cost, then we can write $C = C(x) = \int (MC)\,dx + k$, where k is the constant of integration, k, being the constant, is the fixed cost.

Example 8.C.1. The marginal cost function of manufacturing x units of a product is $5 + 16x – 3x^2$. The total cost of producing 5 times is Rs. 500. Find the total cost function.

Solution: Given, $MC = 5 + 16x – 3x^2$

$$\therefore \quad C(x) = \int \left(5+16x-3x^2\right) dx$$

$$= 5x + 16\frac{x^2}{2} - 3.\frac{x^3}{3} + k$$

$$C(x) = 5x + 8x^2 - x^3 + k$$

When x = 5, C(x) = C(5) = Rs. 500

or, 500 = 25 + 200 – 125 + k

This gives, k= 400

$$\therefore \quad C(x) = 5x + 8x^2 - x^3 + 400$$

Example 8. C.2. The marginal cost (MC) of a product is given to be a constant multiple of number of units (x) produced. Find the total cost function, if fixed cot is Rs. 5000 and the cot of producing 50 units is Rs. 5625

Solution: Here MC á x i.e $MC = k_1x$ (k_1 is a constant)

$$\therefore \quad \frac{dC}{dx} = k_1x \Rightarrow C=\int k_1x\,dx + k_2$$

$$\therefore \quad C = k_1\frac{x^2}{2} + k_2$$

Since fixed cost = Rs 5000 ∴ $x = 0 \Rightarrow C = 5000$

$$\therefore \quad 5625 = k_1\frac{2500}{2} + 5000$$

$\Rightarrow \qquad 625 = 1250\ k_1 \Rightarrow k_1 = \frac{1}{2}$

Hence $C = \frac{x^2}{4} + 5000$, is the required cost function.

8.C.3.2 Determination of Total Revenue Function

If R (x) denote the total revenue function and MR is the marginal revenue function, then

$$MR = \frac{d}{dx}[R(x)]$$

$\therefore \qquad R(x) = \int (MR)dx + k$ Where k is the constant of integration.

Also, when R (x) is known, the demand function can be found as $p = \frac{Rx}{x}$

Example 8.C.20 The marginal revenue function for a product is given by

$$MR = \frac{6}{(x-3)^2} - 4.$$

Find the total revenue function and the demand function.

Solution: $MR = \frac{6}{(x-3)^2} - 4$

$$\therefore \qquad R = \int \left| \frac{6}{(x-3)^2} - 4 \right| dx = \frac{6}{x-3} - 4x + k$$

$X = 0, R = 0 \Rightarrow k = -2$

$R = \frac{6}{x-3} - 4x - 2$, which is the required revenue function.

Now, $PR = \frac{R}{x} = \frac{6}{x(x-3)} - 4 - \frac{2}{x}$

$$= -\frac{6}{x(x-3)} - \frac{2}{x} - 4$$

$$= \frac{-6 - 2x + 6}{x(x-3)} - 4$$

$$= \frac{-2}{x-3} - 4 = \frac{2}{3-x} - 4$$

$\therefore$ The demand function is given by $p = \frac{2}{3-x} - 4.$

EXERCISE -8(C)

Choose the most appropriate option (a) (b) (c) or (d)

1. The fixed cost of a new product is Rs. 18000 and the variable cost per unit is Rs 550. If demand function p(x)=4000 -150x, find the break-even values.

   (a) 15, 8 (b) 7, 12 (c) 3, 17 (d) 5, 15

   Using the data (2-4) A company sells its product at Rs.60 per unit. Fixed cost for the company is Rs.18000 and the variable cost is estimated to be 25% of total revenue.

2. Determine: the total revenue function .

   (a) 70x (b) 60x (c) 90x (d) 100x

3. Determine the total cost function

   (a) 19000 + 6x (b) 20000+ 10 x (c) 18000 + 15x (d) 4000 + 5x

4. Determine the breakeven point

   (a) 600 (b) 400 (c) 700 (d) 1000

   Using the data (5-8) The total cost C(x) of a company as $C(x) = 1000 + 25x + 2x^2$ where x is the output.

5. Determine: the average cost

   (a) 1000/x + 25 + 2x (b) 1000/x + 20 + 2x

   (c) 1000/x + 30 + 3x (d) 1000/x + 25 + x

6. Determine the marginal cost.

   (a) 30 + 4x (b) 25 + 4x (c) 50 + 4x (d) 50 + 5x

7. Find the marginal cost when 15 units are produced,

   (a) 60 (b) 90 (c) 80 (d) 85

8. Find the actual cost of producing 15th unit.

   (a) 80 (b) 70 (c) 83 (d) 90

   Using the data (9-12).The total cost function of a firm is given as

   $C(x) = 0.002x^3 - 0.04x^2 + 5x + 1500$, *where x is the output*.

9. Determine: the average cost

   (a) $0.002x^2 - 0.04x + 5 + 1500/x$ (b) $0.002x^2 - 0.05x + 5 + 1500/x$

   (c) $0.002x^2 - 0.05x + 5 + 1000/x$ (d) $0.002x^2 - 0.05x + 5 + 500/x$

10. Determine the marginal average cost (MAC)

(a) $0.004x - 0.08 - 1500/x^2$ (b) $0.004x - 0.04 - 1500/x^2$

(c) $0.004x - 0.04 - 1000/x^2$ (d) $0.001x - 0.04 - 1500/x^2$

(11) Find the marginal cost.

(a) $0.06x^2 - 0.10x + 5$ (b) $0.06x^2 - 0.16x + 5$

(c) $0.06x^2 - 0.08x + 5$ (d) $0.05x^2 - 0.08x + 5$

(12) Find the rate of change of MC with respect to x.

(a) $0.012x - 0.10$ (b) $0.010x - 0.08$

(c) $0.012x + 0.08$ (d) $0.012x - 0.08$

13. The total cost function for a company is given by $C(x) = \frac{3}{4}x^2 - 7x + 27$. Find the level of output for which MC = AC

(a) 8 (b) 6 (c) 9 (d) 10

Using the data (14-17) The demand function for a monopolist is given by $x = 100 - 4p$, where x is the number of units of product produced and old and p is the price per unit.

14. Find total revenue function

(a) $25x - x^2/4$ (b) $25x + x^2/4$

(c) $25x - x^2/2$ (d) $5x - x^2/4$

15. Find average revenue function

(a) 25-x/6 (b) 25-x/4

(c) 5-x/4 (d) 25 + x/4

16. Find marginal revenue function

(a) 25-x/3 (b) 25-x/4 (c) 5-x/2 (d) 25-x/2

17. Find price and quantity at which MR= 0.

(a) 50, 12.5 (b) 70, 12.5 (c) 100, 12.5 (d) 70, 10

Using the data (18-21) A firm knows that the demand function for one of its products is linear. It also knows that it can sell 1000 units when the price is Rs. 4 per unit and it can sell 1500 units when the price is Rs. 2 per unit.

18. Find the demand function.

(a) 2000-250p (b) 2000-5p (c) 2000+5p (d) 2000-25p

19. Find the total revenue function

(a) $8 - x^2/250$ (b) $8x - x^2/50$ (c) $8x - x^2/250$ (d) $8x - x^2/25$

20. Find the average revenue function.

(a) 8-x/50 (b) 8-x/25 (c) 8+x/250 (d) 8-x/250

21. Find the marginal revenue function.

(a) 8 – x/12 (b) 8 – x/25 (c) 8 – x/125 (d) 8 + x/125

22. A company charge Rs. 15000 for a refrigerator on orders of 20 or less refrigerator. The charge is reduced on every set by Rs. 100 per piece for each piece ordered in excess of 20. Find the largest size order the company should allow so as to receive a maximum revenue.

(a) 85 (b) 80 (c) 100 (d) 70

23. A firm has the following demand and the average cost-functions:

$x = 480 - 20p$ and $AC = 10 + \frac{x}{15}$. Determine the profit maximizing output and price of the monopolist.

(a) 70, 25 (b) 60, 30 (c) 60, 25 (d) 70, 30

Using the data (24-25) The marginal cost of production is $MC = 20 - 0.04x + 0.003x^2$ where x is the number of units produced. The fixed cost is Rs. 7000.

24. Find the total cost function.

(a) $C = 20x - 0.02x^2 + 0.001x^3 + 7000$ (b) $C = -20x - 0.04x^2 + 0.001x^3 + 7000$

(c) $C = 20x + 0.02x^2 + 0.001x^3 + 7000$ (d) $C = 20x - 0.02x^2 + 0.001x^3 - 7000$

25. Find the average cost function.

(a) $AC = 20 - 0.02x + 0.001x^2 + \frac{7000}{x}$ (b) $AC = 20 - 0.02x + 0.001x^2 + \frac{7000}{x}$

(c) $AC = 20 - 0.02x + 0.001x^2 - \frac{7000}{x}$ (d) $AC = 20 + 0.02x + 0.001x^2 + \frac{7000}{x}$

Using the data (26-27) The marginal cost function of manufacturing x units of a product is given by $MC = 3x^2 - 10x + 3$. The total cost of producing one unit of the product is Rs. 7.

26. Find the total cost function

(a) $C = x^3 + 5x^2 + 3x + 7$ (b) $C = x^3 - 5x^2 + 3x + 7$

(c) $C = x^3 + 5x^2 - 3x + 7$ (d) $C = x^3 - 5x^2 - 3x - 7$

27. Find the average cost function.

(a) $AC = x^2 - 5x + 3 + \frac{7}{x}$ (b) $AC = x^2 - 5x + 3 - \frac{7}{x}$

(c) $AC = x^2 - 5x - 3 + \frac{7}{x}$ (d) $AC = x^2 + 5x - 3 + \frac{7}{x}$

Using the data (28-29). The marginal cost function of a commodity is given by $MC = \frac{14000}{\sqrt{7x+4}}$ and the fixed cost is Rs. 18000.

28. Find the total cost function.

(a) $C = 4000\sqrt{7x+4} + 10000$ (b) $C = 4000\sqrt{7x+4} - 10000$

(c) $C = 400\sqrt{7x+4} + 10000$ (d) $C = 4000\sqrt{7x^2+4} + 1000$

29. Find average cost of producing 3 units of the products.

(a) $AC = \frac{4000}{x}\sqrt{7x+2}\ \frac{10000}{x}$ (b) $AC = \frac{4000}{x}\sqrt{7x+4} + \frac{10000}{x}$

(c) $AC = \frac{4000}{x}\sqrt{7x+4} + \frac{10000}{x^2}$ (d) $AC = \frac{4000}{x}\sqrt{7x+4} + \frac{1000}{x}$

30. The marginal revenue of a function MR = 7-4x-x². Find the total Revenue.

(a) $R = 7x - \frac{4x^2}{2} - \frac{x^3}{3}$ (b) $R = 7x + \frac{4x^2}{2} - \frac{x^3}{3}$

(c) $R = 7x - \frac{4x^2}{2} + \frac{x^3}{3}$ (d) $R = 7x + \frac{4x^2}{2} + \frac{x^3}{3}$

ANSWERS

Set C

1 (a)	2 (b)	3 (c)	4 (b)	5 (a)	6 (b)
7 (d)	8 (c)	9 (a)	10 (b)	11 (c)	12 (d)
13 (b)	14 (a)	15 (b)	16 (d)	17 (a)	18 (a)
19 (c)	20 (d)	21 (c)	22 (a)	23 (c)	24 (a)
25 (a)	26 (b)	27 (a)	28 (a)	29 (b)	30 (a)

NOTES

CHAPTER 9

NUMBER SERIES, CODING DECODING AND ODD MAN OUT SERIES

LEARNING OBJECTIVES

- This Section deals with questions on which series or letters in some order, Coding and decoding
- These terms of the series or letters are follows certain pattern throughout

9.1 SERIES

Series Classified into Two Types, Namely

A. Number Series

B. Alphabet Series

A. NUMBER SERIES

Case 1: Missing terms of the series

In these type the questions we have to identify the missing term of the series according to a specific pattern of the series rule to form its code. The students are required to detect the missing number of the series and answer the questions accordingly.

Example 1: Find the missing term of the series 2, 7, 16, ______ , 46, 67, 92

Explanation: Here the terms of the series are +5, +9, +13, +17, +21 , +25…

Thus, 2 + 5 = 7; and 7 + 9 = 16 …

So missing term = 16 + 13 = 29

Example 2: Find the wrong terms of the series 9, 29, 65, 126, 217, 344

Explanation: 2^3+1; $3^3 + 1$; $4^3 + 1$; $5^3 + 1$; $6^3 + 1$; $7^3 + 1$

Here 29 is wrong term of series

Example 3: Find the missing term of the series 1,9, 25, 49, 81, 121, ...

Solution: The given terms of the series consists square of consecutive odd number 1^2, 3^2, 5^2, 7^2, ...

So missing value = 13^2 = 169

B. ALPHABET SERIES

Alphabet series consists of letters of the alphabets placed in a specific pattern. For example, the series are in the following order of the numbers.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Example 4: Find the next term of the series BKS, DJT, FIU, HHV?

Explanation: In each term, the first letter is moved two steps forward, the second letter one step backward and third letter one step forward to obtain the corresponding letter of the next term. So the missing term is JGW.

C. LETTER SERIES:

These type of question usually consist of a series of small letters which follow a certain pattern. However some letters are missing from the series. These missing letters are then given in a proper sequence as one of the alternatives.

Example 5: aab, _____, aaa, bba, _____

(a) baa (b) abb (c) bab (d) aab

1) The first blank space should be filled in by 'b' so that we have two a's by two b's.

2) The second blank place should be `a', so that we have three a's followed by three b's.

3) The last space must be filled in by 'a'.

4) Thus we have two possible answers – 'baa'and 'bba'.

5) But only 'baa'appears in the alternatives.

So the answer (a) is correct.

9.2 CODING AND DECODING

Before transmitting, the data is encoded and at receiver side encode data is decoded in order to obtain original data by determining common key in encoded data.

The Coding and Decoding is classified into the following types.

Type 1: Letter Coding

Type 2: Number Coding

Type 1: Letter Coding

In these type the real alphabets in a word are replaced by certain other alphabets according to a specific rule to form its code. The candidate is required to detect the common rule and answer the questions accordingly.

Case1: To form the code for another word

Example 6: If in a certain language MYSTIFY is coded as NZTUJGZ, how is MENESIS coded in that language?

Explanation: Clearly, each letter in the word MYSTIFY is moved one step forward to obtain the corresponding letter of the code.

M Y S T I F Y

+1 ↓

N Z T U J G Z

So, in MENESIS, N will be coded as O, E as F, M as N and so on. Thus, the code becomes NFOFTJT.

Example 7: If TAP is coded as SZO, then how is FRIEND coded?

Explanation: Clearly each letter in the word TAP is moved one step backward to obtain the corresponding letter of the code.

S Z O

-1 ↑

T A P

Thus, in FRIEND, F will be coded as E, R as Q , I as H, E as D, N as M and D as C. So, the code becomes EQHDMC.

Example 8: In a certain code, MENTION is written as LNEITNO. How is PRESENT written in that code?

Explanation: Clearly, to obtain the code, the first letter of the word MENTION is moved one step backward and the remaining letters are Reversed in order, taking two at a time. So, in PRESENT, P will be coded as O, and the sequence of the remaining letter in the code would be ERESTN. Thus the code becomes OERESTN. Hence, The answer is OERESTN.

Case 2: To find the word by analysing the given code (DECODING)

Example 9: If in a certain language CARROM is coded as BZQQNL, which word will be coded as HORSE?

Explanation: Each letter of the word is one step ahead of the corresponding letter of the code

B Z Q Q N L	H O R S E
| | | | | |	| | | | |
C A R R O M	I P S T F

So, H is coded as I, O as P, R as S, S as T and E as F. HORSE is coded a IPSTF.

Type 2: Number Coding

In these questions, either numerical code values are assigned to a word or alphabetical code letters are assigned to the numbers. The candidate is required to analyse the code as per the directions.

Case 1: When a numerical code values are assigned to words.

Example 10: If in a certain language A is coded as 1, B is coded as 2, and so on, how is AICCI is coded in that code?

Explanation: As given the letters are coded as

A	B	C	D	E	F	G	H	I
1	2	3	4	5	6	7	8	9

So in AICCI, A is coded as 1, I as 9,and C as 3. Thus, AICCI is coded as 19339.

Example 11: If PAINT is coded as 74128 and EXCEL is coded as 93596, then how would you encode ANCIENT ?

Explanation: Clearly, in the given code, the alphabets are coded as follows:

P	A	I	N	T	E	X	C	L
7	4	1	2	8	9	3	5	6

So, in ANCIENT, A is coded as 4, N is coded as 2, C as 5, I is coded as 1, E as 9, and T as 8. Hence, the correct code is 4251928.

Case 2: Number to letter coding.

Example 12: In a certain code, 2 is coded as P, 3 as N, 9 as Q, 5 as R, 4 as A and 6 as B. How is 423599 coded in that code?

Explanation: Clearly as given, 4 as A, 2 as P, 3 as N and 5 is coded as R, 9 as Q. So, 423599 is coded as APNRQQ.

9.3 ODD MAN OUT

Classification means 'to assort the items' of a given group on the basis of a certain common quality they possess and then spot the stranger or 'odd one out'.

These questions are based on words, letters and numerals. In these types of problems, we consider the defining quality of particular things. In these questions, four or five elements are given, out of which one does not belong to the group. You are required to find the 'odd one'.

Example 13: January, May, July, November

(a) January (b) May (c) July (d) November

Explanation: All the months above are 31 days , whereas , November 30 days

Answer: (d)

Example 14: 10, 14, 16, 18, 23, 24 and 26

(a) 26 (b) 17 (c) 23 (d) 9

Explanation: Each of the above series are even number, except 23.

Answer: (c)

Example 15: 6, 9, 15, 21, 24, 26, 30

(a) 9 (b) 26 (c) 24 (d) 30

Explanation: All are multiples of 3, except 26, answer (b)

Answer: (b)

Example 16: 1, 5, 14, 30, 51, 55, 91

(a) 5 (b) 55 (c) 51 (d) 91

Explanation: Each pattern is 1^2, $1^2 + 2^2$, $1^2 + 2^2 + 3^2$, $1^2 + 2^2 + 3^2 + 4^2$, $1^2 + 2^2 + 3^2 + 4^2 + 5^2$, $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$

But 51, is not of the form.

Answer: (c)

Example 17: 16, 25, 36, 62, 144, 196, 225

(a) 36 (b) 62 (c) 196 (d) 144

Explanation:

Each of the number except 62, is a perfect square.

Answer: (b)

EXERCISE 9(A)

(Note: Questions are taken from previous exam questions papers of Competitive exams like SSC, RRB, MAT, UPSC etc.)

Choose the most appropriate answer (a) or (b) or (c) or (d).

1) 6, 11, 21, 36, 56 ?
   (a) 42 (b) 51 (c) 81 (d) 91
2) 10 , 100, 200, 310 ?
   (a) 400 (b) 410 (c) 420 (d) 430
3) 11, 13, 17, 19, 23, 25, 29 ?
   (a) 33 (b) 27 (c) 31 (d) 49
4) 6, 12, 21, 33 ?
   (a) 33 (b) 38 (c) 40 (d) 48
5) 2, 5, 9, 14, ? , 27
   (a) 20 (b) 16 (c) 18 (d) 24
6) 6, 11, 21, ? , 56, 81
   (a) 42 (b) 36 (c) 91 (d) 51
7) 10, 18, 28, 40, 54, ?, 88
   (a) 70 (b) 86 (c) 87 (d) 98
8) 120, 99, ?, 63, 48, 35
   (a) 80 (b) 36 (c) 45 (d) 40

9) 22, 24, 28, 36, ? , 84

(a) 44 (b) 52 (c) 38 (d) 54

10) 4832, 5840, 6848, 7856 ?

(a) 8864 (b) 8815 (c) 8846 (d) 8887

11) 10, 100, 200, 310, 430 ?

(a) 560 (b) 540 (c) 550 (d) 590

12) 28, 33, 31, 36, 34 ?

(a) 38 (b) 39 (c) 40 (d) 42

13) 120, 80, 40, 45, ?, 15

(a) 15 (b) 20 (c) 25 (d) 30

14) 2, 15, 41, 80, 132 ?

(a) 184 (b) 144 (c) 186 (d) 197

15) 6, 17, 39, ?, 116

(a) 72 (b) 75 (c) 85 (d) 80

16) 1, 4, 10, 22, ?, 94

(a) 46 (b) 48 (c) 49 (d) 47

17) 4, 9, 25, 49, ? , 169, 289, 361

(a) 120 (b) 121 (c) 122 (d) 164

18) 4, 12, 36, ? , 324

(a) 107 (b) 109 (c) 108 (d) 110

19) 1, 1, 4, 8 , 9, ? , 16, 64

(a) 27 (b) 28 (c) 32 (d) 40

20) 5760, 960, 192, ? 16, 8

(a) 47 (b) 48 (c) 52 (d) 50

21) 1, 2, 6, 7, 21, 22, 66, ? , 201

(a) 69 (b) 68 (c) 67 (d) 69

22) 48, 24, 96 , ? 192

(a) 48 (b) 47 (c) 44 (d) 54

23) 165, 195, 255, 285, ?, 375

(a) 345 (b) 390 (c) 335 (d) 395

24) 2, 3, 3, 5, 10, 13, 39, ?, 172, 177

(a) 42 (b) 44 (c) 43 (d) 40

25) 7, 26, 63, 124, 215, ?, 511

(a) 342 (b) 343 (c) 441 (d) 421

26) 3, 7, 15, 31, ? 127

(a) 62 (b) 63 (c) 64 (d) 65

27) 8, 28, 116, 584, ?

(a) 1752 (b) 3502 (c) 3504 (d) 3508

28) 6, 13, 28, 59, ?

(a) 122 (b) 114 (c) 113 (d) 112

29) 2, 7, 27, 107, 427, ?

(a) 1707 (b) 4027 (c) 4207 (d) 1207

30) 5, 2, 7, 9, 16, 25, 41, ?

(a) 65 (b) 66 (c) 67 (c) 68

31) In a certain language, MADRAS is coded NBESBT, how DELHI is coded in that code?

(a) EMMJI (b) EFMIJ (c) EMFIJ (d) JIFEM

32) If RAMAN is written as 12325 and DINESH as 675489 how HAMAM is written?

(a) 92323 (b) 92233 (c) 93233 (d) 93292

33) If RED is coded as 6720 then GREEN would be coded as

(a) 9207716 (b) 167129 (c) 1677209 (d) 1672091

34) If A = 1, FAT = 27, FAITH = ?

(a) 44 (b) 45 (c) 46 (d) 36

35) If BROTHER is coded 2456784, SISTER coded as 919684, what is coded for BORBERS?

(a) 2542849 (b) 2542898 (c) 2454889 (d) 2524889

36) If DELHI is coded 73541 and CALCUTTA as 82589662, How can CALICUT be coded?

(a) 5279431 (b) 5978213 (c) 8251896 (d) 8543962

37) If CLOCK is coded 34235 and TIME is 8679, what will be code of MOTEL?

(a) 72894 (b) 77684 (c) 72964 (d) 27894

38) If PALE is coded as 2134 and EARTH is coded as 41590, how is PEARL is coded ?

(a) 29530 (b) 24153 (c) 25430 (d) 254313

39) If LOSE is coded as 1357 and GAIN is coded as 2468, what do figure 82146 stands for?

(a) NGLAI (b) NGLIA (c) GNLIA (d) GNLIA

40) If MEKLF is coded as 91782 and LLLJK as 88867, how can IHJED is coded as?

(a) 97854 (b) 64512 (c) 54610 (d) 75632

41) If in a certain code language NAME is written as 4258 then what is coded as MEAN ?

(a) 2458 (b) 5842 (c) 8524 (d) 5824

42) If GOLD is written as IQNF, how WIND can be written as code?

(a) YKPF (b) VHCM (c) XJOE (d) DNIW

43) If ROSE is written as TQUG, how BISCUIT can be written in that code?

(a) DKUEWKV (b) CJTDVJU (c) DKVEWKV (d) DKUEWKY

LETTER: C Z N V R S W F D

CODE DIGIT: 8 6 4 7 2 9 3 5 1

(Q. No. 44-46) In each of the following questions find out the correctly coded alternative from amongst the given four alternatives (a), (b), (c), (d).

44) ZDRCVF

(a) 612875 (b) 619875 (c) 612845 (d) 612835

45) WNCSZV

(a) 348267 (b) 318267 (c) 348957 (d) 348967

46) RDNFVS

(a) 21679 (b) 216549 (c) 214579 (d) 218579

47) If DELHI is coded as CCIDD, how would you encode BOMBAY?

(a) AJMTVT (b) AMJXVS (c) MJXVSU (d) WXYZAX

48) In a certain code, RIPPLE is written as 613382 and LIFE is written as 8192. How is PILLER written in that code?

(a) 318826 (b) 318286 (c) 618826 (d) 338816

49) If PALAM could be given the code number 43, what code number can be given to SANTACRUZ?

(a) 123 (b) 85 (c) 120 (d) 125

Directions: The number in each question below is to be codified in the following code:

Digit	7	2	1	5	3	9	8	6	4
Letter	W	L	M	S	I	N	D	J	B

50) 184632

(a) MDJBSI (b) MDJBIL (c) MDJBWL (d) MDBJIL

51) In a certain code '256' means 'you are good', '637' means 'we are bad' and '358' means 'good and bad'. Which of the following represents 'and' in that code?

(a) 2 (b) 5 (c) 8 (d) 3

Directions: Find odd man out of the following (52-61):

52) 3, 5, 7, 15, 17, 19

(a) 15 (b) 17 (c) 19 (d) 7

53) 10, 14, 16, 18, 23, 24, 26

(a) 26 (b) 23 (c) 24 (d) 18

54) 1, 4, 9, 16, 24, 25, 36

(a) 9 (b) 24 (c) 25 (d) 36

55) 41, 43, 47, 53, 61, 71, 83, 75

(a) 75 (b) 73 (c) 71 (d) 53

56) 16, 25, 36, 73, 144, 196, 225

(a) 36 (b) 73 (c) 196 (d) 225

57) 1, 4, 9, 16, 19, 36, 49

(a) 19 (b) 9 (c) 49 (d) 16

58) 1, 5, 14, 30, 49, 55, 91

(a) 49 (b) 30 (c) 55 (d) 91

59) 835, 734, 642, 751, 853, 981, 532

(a) 751 (b) 853 (c) 981 (d) 532

60) 4, 5, 7, 10, 14, 18, 25, 32

(a) 7 (b) 14 (c) 18 (d) 33

61) 52, 51, 48, 43, 34, 27, 16

(a) 27 (b) 34 (c) 43 (d) 48

ANSWERS

EXERCISE-9 A

1. (c)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (a)
11. (a)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (a)	**16.** (a)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (b)
21. (c)	**22.** (a)	**23.** (a)	**24.** (c)	**25.** (a)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (b)
31. (b)	**32. (a)**	**33. (c)**	**34. (a)**	**35. (a)**	**36. (c)**	**37. (a)**	**38. (b)**	**39. (a)**	**40. (c)**
41. (d)	**42. (a)**	**43. (a)**	**44. (a)**	**45. (d)**	**46. (c)**	**47. (b)**	**48. (a)**	**49. (a)**	**50. (d)**
51. (c)	**52. (a)**	**53. (b)**	**54. (b)**	**55. (a)**	**56. (b)**	**57. (a)**	**58. (a)**	**59. (a)**	**60. (c)**
61. (b)									

NOTES

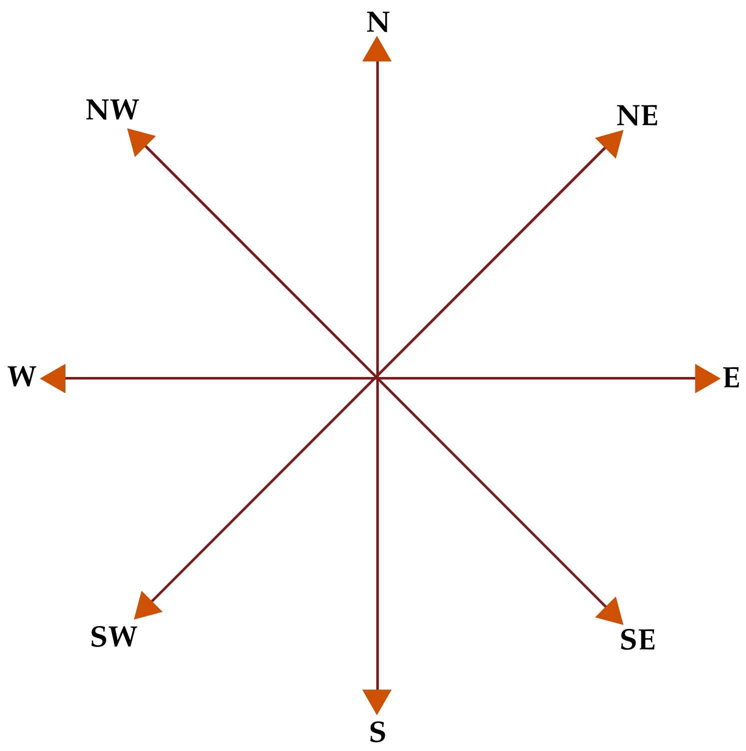
CHAPTER BA

DIRECTION SENSE TEST

INTRODUCTION

After reading this chapter, students will be able to understand:

- In this test, the questions consist of a sort of direction puzzle. A successive follow-up of direction is formulated and the student is required to ascertain the final direction. The test is meant to judge the ability to trace and follow correctly and sense the direction correctly.
- The adjoining figure shows the four main directions (North N, South S, East E, and West W) and four cardinal directions (North East (NE), North West (NW), South East (SE), South West (SW) to help the students know the directions.



Always Remember:

Direction mentioned (given)	Direction indicated
Left + left	Down
Left + right	Up
Right + left	Up
Right + right	Down
Up + left	Left
Up + right	Right
Down + left	Right
Down + right	Left

Examples:

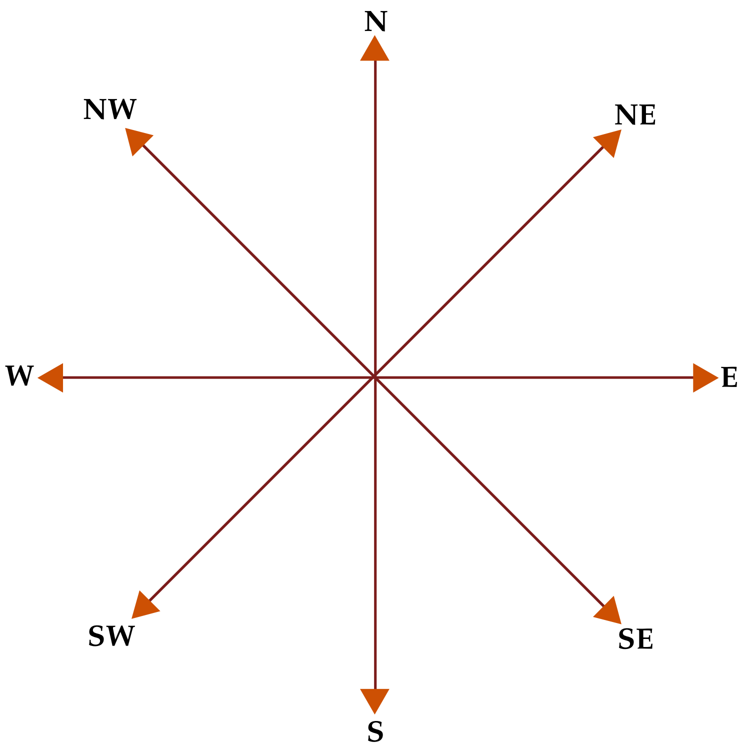
1. A man starts from a point and walks 2 km towards North, turns towards his right and walks 2 km, turns right again and walks. What is the direction now be is facing?

   (a) South (b) South-East

   (c) North (d) West

   Explanation: (a) The diagram given below helpful solving the questions and Direction Test.

   South.

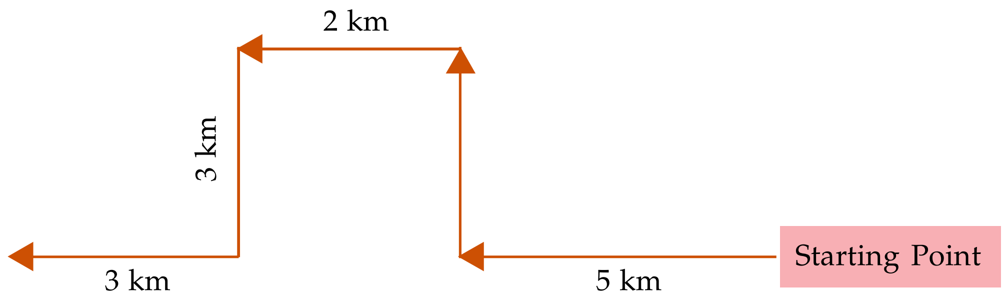


2. Ramu walks 5 kms starting from his house towards west then turns right and walks 3 km. Thereafter she takes left turn and walks 2 km. Further, he turn left and walks 3 km. Finally, he turns right and walks 3 kms. In what direction he is now from his house?

   (a) West (b) North

   (c) South (d) East

   Explanation



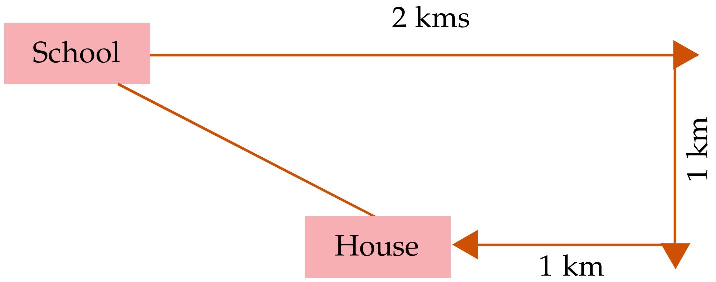
It's clear from the diagram Ramu is to the West of his house.

3. Gopal started walking 2 kms straight from his school. Then he turned right and walked 1 km. Again he turned right and walked 1 km to reach his house. If his house is sourth-east from his school, then in which direction did Gopal start walking from the school?

   (a) East (b) West

   (c) South (d) North

   Explanation



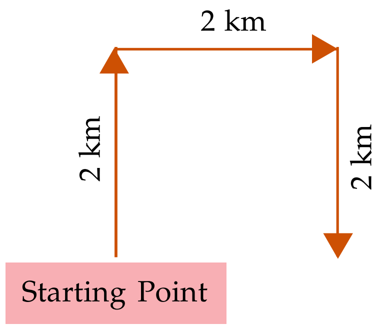
From the diagram that Gopal Started walking towards East from the school.

4. A man starts from a point, walks 2 kms towards north, turns towards his right and walks 2 kms, turns right again and walks. What is the direction now he is facing?

   (a) South (b) East

   (c) North (d) West

Explanation:



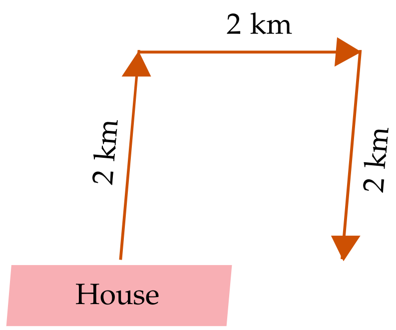
Based on the diagram the person facing towards south.

5. Janki started from her house and walked 2 km towards North. Then she took a right turn and covered one kilometre. Then she took again a right turn and walked for 2 kms. In what direction is she going?

   (a) North (b) East

   (c) South (d) West

Explanation:



Based on the diagram; it's clear that Janki is going towards South.

EXERCISE – 10(A)

(Note: Questions are taken from previous exam questions papers of Competitive exams like SSC, RRB, MAT, UPSC etc.)

Choose the appropriate answer (a) or (b) or (c) or (d)

1. Mohan starts from point A and walks 1 km towards south, turns left and walks 1km. Then he turns left again and walks 1 km. Now he is facing.

   (a) East (b) West (c) North (d) South-West

2. Suresh starts from a point, walks 2 miles towards south, turns right and walks $1^1/_2$ miles, turns left and walks ½ miles and then he turns back. What is the direction he is facing now?

   (a) East (b) West (c) South (d) North

3. A man starts from a point, walks 4 miles towards north and turns left and walks 6 miles, turns right and walks for 3 miles and again turns right and walks 4 miles and takes rest for 30 minutes. He gets up and walks straight 2 miles in the same direction and turns right and walks one mile. What is the direction he is facing?

   (a) North (b) South (c) South-East (d) West

4. Arun started from point A and walked 10 kms East to point B, then turned to North and walked 3 kms to point C and then turned West and walked 12 kms to point D, then again turned South and walked 3 kms to point E. In which direction is he from his start point?

   (a) East (b) South (c) West (d) North

5. A starts from a point and walks 5 kms north, then turns left and walks 3 kms. Then again turns left and walks 5 km. Point out the direction in which he is going now.

   (a) North (b) South (c) East (d) West

6. A rat run 20 kms towards East and turns to right runs 10 kms and turns to right runs 9 kms and again turns to left runs 5 kms and then turns to left runs 12 kms and finally turns to left and runs 6 kms. Now what direction is the rat facing?

   (a) East (b) North (c) West (d) South

7. A driver left his village and drove North for 20 kms, after which he stopped for breakfast. Then he turned left and drove another 30 kms, when he stopped for lunch. After some rest, he again turned left and drove 20 kms before stopping for evening tea. Once more he turned left and drove 30 kms to reach the town where he had supper. After evening tea in which direction did he drive?

   (a) West (b) East (c) North (d) South

8. A man is facing East, then he turns left and goes 10 m, then turns right and goes 5 meters then goes 5 meters to the South and from there 5 meters to West. In which direction is be from his original place?

   (a) East (b) West (c) North (d) South

9. From her home Prerna wishes to go to school. From home she goes towards North and then turns left and then turns right, and finally she turns left and reaches school. In which direction her school is situated with respect to her home?

   (a) North-East (b) North-West (c) South-East (d) South-West

10. A child walks 25 feet towards North, turns right and walks 40 feet, turns right again and walks 45 feet. He then turns left and walks 20 feet. He turns left again walks 20 feet. Finally, he turns to his left to walks another 20 feet. In which direction is the child from his starting point?

    (a) North (b) South (c) West (d) East

11. Raju facing North and moves 20 kms, then he turned to his right and moves 20 kms and then he moves 10 kms in North-East, then he turned to his right and moves 20 kms and then he turned to his right and moves 20 kms and again he turned to his left and moves 20 kms. Now in which direction Raju is facing?

    (a) South-East (b) North-East (c) South-West (d) North-West

12. K is a place which is located 2 kms away in the north-west direction from the capital P. R is another place that is located 2 kms away in the south-west direction from K. M is another place and that is located 2 kms away in the north-west direction from R. T is yet another place that is located 2 kms away in the south-west direction from M. In which direction is T located in relation to P?

    (a) South-West (b) North-West (c) West (d) North

13. Babu is Rahim's neighbour and his house is 200 meters away in the north-west direction. Joseph is Rahim's neighbour and his house is located 200 meter away in the south-west direction. Gopal is Joseph's neighbour and he stays 200 meters away in the south-east direction. Roy is Gopal's neighbour and his house is located 200 meters away in the north-east direction. Then where is the position of Roys' house in relation to Babu's ?

    (a) South-East (b) South-West (c) North (d) North-East

14. A tourist drives 10 km towards west and turns to left and takes a drive of another 4 km. He then drives towards east another 4 km and then turns to his right and drives 5 km . Afterwards he turns to his left and travels 6 km. In which direction is je from the starting point?

    (a) North (b) East (c) West (d) South

15. A man started walking West. He turned right, then right again and finally turned left. Towards which direction was he walking now?

    (a) North (b) South (c) West (4) East

16. One evening, Raja started to walk toward the Sun. After walking a while, he turned to

his right and again to his right. After walking a while, he again turned right. In which direction is he facing?

(a) South (b) East (c) West (d) North

17. Five boys A, B, C, D and E are sitting in a park in a circle. A is facing South-West, D is facing South-East, B and E are right opposite A and D respectively and C is equidistant between D and B. Which direction is C facing?

(a) West (b) South (c) North (4) East

18. If a man on a moped starts from a point and rides 4 kms South then turns left and rides 2 kms and turn again to the right to ride to go more towards which direction is he moving ?

(a) North (b) West (c) East (d) South

19. A man starts from a point, walk 8 kms towards North, turns right and walks 12 kms, turns left and walks 7 km turns and walks 20 kms towards South, turns right and walks 12 kms. In which direction is he from the starting point?

(a) North (b) South (c) West (d) East

20. Daily in the morning the shadow of Gol Gumbaz falls on Bara Kaman and in the evening the shadow of Bara Kaman falls on Gol Gumbaz exactly. So in which direction is Gol Gumbaz to Bara Kaman?

(a) Eastern side (b) Western side (c) Northern side (d) Southern side

21. Ashok went 8 kms South and turned West and walked 3 kms again he turned North and walked 5 kms. He took a final turn to East and walked 3 kms. In which direction was Ashok from the starting point?

(a) East (b) North (c) West (d) South

22. If X stands on his head with his face towards south, to which direction will his left hand point ?

(a) East (b) West (c) North (d) South

23. I drove East for 5 miles then drove North 3 miles, then turned to my left and drove for 2 miles and again turned to my left. Which direction am I going now?

(a) South (b) North (c) West (d) North-west

24. If A stands on his head with his face towards north. In which direction will his left hand point ?

(a) North-East (b) North (c) East (d) North-West

25. A car travelling from south covers a distance of 8 kms, then turns right and runs another 9 kms and again turns to the right and was stopped. Which direction does it face now?

(a) South (b) North (c) West (d) East

26. A taxi driver commenced his journey from a point and drove 10 kms toward north and turned to his left and drove another 5 kms. After waiting to meet a friend here, he turned to his right and continued to drive another 10 km. He has covered a distance of 25 kms so far, but in which direction would he be now?

(a) South (b) North (c) East (d) South-east

27. A walks 3 kms northward and then he turns left and goes 2 kms. He again turns left and goes 3 kms. He turns right and walks straight. In which direction is he walking now?

(a) East (b) West (c) North (d) South

28. Áwalks southeards, then turns right, then left and then right. In which direction is he from the starting point?

(a) South (b) East (c) West (d) North

29. A man starts from a point, walks 15 metres towards East, turns left and walks 10 metres, turns right again and walks. Towards which direction is he now waking?

(a) North (b) East (c) West (d) South

30. 30. A boy starts walking towards west, after a while walking he turns right and after walking for 20 minutes. Towards which direction he is walking now ?

(a) West (b) North (c) South (d) East

31. I stand with my right hand extended side-ways towards South. Towards which direction will his back be ?

(a) North (b) West (c) East (d) South

32. If a person moves 4 kms towards west, then turns right and moves 3 kms and then turns right and moves 6 kms, which is the directions in which he is now moving ?

(a) East (b) West (c) North (d) South

33. If Mohan sees the rising sun behind the temple and the setting sun behind the railway station from his house, what is the direction of the temple from the railway station?

(a) South (b) North (c) East (d) West

34. Laxman went 15 kms to North then he turned West and covered 10 kms. Then he turned south and covered 5 kms. Finally turning to East he covered 10 kms. In which direction he is from his house?

(a) East (b) West (c) North (d) South

35. A man starts from a point, walks 4 miles North, turns to his right and walks 2 miles, again turns to his right and walks 2 miles, again turns to his right and walks 2 miles. In

which direction would he be now from into starting point.

(a) North (b) South (c) East (d) West

36. I started walking down a road in the morning facing the Sun. After walking for sometime I turned to my left. Then I turned to my right. In which direction was I going then ?

(a) East (b) West (c) North (d) South

37. Lakshmi walked 2 kms north from her house and took a turn to left and continued to walk another one kilometre and finally she turned left and reached the school. Which direction is she facing now?

(a) West (b) North (c) South (d) North

38. You are going straight, first eastwards, then turn to the right, then right again, then left. In which direction would you be going now?

(a) East (b) West (c) South (d) North

39. If Ahmed travels towards North from his house, then to left, then to South covering equal distances in each direction to reach Sohan's house, in which direction is Ahmed's house now?

(a) East (b) South (c) North (d) West

40. Raja comes towards North, turn right, then right again and then go to the left. In which direction is Raja now?

(a) South (b) East (c) West (d) North

41. Roopa starts from a point and walks 15 metres towards west, turns left and walks 12 metres, turns right again and walks. What is the direction she is now facing?

(a) South (b) West (c) East (d) North

42. A man starts his journey facing the sun early morning. He then turns right and walks 2 kms. He then walks 3 kms after turning right again. Which is the direction he is facing now?

(a) North-East (b) North (c) West (d) South

43. Roy walks 2 kms to East, then turns North-West and walks 3 kms. Then he turns South and walks 5 kms. Then again he turns West and walks 2 kms. Finally he turns North and walks 6 kms. In which direction, is he from the starting point?

(a) South-West (b) South-East (c) North-West (d) North-East

44. Seeta starts from a point, walks 2 kms towards north, turns towards her right and walks 2 kms, turns right again and walks. What is the direction she is facing now?

(a) East (b) West (c) South (d) North

45. Shyam was facing East. He walked 5 kms forward and then after turning to his right walked 3 kms. Again he turned to his right and walked 4 km. After this he turned back. Which direction was he facing at that time?

(a) East (b) West (c) North (d) South

46. Raju is standing facing north. He goes 30 metres ahead and turns left and goes for 15 metres. Now he turns right and goes for 50 metres and finally turns to his right and walks. In which direction is he heading?

(a) North (b) East (c) South (d) West

47. Sanmitra starts from his house and walks 3 kms towards north. Then he turns right and walks 2 kms and then turns right and walks 5 kms, then turns right and walks 2 kms and then again turns right and walks 2 kms. Which direction is he facing now?

(a) North (b) South (c) West (d) East

48. Raju is Ramu's neighbour and he stays 100 metres away towards southeast. Venu is Raju's neighbour and he stays 100 metres away towards southwest. Khader is Venu's neighbour and he stays 100 metres away towards, north-west. Then where is the position of Khader's home in relation to Ramu's?

(a) South-East (b) South-West (c) North-West (d) East

49. Ramesh walked 3 kms, towards West and turned to his left and walked 2 kms. He, then turned to his right and walked 3 kms. Finally, he turned to his right again and walked another 2 kms. In which direction is Ramesh from his starting point now?

(a) East (b) West (c) North (d) South

50. Deepa starts walking north towards and after a while she turns to her right. After walking some distance, she turns to his left and walks a distance of 1 km. She then urns to her left again. In which direction she moving now?

(a) North (b) West (c) East (d) South

51. Raman starts walking in the morning facing the Sun. After sometime, he turned to the left later again he turned to his left. At what direction is Raman moving now?

(a) East (b) West (c) South (d) North

52. Kamal starts walking towards North, then turns left and cover some disstance, then he turns towards right and walks. After some time he turns to his right and then turns left. In which direction Kamal is walking now

(a) East (b) South (c) North (d) South-East

53. X walks southwards and then turns right, then left and then right,. In which direction is he moving now?

(a) South (b) North (c) West (d) South-West

54. A man started to walk East. After moving a distance, he turned to his right. After moving a distance, he turned to his right again. After moving a little he turned in the end to his left. In which direction was he going now.?

(a) North (b) South (c) East (d) West

ANSWERS: EXERCISE 10(A)

1. (c)	**2.** (d)	**3.** (b)	**4.** (c)	**5.** (b)
6. (b)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (d)
11. (a)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (a)
16. (d)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (a)
21. (d)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (a)
26. (b)	**27.** (b)	**28.** (a)	**29.** (b)	**30.** (b)
31. (b)	**32.** (a)	**33.** (c)	**34.** (c)	**35.** (a)
36. (a)	**37.** (c)	**38.** (c)	**39.** (a)	**40.** (b)
41. (b)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (a)
46. (b)	**47.** (a)	**48.** (b)	**49.** (b)	**50.** (b)
51. (b)	**52.** (c)	**53.** (c)	**54.** (b)	

NOTES

CHAPTER 11

SEATING ARRANGEMENTS

LEARNING OBJECTIVES

- To understand the Logical statements involved in the Seating Arrangements.
- To understand the types of Seating Arrangements.

The process of making a group of people to sit as per a prefixed manner is called Seating Arrangement. In these of questions, some conditions are given on the basis of which students are required to arrange objects, either in a row or in a circular order.

INTRODUCTION

11.1 BASED ON VARIOUS PATTERN OF SITTING ARRANGEMENTS ARE CLASSIFIED INTO

1) Linear Arrangements
2) Circular Arrangements
3) Polygon Arrangements

Here we are limited to our topic linear and circular arrangements only. While making arrangements, it should be noted that all the conditions given are complied with. These type of questions generally involve five to eight individuals arranged in a certain manner or pre-conditions. They may have to be arranged in a circle or in a row accordingly.

Sometimes these questions are made more difficult by allowing an individual to a particular position with some conditions.

General instructions to Solve Seating Arrangement Questions are as follows.

1) First of all take a review on the given information. After performing this step, you would get an idea of the situation of people or objects.
2) Next, determine the usefulness of each information's and classify them accordingly into 'definite information', 'comparative information' and 'negative information'.
3) When the place of any objects or persons is definitely mentioned then we say that it is a definite information, X is sitting on the right end of the bench.
4) When the place of any object or person is not mentioned definitely but mentioned only in the comparison of another person or object, then we say that it is a comparative information.

   Example 1: A is sitting second to the right of E. This type of information can be helpful when we can get the definite information about E.

5) A part of definite information may consist of negative information. A negative information does not tell us anything definitely but it gives an idea to eliminate a possibility.

Example 2: C is not sitting on the immediate left of A.

11.2 TYPE-1 LINEAR ARRANGEMENT

In this type of arrangement, we arrange objects or persons in a line or row. The arrangement is done only on one 'axis' and hence, the position of persons or objects assumes importance in terms of order like positions. In this type of arrangement, we take directions according to our left and right.

Steps to Solve the Linear Arrangements:

(a) Identify the number of objects and their names.

(b) Use pictorial method to represent the people or objects and their positions.

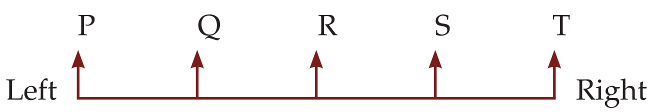
(c) Arrange the information with relevant facts and their positions and try to find out the solution.

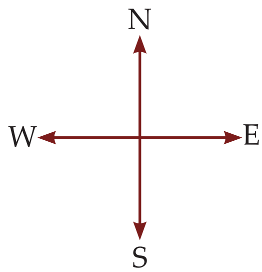
(d) Answer the questions based on the arrangement having made.

There are few words which must be paid adequate attention, i.e., 'between' means sandwiched, 'immediate left' is different from 'to the left'. To understand it let us see some pictorial representation.

When direction of face is not clear, then we take **One Row Sequence**

(A) When direction of face is not clear, then we take based on diagram will be as follows:





From the above diagram, it is clear that

(i) Q, R, S, T are right of P but only Q is the immediate right of P.

(ii) S, R, Q, P are left of T but only S is the immediate left of T.

(iii) R, S, T are right of Q only R is the immediate right of Q.

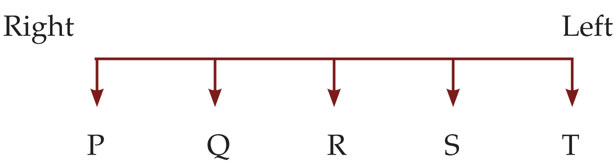
(iv) R, Q, P are left of S but only R is the immediate left of S.

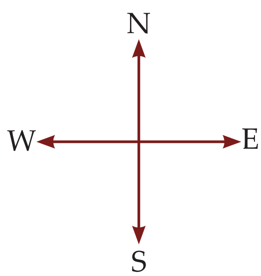
(v) S and T are right of R but only S is the immediate right of R.

(vi) Q and P are left of R but only Q is the immediate left of R.

(vii) P is the immediate left of Q while T is the immediate right of S.

(B) When direction of face is towards you, then the diagram will be as follows:





From the above diagram, it is clear that

(i) Left of P = Q, R, S and T

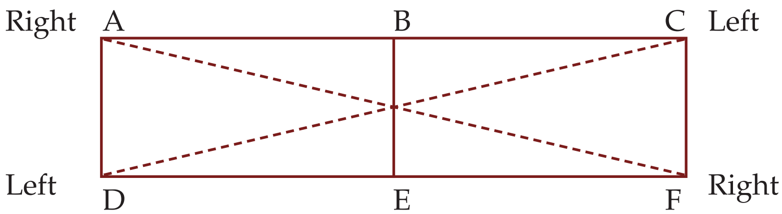
(ii) Right of T = S, R, Q and P

(iii) Q is immediate left of P; R is immediate left of Q; S is immediate left of R and T is immediate left of S.

(iv) S is immediate right of T; R is immediate right of S; Q is immediate right of R; and P is immediate right of Q.

Two Rows Sequence

Let us see 6 persons seating in two rows.



From the above diagram, it is clear that

(i) A is sitting opposite D

(ii) B is sitting opposite E

(iii) C is sitting opposite F

(iv) D and C are sitting at diagonally opposite positions

(iv) A and F are sitting at diagonally opposite positions.

Example 3: Four Children's are sitting in arrow. A is occupying seat next to B but not next to C. If C is not sitting next to D ? Who is occupying seat adjacent to D.

(a) B (b) B and A (c) Impossible to tell (d) A

Solution: (d) The arrangements as per given information is possible only if C is sitting next to B and D is sitting next to A.

Therefore, two possible arrangements are C, B, A, D, or D, A, B, C

Clearly, only A is sitting adjacent to D:

Example 4: P, Q, R, S, T, U, V and W are sitting in a row facing North.

(i) P is fourth to the right of T

(ii) W is fourth to the left of S

(iii) R and U, which are not at the ends, are neighbours of Q and T respectively.

(iv) W is next to the left of P and P is the neighbour of Q. Who are sitting at the extreme ends ?

Solution:

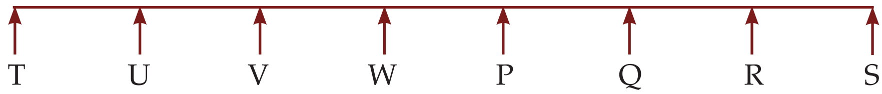
From information

(i) We get that there are three persons between P and TXXXP.

In the information (iv), it is given that W is next to the left of P and Q is the neighbour of P. Using the information with (i), we get TXXWPQ.

By the information (ii), TXXWPQXS

By the information (iii),

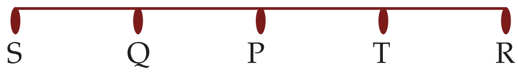


So, T and S are sitting at the extreme ends.

Example 5: There are Five houses P, Q , R, S, T . P is immediate right of Q and T is immediate left of R and immediate right of P . Q is right of S. Which house in the middle.

(a) P (b) Q (c) R (d) T

Solution: According to the question the houses can be arranged as follows. Assuming all houses are facing towards North.

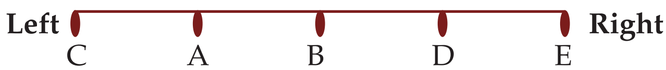


Therefore, house P is middle.

Example 6: Five friends are sitting on a bench. A is to the immediate left of B but on the immediate right of C, D is to the immediate right of B but on the immediate left of E. Who are at the extremes?

(a) A, B (b) A, D (c) C, E (d) B, D

Solution: Arrangements according to the question as follows. Assuming all students are facing towards North.



Clearly C and E are the extremes.

Example 7: In a college party, 5 girls are sitting in a row. P is to the immediate left of M and to the immediate right of O. R is sitting to the immediate right of N but to the left of O. Who is sitting in the middle?

(a) O (b) R (c) P (d) M

Solution: (a) arrangements of the question as follows.

Therefore, O is sitting in the middle.

Example 8: Five boys A, B, C, D and E are standing in a row. D is on the immediate right of E, B is on the immediate left of E but on the immediate right of A. D is one the immediate left of C, who is standing on the extreme right. Who is standing in the middle?

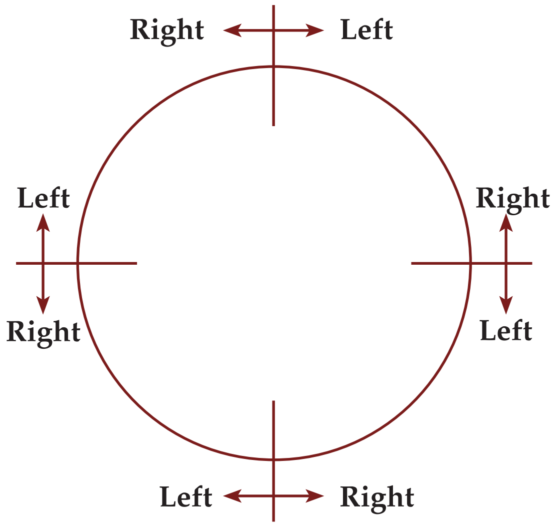
(a) B (b) C (c) D (d) E

Solution: The sequence of Boys as follows. Assuming all boys are sitting towards North.

There E is standing in the middle.

Circular Arrangement:

In this arrangement, some persons are sitting around a circle and they are facing the centre.



1. Left movement is called clockwise rotation.
2. Right movement is called anti–clockwise rotation.
   (i) The above presentation is for 4 persons but for any number of persons, the direction is taken in the same manner.
   (ii) For rectangular and sequence arrangement, directions are taken as discussed in two rows sequence.

Example 9: (Q Nos. 1 to 3) **Study the following Question carefully and answer the given questions.**

Four ladies A, B, C and D and Four Gentlemen E, F, G and H are sitting in a circle around a table facing each other .

I. No two ladies or gentlemen are sitting side by side.

II. C, who is sitting between G and E, is facing D.

III. F is between D and A and facing G.

IV. H is to the right of B.

(1) Who is sitting left of A?

(a) E (b) F (c) G (d) H

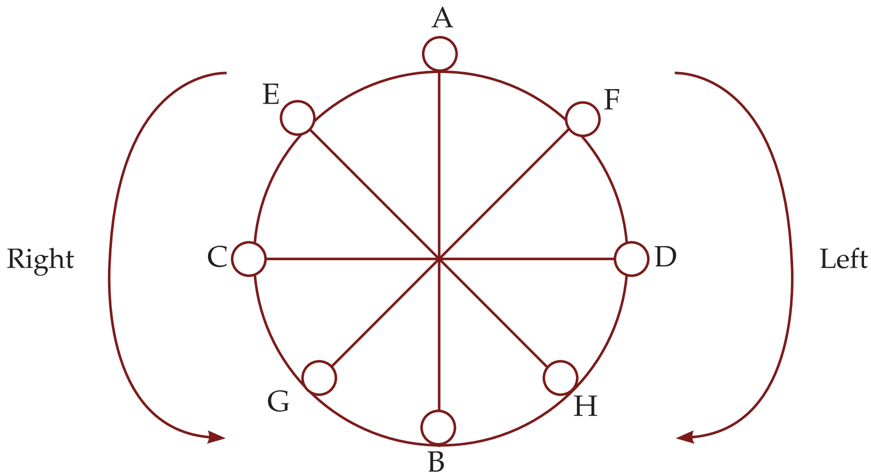
(2) E is facing whom?

(a) F (b) B (c) G (d) H

(3) Who is immediate neighbour of B?

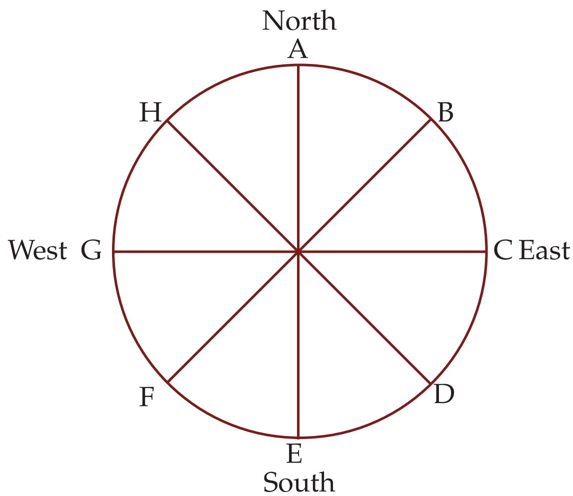
(a) G and H (b) E and F (c) E and H (d) F and H

Solution: On the basis of given information in the question, the seating arrangements of the persons are as follows.



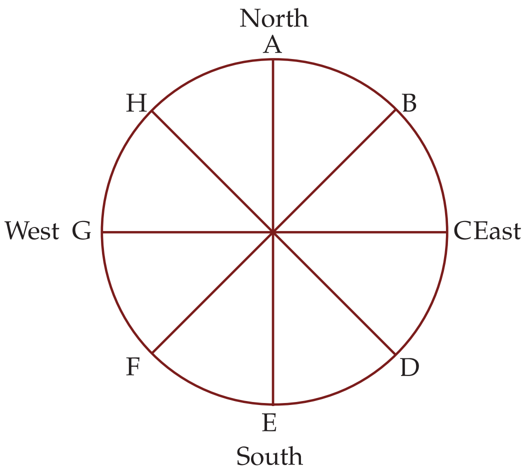
1) (b) Clearly , F is sitting left of A.
2) (d) Clearly E is facing H.
3) (a) G and H are neighbours of B.

Example 10: Eight persons A, B, C, D, E, F, G and H are sitting around the circle as given in the figure. They are facing the direction opposite to centre. If they move upto three places anti-clockwise, then.

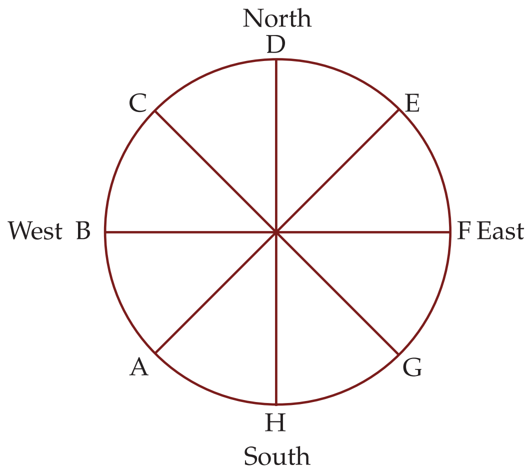


(a) B will face West
(b) E will face East
(c) H will face North-West
(d) A will face South

Solution: Following Seating arrangement is formed from the given in formation.



Original position



New Position

Clearly B will Face west

Example 11: Five People A, B, C, D and E are seated around a round table. Every chair is spaced equidistant from adjacent chairs. (UPSC (CSAT) 2013)

I. C is seated next to A .

II. A is seated two seats from D.

III. B is not seated next to A.

Which of the following must be true?

I. D is seated next to B.

II. E is seated next to A.

Select the correct from the options given below:

(a) Only I

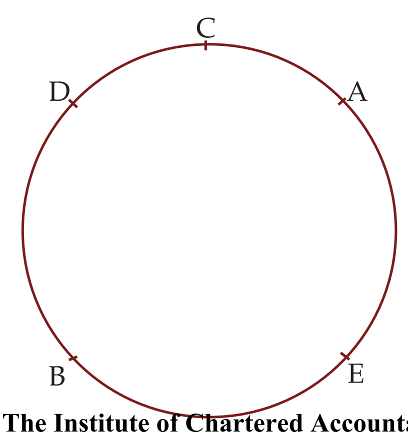
(b) Only II

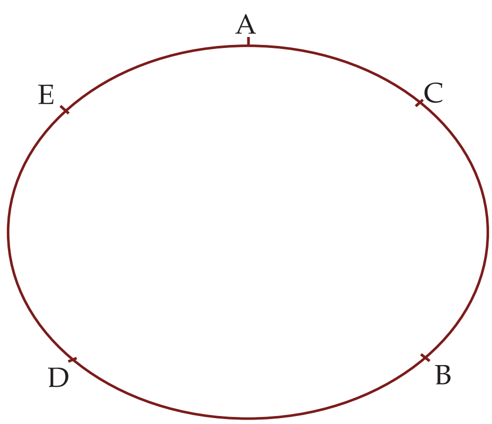
(c) Both I and II

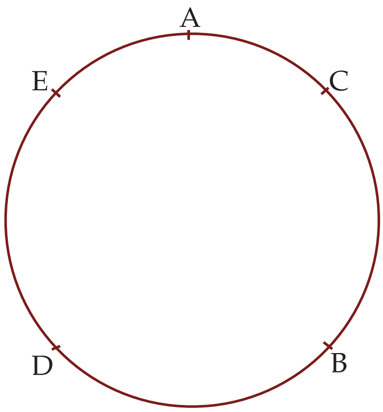
(d) Neither I nor II

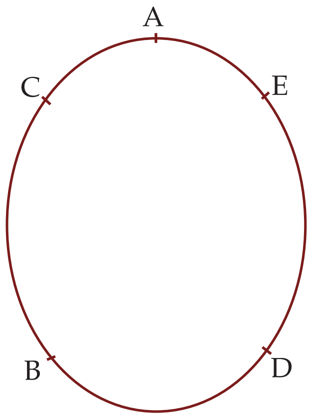
Solution:

According to the given information there are possible Seating arrangements.









From the above arrangements. It is clear that D is seated next to B . Also E is next to A.

Clearly both statements I and II are true.

Example 12: Study the following Question carefully and answer the given questions.

Eight friends A, B, C, D,E, F, G and H are sitting in a circle facing the centre, not necessarily in the same order. D sits third to the left of A. E sits to the immediate right of A. B is third to left of D. G is second to the right of B. C is neighbour of B. C is third to left of H. (GIC 2012)

1) Who amongst the following is sitting exactly between F and D?

(a) C (b) E (c) H (d) A

2) Three of the following four are alike in a certain way based on the information given above and so form a group. Which is does not belong to that group.

(a) DC (b) AH (c) EF (d) CB

3) Who amongst the following second to the left of H?

(a) E (b) B (c) A (d) None of these

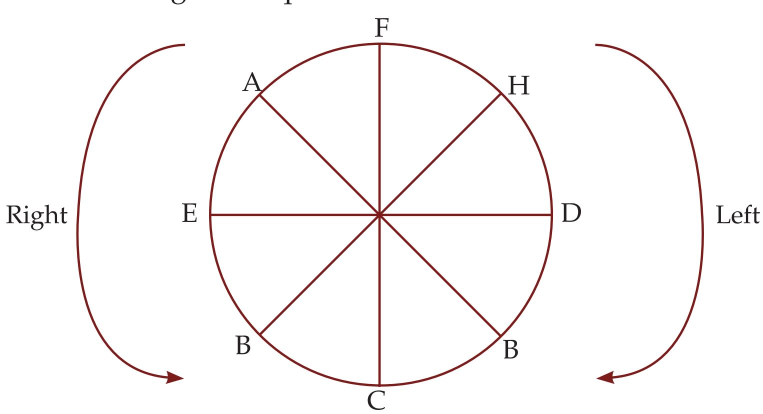
4) Who amongst the following are immediate neighbours of G?

(a) CA (b) AF (c) DC (d) DF

5) Who amongst the following is sitting third to the right of A?

(a) F (b) B (c) H (d) C

Solution: Arrangements according to the question is as follows.



1) (c), Clearly H is sitting exactly F and D

2) (d)

DGC	AFH	EAF	C none B
↑	↑	↑	↑
Skipped	Skipped	Skipped	No member is skipped in between

So, CB does not belong to the group.

3) (d) Clearly, G is sitting second to the left of H

4) (c) Clearly D and C immediate neighbours of G

5) (d) Clearly, C is sitting third to the right of A

EXERCISE 11.A

(Note: Questions are taken from previous exam questions papers of Competitive exams like SSC, RRB, MAT, UPSC etc.)

Choose the appropriate answer (a) or (b) or (c) or (d)

1. Five boys A, B, C, D and E are sitting in a row A is to the right of B and E is to the left of B but to the right of C. A is to the left of D. Who is second from the left end? (U.P.B.Ed 2013)

   (a) D (b) A (c) E (d) B

2. There are five different houses, A to E, in a row. A is to the right of B and E is to the left of C and right of A, B is to the right of D. Which of the houses is in the middle? IB CA (IO) 2013)

   (a) A (b) B (c) C (d) D

3. Five friends P, Q, R, S and T are sitting in a row facing North. Here, S is between T and Q and Q is to the immediate left of R. P is to the immediate left of T. Who is in the middle? (SSC (Multi Task) 2014)

   (a) S (b) T (c) Q (d) R

4. Six children A, B, C, D, E and F are standing in a row. B is between F and D. E is between A and C. A does not stand next to F or D. C does not stand next to D. F is between which of the following pairs of children? (SSC (FCI) 2012)

   (a) B and E (b) B and C (c) B and D (d) B and A

5. There are eight books kept one over the other. Two books are on Organisation Behaviour, two books on TQM, three books on Industrial Relations and one book is on Economics. Counting from the top, the second, fifth and sixth books are on Industrial Relations. Two books on Industrial Relations are between two books on TQM. One book of Industrial Relations is between two books on Organizational Behaviour while the book above the book of Economics is a book of TQM. Which book is the last book from the top? (MAT 2011)

   (a) Economics (b) TQM

   (c) Industrial Relations (d) Organizational Behaviour

6. Five boys are standing in a row facing East . Pavan is to the left of Tavan , Vipin and Chavan. Tavan , Vipin and Chavan are to the left of Nakul. Chavan is between Tavan and Vipin. If Vipin is fourth from the left , then how far is Tavan from the right ?

(a) First (b) Second (c) Third (d) Fourth

7. Six persons M, N, O, P, Q and R are sitting in two rows with three persons in each row. Both the row are in front of each other. Q is not at the end of any row. P is second the left of R. O is the neighbour of Q and diagonally opposite to P. N is the neighbour of R. Who is in fronts N? (UPSC (CSAT) 2011)

(a) R (b) Q (c) P (d) M

8. Six persons A, B, C, D, E and F are sitting in two rows, three in each row. (MAT 2011)

(I) E is not at the end of any row

(II) D is second to the left of F

(III) C, the neighbor of E, is sitting diagonally opposite to D

(IV) B is the neighbor of F.

Which of the following are in one of the two rows?

(a) D, B and F (b) C, E and B (c) A, E and F (d) F, B and C

Direction (Q.No.9): Read the following information carefully and answer the question that follows.

Five boys A_1, A_2, A_3, A_4 and A_5 are sitting in a stair in the following way. (RRB (TC/CC) 2010)

I. A_5 is above A_1

II. A_4 is under A_2

III. A_2 is under A_1

IV. A_4 is between A_2 and A_3.

9. Who is at the lowest position of the stair?

(a) A_1 (b) A_3 (c) A_5 (d) A_2

10. Five children are sitting in a row. S is sitting next to P but not T. K is sitting next to R, who is sitting on the extreme left and T is not sitting next to K. Who is/are adjacent to S? (NIFT (UG) 2014)

(a) K and P (b) R and P (c) Only P (d) P and T

11. Five senior citizens are living in a multi-storeyed building. Mr. Muan lives in a flat above Mr. Ashokan, Mr. Lokesh in a flat below Mr. Gaurav, Mr. Ashokan lives in a flat above Mr. Gaurav and Mr. Rakesh lives in a flat below Mr. Lokesh. Who lives in the topmost flat? (MAT 2013).

(a) Mr. Lokesh (b) Mr. Gaurav (c) Mr. Muan (d) Mr. Rakesh

12. In a gathering seven members are sitting in a row. 'C' is sitting left to 'B' but on the right to 'D'. 'A' is sitting right to 'B', 'F; is sitting right to 'E' but left to 'D'. 'H' is sitting left to 'E'. Find the person sitting in the middle. (SSC (10+2) 2013)

(a) C (b) D (c) E (d) F

Directions (No: 13-17): Study the following information carefully to answer the given questions.

A to H are seated in straight line facing North. C sits fourth left of G. D sits second to right of G. Only two people sit between D and A. B and F are immediate neighbours of each other. B is not an immediate neighbour of A. H is not neighbour of D. (GIC 2012)

13. Who amongst the following sits exactly in the middle of the persons who sit fifth from the left and the person who sit sixth from the right?

    (a) C (b) H (c) E (d) F

14. Who amongst the following sits third to the right of C?

    (a) B (b) F (c) A (d) E

15. Which of the following represents persons seated at the two extreme ends of the line?

    (a) C, D (b) A, B (c) B, G (d) D, H

16. What is the position of H with respect to F?

    (a) Third to the left (b) Immediate right (c) Second to right (d) Fourth to left

17. How many persons are seated between A and E?

    (a) One (b) Two (c) Three (d) Four

Directions (Q. No. 18-22) (MAT 2012)

Study the following information carefully to answer the given questions.

Ten students are A to J are sitting in a row facing west.

I. B and F are not sitting on either of the edges.

II. G is sitting left of D and H is sitting to the right of J.

III. There are four persons between E and A.

IV. I is the north of B and F is the south of D.

V. J is between A and D and G is in between E and F.

VI. There are two persons between H and C.

18. Who is sitting at the seventh place counting from left?

    (a) H (b) C (c) J (d) Either H or C

19. Who among the following is definitely sitting at one of the ends?

    (a) C (b) H (c) E (d) Cannot be determined

20. Who are immediate neighbours of I?

    (a) BC (b) BH (c) AH (d) Cannot determined

21. Who is sitting second left of D?

    (a) G (b) F (c) E (d) J

22. If G and A interchange their positions, then who become the immediate neighbours of E?

    (a) G and F (b) Only F (c) Only A (d) J and H

Directions (Q. Nos. 23-24) Read the following information carefully and then answer the questions that follow.

A group of seven singers, facing the audience, are standing in line on the stage as follows.

I. D is to the immediate right to C

II. F is standing beside G.

III. B is to the immediate left of F

IV. E is to the immediate left of A

V. C and B have one person between them

VI. A and D have oneperson between them

23. Who is on the Second extreme right?
    (a) D (b) F (c) G (d) E

24. If we start counting from the left, on which number is B ?
    (a) 1^{st} (b) 2^{nd} (c) 3^{rd} (d) 5^{th}

Directions (Q. No. 25- 27): Study the following information carefully to answer the given questions.

Eight persons P to W are sitting in front of one another in two rows. Each row has four persons. P is between U and V and facing North. Q, who is to the immediate left of M is facing W. R is between T and M and W is to the immediate right of V.

(UCO Bank 2011)

25. Who is sitting in front of R?
    (a) U (b) Q (c) V (d) P

26. Who is to the immediate right of R?
    (a) M (b) U (c) M or T (d) None of these

27. In which of the following pairs, persons are sitting in front of each other?
    (a) MV (b) RV (c) TV (d) UR

28. Four girls A, B, C, D are sitting around a circle facing the centre. B and C infront of each other, which of the following is definitely true ? (MAT 2009)
    (a) A and D infront of each other (b) A is not between B and C
    (c) D is left of C (d) A is left of C

ANSWERS

1. (c)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (d)	**7.** (b)
8. (a)	**9.** (b)	**10.** (d)	**11.** (c)	**12.** (b)	**13.** (d)	**14.** (c)
15. (d)	**16.** (a)	**17.** (a)	**18.** (d)	**19.** (c)	**20.** (d)	**21.** (a)
22. (c)	**23.** (b)	**24.** (d)	**25.** (d)	**26.** (d)	**27.** (a)	**28.** (a)

NOTES

CHAPTER 12

BLOOD RELATIONS

LEARNING OBJECTIVES

- Blood relations of a group of persons are given in jumbled form. In these tests, the questions which are asked in this section depend upon Relation.

12.1 DEFINITION

A person who is related to another by birth rather than by marriage.

Prerequisites:

To remember easily the relations may be divided into two sides as given below:

(i) Relations of Paternal side:

Father's father → Grandfather

Father's mother → Grandmother

Father's brother → Uncle

Father's sister → Aunt

Children of uncle → Cousin

Wife of uncle → Aunt

Children of aunt → Cousin

Husband of aunt → Uncle

(ii) Relations of Maternal side:

Mother's father → Maternal grandfather

Mother's mother → Maternal grandmother

Mother's brother → Maternal uncle

Mother's sister → Aunt

Children of maternal uncle → Cousin

Wife of maternal uncle → Maternal aunty

Relations:

1.	Grandfather's son	• Father or Uncle
2.	Grandmother's son	• Father or Uncle
3.	Grandfather's only son	• Father
4.	Grandmother's only son	• Father
5.	Mother's or father's mother	• Grandmother
6.	Son's wife	• Daughter-in-Law
7.	Daughter's husband	• Son-in-Law
8.	Husband's or wife's sister	• Sister-in-Law
9.	Brother's son	• Nephew
10.	Brother's daughter	• Niece
11.	Uncle or aunt's son or daughter	• Cousin
12.	Sister's husband	• Brother-in-Law
13.	Brother's wife	• Sister-in-Law
14.	Granson's or grand daughter's daughter	• Great grand Daughter

The efficiency in doing the problems of blood relations depends upon the knowledge of the blood relations. Some of the important relations are given below:

a) My mother's or father's son is my Brother.

b) My mother's or father's daughter is my Sister.

c) My mother's or father's father is my Grandfather.

d) My mother's or father's sister is my Aunt.

e) My mother's or father's brother is my Uncle.

f) My son's wife is my Daughter-in-law.

g) My daughter's husband is my Son-in-law.

h) My brother's son is my Nephew.

i) My brother's daughter is my Neice.

j) My sister's husband is my Brother-in-law.

k) My brother's wife is my Sister-in-law.

l) My husband's or wifer's sister is my Sister-in-law.

m) My husband's or wife's brother is my Brother-in-law.

n) My uncle's or aunt's son or daughter is my Cousin.

o) My wife's father or husband's father is my Father-in-law.

p) My wife's mother or husband's mother is my Mother-in-law.

q) My father's wife is my Mother.

r) My mother's husband is my Father.

s) My son's or daughter's son is my Grandson.

t) My son's or daughter's daughter is my Grand-daughter.

Different types of questions with explanation:

(1) A is B's daughter, B is C's mother. D is C's brother. How is D related to A?

(a) Father (b) Grandfather (c) Brother (d) Son

Explanation: A is daughter D .

B is mother of C, 'D' is Brother of 'A'

Therefore, D is Son of B. Thus 'D' is brother of 'A'

(2) P is Q's brother. R is Q's mother. S is R's father. T is S's mother. How is P related to T?

(a) Grand-daughter (b) Great grandson (c) Grandson (d) Grandmother

Explanation: P is brother of Q . Therefore, P is a male.

R is mother of P and Q and R is daughter of S. S is Son of T.

S is grandfather of P. And T is great grandmother of P. Hence, P is greatgrand son of T.

(3) A is B's brother. C is D's father. E is B's mother. A and D are brothers. How is E related to C?

(a) Sister (b) Sister-in-law (c) Niece (d) Wife

Explanation: A is brother of B . Therefore, A is male.

C is father of D. Therefore, C is male.

E is mother of B. Therefore, E is Female.

A and D are brothers.

Therefore, D is male.

Deductions:

(i) A and D are brothers of D

(ii) C is the father of A, B and D

(iii) C is the mother of A, B and D

Thus, E is wife of C

(4) A is the sister of B. B is the brother of C. C is the son of D. How is D related to A?

(a) Mother (b) Daughter (c) Son (d) Uncle

Explanation: (1) B is brother of C

C is son of D.

A is the sister of B and C. A is the daughter of 'D'.

According to the options given, we are left with no choice. But selection option (a) is correct.

5. B is the brother of A. whose only sister is mother of C. D is maternal grandmother of C. How is A related to D?

   (a) Daughter-in-law (b) Daughter (c) Aunt (d) Nephew

   Explanation: Although sex of A is not mentioned clearly in the question. On the basis of information given is A is daughter of D.

6. A and B are sisters. R and S are brothers. A's daughter is R's sister. What is B's relation to S?

   (a) Mother (b) Grandmother (c) Sister (d) Aunt

   Explanation: A's daughter is the sister of R and S.

   B is sister of A. B is aunt of S.

7. E is the sister of B. A is the father of C. B is the son of C. How is A related to E?

   (a) Grandfather (b) Grand-daughter (c) Father (d) Great-grandfather

   Explanation: B' is the son of C and Grandson of A.

   E is sister of B.

   Therefore, A is Grandfather of E.

8. Given that:

   A is the mother of B.

   C is the son of A.

   D is the brother of E.

   E is the daughter of B.

   Who is grandmother of D?

   (a) A (b) B (c) C (d) D

   Explanation: E is the daughter of B and D is brother of E. Therefore D is son B and A is son of A and A is mother of B.

   C is son of A and A is mother of B.

   Thus A, is Grandmother of D. Therefore, D is son of B.

9. A is D' brother. D is B's father. B and C are sisters. How is A related to C?

   (a) Son (b) Grandson (c) Father (d) Uncle

   Explanation: B and C daughters of D .

   A is brother of D .

   Therefore A is uncle of C.

10. A is B's sister, C is B's mother, D is C's father, E is D's mother, then how A is related to D?

    (a) Grandfather (b) Daughter (c) Grandmother (d) Granddaughter

    Explanation: D is Father of C and C is mother of A and B.

    Thus, A is granddaughter of D

11. (i) F is the brother of A.

    (ii) C is the daughter of A.

    (iii) K is the sister of F.

    (iv) G is the brother of C.

    Who is the uncle of G?

    (a) A (b) C (c) K (d) F

    Explanation: G is son of A and F is brother of A. Hence, F is uncle of G.

12. A is father of C and D is son of B. E is brother of A. If C is sister of D how is B related to E?

    (a) Sister-in-law (b) Sister (c) Brother (d) Brother-in-law

    Explanation: C and D are Children of A and B.

    B is mother of C and D.

    Therefore, B is Sister-in-law of E.

13. C is wife of B. E is the son of C, A is the brother of B and father of D. What is the relationship of E to D?

    (a) Mother (b) Sister (c) Brother (d) Cousin

    Explanation: E is son of B and C.

    A is uncle of E and Father of D.

    Therefore E is cousin of D.

14. M is the son of P. Q is the grand-daughter of O, who is the husband of P. How is M related to O?

    (a) Son (b) Daughter (c) Mother (d) Father

    Explanation: O is the Husband of P. M is the son of P.

    Therefore, M is son of O.

15. X and Y are brothers. R is the father of Y. S is the brother of T and maternal uncle of X. What is T to R?

    (a) Mother (b) Wife (c) Sister (d) Brother

    Explanation: R is the Father of X and Y.

    S is the maternal uncle of X and Y.

    Considering the option (b), T is wife of R.

EXERCISE 12 (A)

(Note: Questions are taken from previous exam questions papers of Competitive exams like SSC, RRB, MAT, UPSC etc.)

Choose the appropriate answer (a) or (b) or (c) or (d)

1 A is B's brother. C is A's mother. D is C's father, E is B's son. How is D related to A?

(a) Son (b) Grandson (c) Grandfather (d) Great Grandfather

2. As is B's brother. C is A's father. D is C's sister and E is D's mother. How is B related to E?

(a) Grand-daughter (b) Great grands daughter (c) Grandaunt (d) Daughter

3. A is B's Sister. C is B's Mother. D is C's Father. E is D's Mother. Then how is A related to D?

(a) Grandmother (b) Grandfather (c) Daughter (d) Grands-daughter

4. A is the father of B. C is the daughter of B. D is the brother of B. E is the son of A. What is the relationship between C and E?

(a) Brother and sister (b) Cousins (c) Niece and uncle (d) Uncle and aunt

5. If P is the husband of Q and R is the mother of S and Q. What is R to P?

(a) Mother (b) Sister (c) Aunt (d) Mother-in-law

6. P and Q are brothers. R and S are sister. P's son is S's brother. How is Q related to R?

(a) Uncle (b) Brother (c) Father (d) Grandfather

7. X is the husband of Y. W is the daughter of X. Z is husband of W. N is the daughter of Z. What is the relationship of N to Y?

(a) Cousin (b) Niece (c) Daughter (d) Grand-daughter

8. A reads a book and find the name of the author familiar. The author 'B' is the paternal uncle of C. C is the daughter of A. How is B related to A?

(a) Brother (b) Sister (c) Father (d) Uncle

9. A's mother is sister of B and she has a daughter C, who is 21 years old. How is B related to C?

(a) Uncle (b) Maternal Uncle (c) Niece (d) Daughter

10. A is B's brother. C is A's mother. D is C's father. F is A's son. How is F related to D?

(a) Son (b) Grandson (c) Great-grandson (d) Grand-daughter

11. A is B's brother. C is A's mother. D is C's father. E is B's son. How is B related to D?

(a) Son (b) Grand-daughter (c) Grandfather (d) Great grandfather

12. A is B's brother. C is A's mother. D is C's father. F is A's son. How is B related to F's child?

(a) Aunt (b) Cousin (c) Nephew (d) Grandfather

13. A is B's daughter. B is C's mother. D is C's brother. How is D related to A?

(a) Father (b) Grandfather (c) Brother (d) Son

14. A is D's brother. D is B's father. B and C are sisters. How is C related to A?

(a) Cousin (b) Niece (c) Aunt (d) Nephew

15. A is B's brother. C is A's mother, D is C's father. E is B's son. How is D related to E ?

(a) Grandson (b) Great Grandson (c) Great Grandfather (d) Grandfather

16. X and Y are the children of A. A is the father of X but Y is not his son. How is Y related to A?

(a) Sister (b) Brother (c) Son (d) Daughter

17. A is B's brother. C is A's mother. D is C's father. E is B's son. How is E related to A?

(a) Cousin (b) Nephew (c) Uncle (d) Grandson

18. Based on the statements given below, find out who is the uncle of P?

(i) K is the bother of J

(ii) M is the sister of K

(iii) P is the brother of N

(iv) N is the daughter of J

(a) K (b) J (c) N (d) M

19. A and B are sisters. A is mother of D. B has a daughter C, who is married to F. G is the husband of A. How is C related to D?

(a) Cousin (b) Niece (c) Aunt (d) Sister-in-law

20. R and S are brothers. X is the sister of Y and X is mother of R. What is Y to S?

(a) Uncle (b) brother (c) Father (d) Mother

21. A is B's brother. C is A's mother. D is C's father. B is D's grand-daughter. How is B related to F. Who is A's son?

(a) Aunt (b) Cousin (c) Niece (d) Grandaunt

22. A is the son of B while B and C are sisters to one another. E is the mother of C. If D is the son of E, which of the following statements is correct?

(a) D is the maternal uncle of A (b) E is the brother of B

(c) D is the cousin of A (d) B and D are brothers

23. P is the father of T. T is the daughter of M. M is the daughter of K. What is P to K?

(a) Father (b) Father-in-law (c) Brother (d) Son-in-law

24. A and B are brothers. E is the daughter of F. F is the wife of B. What is the relation of E to A?

(a) Sister (b) Daughter (c) Niece (d) Sister-in-law

25. M and F are a married couple. A and B are sisters. A is the sister of F. Who is B to M?

(a) Sister (b) Sister-in-law (c) Niece (d) Daughter

26. If A is the mother of D. B is not the son of C. C is the father of D, D is the sister of B, then how is A related to B?

(a) Mother (b) Brother (c) Step son (d) Sister

27. A and B are brother and sister respectively. C is A's father. D is C's sister and E is D's mother. How is B related to E?

(a) Grand-daughter (b) Great grand-daughter

(c) Aunt (d) Daughter

28. Q is the son of P. X is the daughter of Q. R is the aunty (Bua) of X and L is the son of R, then what is L to P?

(a) Grandson (b) Grand-daughter (c) Daughter (d) Nephew

29. P and Q are brothers. R and S are sisters. P's son is S's brother. How is Q related to R?

(a) Uncle (b) Brother (c) Father (d) Grandfather

30. A and B are the young ones of C. If C is the mother of B but A is not the daughter of C, then what is the relationship between C and A?

(a) Nephew and Aunty (b) Brother and Sister

(c) Mother and son (d) Niece and Aunty

31. A is the mother of D and sister of B. B has a daughter C who is married to F. G is the husband of A. How is G related to D?

(a) Uncle (b) Husband (c) Son (d) Father

32. Pointing towards A, B said "your mother is the younger sister of my mother". How is A related to B?

(a) Uncle (b) Cousin (c) Nephew (d) Father

33. A is B's wife's husband's brother. C and D are sisters of B. How is A related to C?

(a) Brother (b) Sister-in-law (c) Wife (d) Sister

34. A and B are brothers. C and D are sisters. A's son is D's brother. How is B related to C?

(a) Father (b) Brother (c) Uncle (d) Son

35. A is B's sister. C is B's mother. D is C's father. E is D's mother. Then how is A related to D?

(a) Grandmother (b) Grandfather (c) Daughter (d) Grand-daughter

36. P, Q, R, S, T, U are 6 members of a family in which there are two married couples. T, a teacher is married to a doctor who is mother of R and U. Q the lawyer is married to P. P has one son and one grandson. Of the two married ladies one is a housewife. There is also one student and one male engineer in the family. Which of the following is true about the grand-daughter of the family?

(a) She is a lawyer (b) She is an engineer

(c) She is a student (d) She is a doctor

37. Six members of a family namely A, B, C, D, E and F are travelling together. 'B' is the son of C but C is not the mother of B. A and C are married couple. E is the brother of C. D is the daughter of A. F is the brother of B. How many male members are there in the family?

(a) 3 (b) 2 (c) 4 (d) 1

38. A's mother is sister of B and has a daughter C. How can A be related to B from among the following?

(a) Niece (b) Uncle (c) Daughter (d) Father

39. Rajiv is the brother of Atul. Sonia is the sister of Sunil. Atul is the son of Sonia. How is Rajiv related to Sonia?

(a) Nephew (b) Son (c) Brother (d) Father

40. Sita is the niece of Ashok. Ashok's mother is Lakshmi. Kalyani is Lakhshmi's mother. Kalyani's husband is Gopal. Parvathi is the mother-in-law of Gopal. How is Sita related to Gopal?

(a) Great grandson's daughter (b) Gopal's Sita's father

(c) Sita is Gopal's great grand-daughter (d) Grand niece

41. Seema is the daughter-in-law of Sudhir and sister-in-law of Ramesh. Mohan is the son of Sudhir and only brother of Ramesh. Find the relation between Seema and Mohan.

(a) Sister-in-law (b) Aunt

(c) Cousin (d) Wife

42. Suresh introduces a man as "He is the son of the woman who is the mother of the husband of my mother". How is Suresh related to the man?

(a) Uncle (b) Son (c) Cousin (d) Grandson

43. Pointing to a lady in a photograph. Meera said. "Her father's only son's wife is my mother-in-law "How is Meera's husband related to that lady in the photo?

(a) Nephew (b) Uncle (c) Son (d) Father

44. Pointing to a photograph Vikas said "She is the daughter of my grandfather's only son". How is lady in the photograph related to Vikas in the photograph?

(a) Father (b) Brother (c) Sister (d) Mother

45. Suresh's sister is the wife of Ram. Ram is Rani's brother. Ram's father is Madhur. Sheetal is Ram's grandmother. Rema is Sheetal is daughter-in-law. Rohit is Rani's brother's son. Who is Rohit to Suresh?

(a) Brother-in-law (b) Son

(c) Brother (d) Nephew

46. Vinod introduces Vishal as the son of the only brother of his father's wife. How is Vinod related to Vishal?

(a) Cousin (b) Brother (c) Son (d) Uncle

47. Among her children, Ganga's favourites are Ram and Rekha. Rekha is the mother of Sharat, who is loved most by his uncle Mithun. The head of the family is Ram Lal, who is succeeded by his sons Gopal and Mohan. Gopal and Ganga have been married for 35 years and have 3 children. What is the relation between Mithun and Mohan?

(a) Uncle (b) Son (c) Brother (d) No relation

48. Rahul and Robin are brothers. Promod is Rohin's father. Sheela is Pramod's sister. Prema is Promod's niece. Shubha is Sheela's grand-daughter. How is Rahul related to Shubha?

(a) Brother (b) Cousin (c) Uncle (d) Nephew

49. Preeti has a son, named Arun. Ram is Preeti's brother. Neeta too has a daughter named Reema. Neeta is Ram's sister. What is Arun's relationship to Reema?

(a) Brother (b) Nephew (c) Cousin (d) Uncle

50. There are 2 film stars. One is the father of the other's son. What is the relationship of the two with each other?

(a) Grandfather and Grandson (b) Grandfather and son

(c) Husband and wife (d) Father and Son

51. Ramu's mother said to Ramu,"My mother has a son whose son is Achyut". How is Achyut relation to Ramu?

(a) Uncle (b) Cousin (c) Brother (d) Nephew

52. Ravi's father has a son Rohit who has an aunt Laxmi who has a husband Rao whose father-in-law is Mohan. What is the relation of Mohan to Ravi?

(a) Nephew (b) Grandfather (c) Son (d) Uncle

53. Vijay says, Ananda's mother is the only daughter of my mother". How is Ananda relation to Vijay?

(a) Brother (b) Father (c) Nephew (d) Grandfather

54. Introducing a man, a woman said, "His wife is the only daughter of my mother." How is the woman related with the man?

(a) Sister-in-law (b) Wife (c) Aunt (d) Mother-in-law

55. A prisoner introduced a boy who came to visit him to the jailor as "Brothers and sisters I have none, he is my father's son's son". Who is the boy?

(a) Nephew (b) Son (c) Cousin (d) Uncle

ANSWERS

EXERCISE 12(A)

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)
5. (d)	**6.** (a)	**7.** (d)	**8.** (a)
9. (b)	**10.** (c)	**11.** (b)	**12.** (d)
13. (c)	**14.** (b)	**15.** (c)	**16.** (d)
17. (b)	**18.** (a)	**19.** (a)	**20.** (a)
21. (a)	**22.** (a)	**23.** (d)	**24.** (c)
25. (b)	**26.** (a)	**27.** (a)	**28.** (a)
29. (a)	**30.** (c)	**31.** (d)	**32.** (b)
33. (a)	**34.** (c)	**35.** (d)	**36.** (c)
37. (c)	**38.** (a)	**39.** (b)	**40.** (c)
41. (d)	**42.** (b)	**43.** (a)	**44.** (c)
45. (d)	**46.** (a)	**47.** (d)	**48.** (c)
49. (c)	**50.** (c)	**51.** (b)	**52.** (b)
53. (c)	**54.** (b)	**55.** (b)	

NOTES

CHAPTER BD

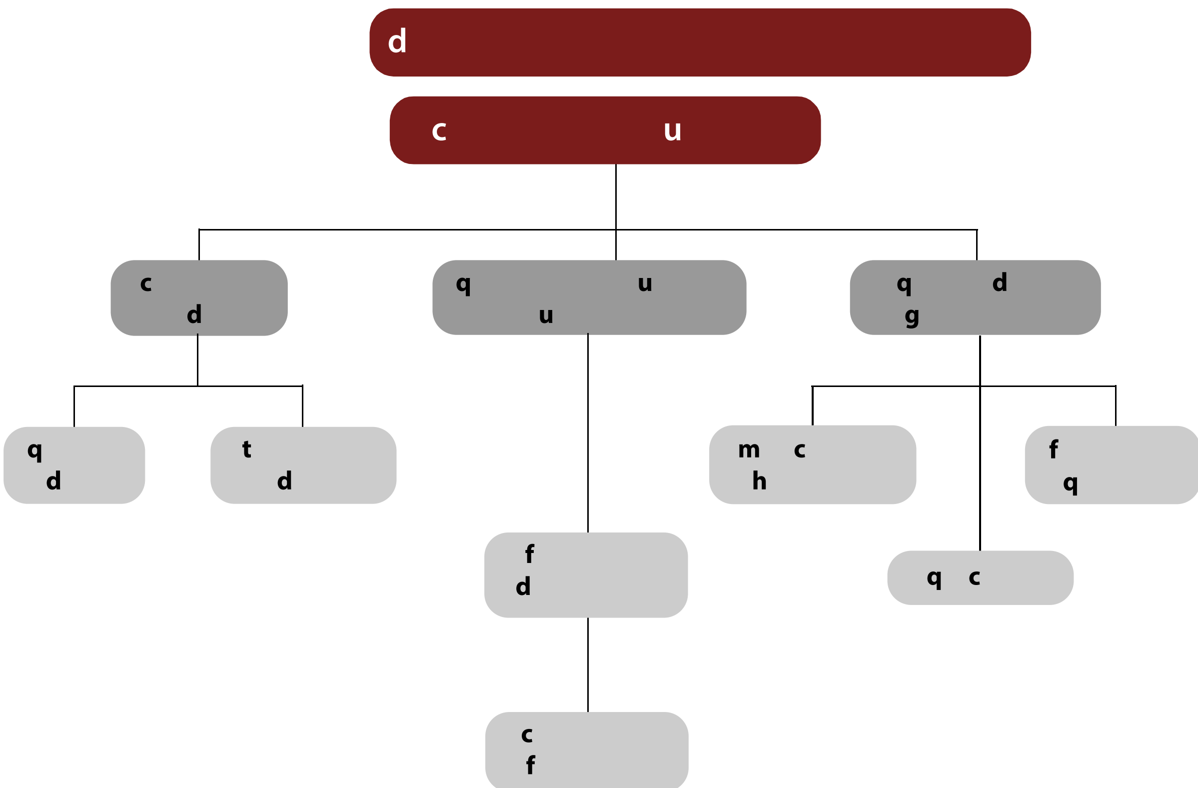
UNIT 1: STATISTICAL DESCRITPION OF DATA

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- Have a broad overview of the subject of statistics and application thereof.
- Know about data collection technique including the distinction of primary and secondary data.
- Know how to present data in textual and tabular format including the technique of creating frequency distribution and working out cumulative frequency.
- Know how to present data graphically using histogram, frequency polygon and pie chart.

UNIT OVERVIEW



13.1.1 INTRODUCTION OF STATISTICS

The modern development in the field of not only Management, Commerce, Economics, Social Sciences, Mathematics and so on but also in our life like public services, defence, banking, insurance sector, tourism and hospitality, police and military etc. are dependent on a particular subject known as statistics. Statistics does play a vital role in enriching a specific domain by collecting data in that field, analysing the data by applying various statistical techniques and finally making statistical inferences about the domain. In the present world, statistics has almost a universal application. Our government applies statistics to make the economic planning in an effective and a pragmatic way. The businessman plan and expand their horizons of business on the basis of the analysis of the feedback data. The political parties try to impress the general public by presenting the statistics of their performances and accomplishments. Most of the research scholars of today also apply statistics to present their research papers in an authoritative manner. Thus the list of people using statistics goes on and on and on. Due to these factors, it is necessary to study the subject of statistics in an objective manner.

History of Statistics

Going through the history of ancient period and also that of medieval period, we do find the mention of statistics in many countries. However, there remains a question mark about the origin of the word 'statistics'. One view is that statistics is originated from the Latin word 'status'. According to another school of thought, it had its origin in the Italian word 'statista'. Some scholars believe that the German word 'statistik' was later changed to statistics and another suggestion is that the French word 'statistique' was made as statistics with the passage of time.

In those days, statistics was analogous to state or, to be more precise, the data that are collected and maintained for the welfare of the people belonging to the state. We are thankful to Kautilya who had kept a record of births and deaths as well as some other precious records in his famous book 'Arthashastra' during Chandragupta's reign in the fourth century B.C. During the reign of Akbar in the sixteenth century A.D. We find statistical records on agriculture in Ain-i-Akbari written by Abu Fazl. Referring to Egypt, the first census was conducted by the Pharaoh during 300 B.C. to 2000 B.C.

Definition of Statistics

We may define statistics either in a singular sense or in a plural sense Statistics, when used as a plural noun, may be defined as data qualitative as well as quantitative, that are collected, usually with a view of having statistical analysis.

However, statistics, when used as a singular noun, may be defined, as the scientific method that is employed for collecting, analysing and presenting data, leading finally to drawing statistical inferences about some important characteristics it means it is 'science of counting' or 'science of averages'.

Application of statistics

Among various applications of statistics, let us confine our discussions to the fields of Economics, Business Management and Commerce and Industry.

Economics

Modern developments in Economics have the roots in statistics. In fact, Economics and Statistics are closely associated. Time Series Analysis, Index Numbers, Demand Analysis etc. are some overlapping areas of Economics and Statistics. In this connection, we may also mention Econometrics – a branch of Economics that interact with statistics in a very positive way. Conducting socio-economic surveys and analysing the data derived from it are made with the help of different statistical methods. Regression analysis, one of the numerous applications of statistics, plays a key role in Economics for making future projection of demand of goods, sales, prices, quantities etc. which are all ingredients of Economic planning.

Business Management

Gone are the days when the managers used to make decisions on the basis of hunches, intuition or trials and errors. Now a days, because of the never-ending complexity in the business and industry environment, most of the decision making processes rely upon different quantitative techniques which could be described as a combination of statistical methods and operations research techniques. So far as statistics is concerned, inferences about the universe from the knowledge of a part of it, known as sample, plays an important role in the development of certain criteria. Statistical decision theory is another component of statistics that focuses on the analysis of complicated business strategies with a list of alternatives – their merits as well as demerits.

Statistics in Commerce and Industry

In this age of cut-throat competition, like the modern managers, the industrialists and the businessmen are expanding their horizons of industries and businesses with the help of statistical procedures. Data on previous sales, raw materials, wages and salaries, products of identical nature of other factories etc are collected, analysed and experts are consulted in order to maximise profits. Measures of central tendency and dispersion, correlation and regression analysis, time series analysis, index numbers, sampling, statistical quality control are some of the statistical methods employed in commerce and industry.

Limitations of Statistics

Before applying statistical methods, we must be aware of the following limitations:

I Statistics deals with the aggregates. An individual, to a statistician has no significance except the fact that it is a part of the aggregate.

II Statistics is concerned with quantitative data. However, qualitative data also can be converted to quantitative data by providing a numerical description to the corresponding qualitative data.

III Future projections of sales, production, price and quantity etc. are possible under a specific set of conditions. If any of these conditions is violated, projections are likely to be inaccurate.

IV The theory of statistical inferences is built upon random sampling. If the rules for random sampling is not strictly adhered to, the conclusion drawn on the basis of these unrepresentative samples would be erroneous. In other words, the experts should be consulted before deciding the sampling scheme.

13.1.2 COLLECTION OF DATA

We may define 'data' as quantitative information about some particular characteristic(s) under consideration. Although a distinction can be made between a qualitative characteristic and a quantitative characteristic but so far as the statistical analysis of the characteristic is concerned, we need to convert qualitative information to quantitative information by providing a numeric descriptions to the given characteristic. In this connection, we may note that a quantitative characteristic is known as a variable or in other words, a variable is a measurable quantity. Again, a variable may be either discrete or continuous. When a variable assumes a finite or a countably infinite number of isolated values, it is known as a discrete variable. Examples of discrete variables may be found in the number of petals in a flower, the number of misprints a book contains, the number of road accidents in a particular locality and so on. A variable, on the other hand, is known to be continuous if it can assume any value from a given interval. Examples of continuous variables may be provided by height, weight, sale, profit and so on. Finally, a qualitative characteristic is known as an attribute. The gender of a baby, the nationality of a person, the colour of a flower etc. are examples of attributes.

We can broadly classify data as

(a) Primary;

(b) Secondary.

Collection of data plays the very important role for any statistical analysis. The data which are collected for the first time by an investigator or agency are known as primary data whereas the data are known to be secondary if the data, as being already collected, are used by a different person or agency. Thus, if Prof. Das collects the data on the height of every student in his class, then these would be primary data for him. If, however, another person, say, Professor Bhargava uses the data, as collected by Prof. Das, for finding the average height of the students belonging to that class, then the data would be secondary for Prof. Bhargava.

Collection of Primary Data

The following methods are employed for the collection of primary data:

(i) Interview method;
(ii) Mailed questionnaire method;
(iii) Observation method;
(iv) Questionnaires filled and sent by enumerators.

Interview method again could be divided into (a) Personal Interview method, (b) Indirect Interview method and (c) Telephone Interview method.

In personal interview method, the investigator meets the respondents directly and collects the required information then and there from them. In case of a natural calamity like a super cyclone or an earthquake or an epidemic like plague, we may collect the necessary data much more quickly and accurately by applying this method.

If there are some practical problems in reaching the respondents directly, as in the case of a rail accident, then we may take recourse for conducting Indirect Interview where the investigator collects the necessary information from the persons associated with the problems.

Telephone interview method is a quick and rather non-expensive way to collect the primary data where the relevant information can be gathered by the researcher himself by contacting the interviewee over the phone. The first two methods, though more accurate, are inapplicable for covering a large area whereas the telephone interview, though less consistent, has a wide coverage.

The nuculer of non-responses is maximum for this third method of data collection.

Mailed questionnaire method comprises of framing a well-drafted and soundly-sequenced questionnaire covering all the important aspects of the problem under consideration and sending them to the respondents with pre-paid stamp after providing all the necessary guidelines for filling up the questionnaire. Although a wide area can be covered using the mailed questionnaire method, the amount of non-responses is likely to be maximum in this method.

In observation nuculer, data are collected, as in the case of obtaining the data on the height and weight of a group of students, by direct observation or using instrument. Although this is likely to be the best method for data collection, it is time consuming, laborious and covers only a small area. Questionnaire form of data collection is used for larger enquiries from the persons who are surveyed. Enumerators collects information directly by interviewing the persons having information : Question are explained and hence data is collected.

Sources of Secondary Data

There are many sources of getting secondary data. Some important sources are listed below:

(a) International sources like WHO, ILO, IMF, World Bank etc.

(b) Government sources like Statistical Abstract by CSO, Indian Agricultural Statistics by the Ministry of Food and Agriculture and so on.

(c) Private and quasi-government sources like ISI, ICAR, NCERT etc.

(d) Unpublished sources of various research institutes, researchers etc.

Scrutiny of Data

Since the statistical analyses are made only on the basis of data, it is necessary to check whether the data under consideration are accurate as well as consistence. No hard and fast rules can be recommended for the scrutiny of data. One must apply his intelligence, patience and experience while scrutinising the given information.

Errors in data may creep in while writing or copying the answer on the part of the enumerator. A keen observer can easily detect that type of error. Again, there may be two or more series of figures which are in some way or other related to each other. If the data for all the series are provided, they may be checked for internal consistency. As an example, if the data for population, area and density for some places are given, then we may verify whether they are internally consistent by examining whether the relation

$$\text{Density} = \frac{\text{Population}}{\text{Area}} \quad \text{holds.}$$

A good statistician can also detect whether the returns submitted by some enumerators are exactly of the same type thereby implying the lack of seriousness on the part of the enumerators. The bias of the enumerator also may be reflected by the returns submitted by him. This type of error can be rectified by asking the enumerator(s) to collect the data for the disputed cases once again.

13.1.3 PRESENTATION OF DATA

Once the data are collected and verified for their homogeneity and consistency, we need to present them in a neat and condensed form highlighting the essential features of the data. Any statistical analysis is dependent on a proper presentation of the data under consideration.

Classification or Organisation of Data

It may be defined as the process of arranging data on the basis of the characteristic under consideration into a number of groups or classes according to the similarities of the observations. Following are the objectives of classification of data:

(a) It puts the data in a neat, precise and condensed form so that it is easily understood and interpreted.

(b) It makes comparison possible between various characteristics, if necessary, and thereby finding the association or the lack of it between them.

(c) Statistical analysis is possible only for the classified data.

(d) It eliminates unnecessary details and makes data more readily understandable.

Data may be classified as -

(i) Chronological or Temporal or Time Series Data;

(ii) Geographical or Spatial Series Data;

(iii) Qualitative or Ordinal Data;

(iv) Quantitative or Cardinal Data.

When the data are classified in respect of successive time points or intervals, they are known as time series data. The number of students appeared for CA final for the last twenty years, the production of a factory per month from 2000 to 2015 etc. are examples of time series data.

Data arranged region wise are known as geographical data. If we arrange the students appeared for CA final in the year 2015 in accordance with different states, then we come across Geographical Data.

Data classified in respect of an attribute are referred to as qualitative data. Data on nationality, gender, smoking habit of a group of individuals are examples of qualitative data. Lastly, when the data are classified in respect of a variable, say height, weight, profits, salaries etc., they are known as quantitative data.

Data may be further classified as *frequency data* and *non-frequency data*. The qualitative as well as quantitative data belong to the frequency group whereas time series data and geographical data belong to the non-frequency group.

Mode of Presentation of Data

Next, we consider the following mode of presentation of data:

(a) Textual presentation;

(b) Tabular presentation or Tabulation;

(c) Diagrammatic representation.

(a) Textual presentation

This method comprises presenting data with the help of a paragraph or a number of paragraphs. The official report of an enquiry commission is usually made by textual presentation. Following is an example of textual presentation.

'In 2009, out of a total of five thousand workers of Roy Enamel Factory, four thousand and two hundred were members of a Trade Union. The number of female workers was twenty per cent of the total workers out of which thirty per cent were members of the Trade Union.

In 2010, the number of workers belonging to the trade union was increased by twenty per cent as compared to 2009 of which four thousand and two hundred were male. The number of workers not belonging to trade union was nine hundred and fifty of which four hundred and fifty were females.'

The merit of this mode of presentation lies in its simplicity and even a layman can present data by this method. The observations with exact magnitude can be presented with the help of textual presentation. Furthermore, this type of presentation can be taken as the first step towards the other methods of presentation.

Textual presentation, however, is not preferred by a statistician simply because, it is dull, monotonous and comparison between different observations is not possible in this method. For manifold classification, this method cannot be recommended.

(b) Tabular presentation or Tabulation

Tabulation may be defined as systematic presentation of data with the help of a statistical table having a number of rows and columns and complete with reference number, title, description of rows as well as columns and foot notes, if any.

We may consider the following guidelines for tabulation :

I A statistical table should be allotted a serial number along with a self-explanatory title.

II The table under consideration should be divided into caption, Box-head, Stub and Body. Caption is the upper part of the table, describing the columns and sub-columns, if any. The Box-head is the entire upper part of the table which includes columns and sub-column numbers, unit(s) of measurement along with caption. Stub is the left part of the table providing the description of the rows. The body is the main part of the table that contains the numerical figures.

III The table should be well-balanced in length and breadth.

IV The data must be arranged in a table in such a way that comparison(s) between different figures are made possible without much labour and time. Also, the row totals, column totals, the units of measurement must be shown.

V The data should be arranged intelligently in a well-balanced sequence and the presentation of data in the table should be appealing to the eyes as far as practicable.

VI Notes describing the source of the data and bringing clarity and, if necessary, about any rows or columns known as footnotes, should be shown at the bottom part of the table.

The textual presentation of data, relating to the workers of Roy Enamel Factory is shown in the following table.

Table 13.1.1

Status of the workers of Roy Enamel factory on the basis of their trade union membership for 2009 and 2010.

Status	Member of TU			Non-member			Total		
Year	M (1)	F (2)	T (3)=(1)+ (2)	M (4)	F (5)	T (6)=(4)+ (5)	M (7)	F (8)	T (9)=(7)+ (8)
2009	3900	300	4200	300	500	800	4200	800	5000
2010	4200	840	5040	500	450	950	4700	1290	5990

Source:

Footnote: TU, M, F and T stand for trade union, male, female and total respectively.

The tabulation method is usually preferred to textual presentation as

(i) It facilitates comparison between rows and columns.

(ii) Complicated data can also be represented using tabulation.

(iii) It is a must for diagrammatic representation.

(iv) Without tabulation, statistical analysis of data is not possible.

(c) Diagrammatic representation of data

Another alternative and attractive representation of statistical data is provided by charts, diagrams and pictures. Unlike the first two methods of representation of data, diagrammatic representation can be used for both the educated section and uneducated section of the society. Furthermore, any hidden trend present in the given data can be noticed only in this mode of representation. However, compared to tabulation, this is less accurate. So, if there is a priority for accuracy, we have to recommend tabulation.

We are going to consider the following types of diagrams :

I Line diagram or Historiagram;

II Bar diagram;

III Pie chart.

I Line diagram or Historiagram

When the data vary over time, we take recourse to line diagram. In a simple line diagram, we plot each pair of values of (t, y_t), y_t representing the time series at the time point t in the $t–y_t$ plane. The plotted points are then joined successively by line segments and the resulting chart is known as line-diagram.

When the time series exhibit a wide range of fluctuations, we may think of logarithmic or ratio chart where Log y_t and not y_t is plotted against t. We use Multiple line chart for representing two or more related time series data expressed in the same unit and multiple – axis chart in somewhat similar situations if the variables are expressed in different units.

II **Bar diagram**

There are two types of bar diagrams namely, Horizontal Bar diagram and Vertical Bar diagram. While horizontal bar diagram is used for qualitative data or data varying over space, the vertical bar diagram is associated with quantitative data or time series data. Bars i.e. rectangles of equal width and usually of varying lengths are drawn either horizontally or vertically. We consider Multiple or Grouped Bar diagrams to compare related series. Component or sub-divided Bar diagrams are applied for representing data divided into a number of components. Finally, we use Divided Bar charts or Percentage Bar diagrams for comparing different components of a variable and also the relating of the components to the whole. For this situation, we may also use Pie chart or Pie diagram or circle diagram.

ILLUSTRATIONS:

Example 13.1.1: The profits in lakhs of Rupees of an industrial house for 2009, 2010, 2011, 2012, 2013, 2014, and 2015 are 5, 8, 9, 6, 12, 15 and 24 respectively. Represent these data using a suitable diagram.

SOLUTION:

We can represent the profits for 7 consecutive years by drawing either a line chart or a vertical bar chart. Fig. 13.1.1 shows a line chart and figure 13.1.2 shows the corresponding vertical bar chart.

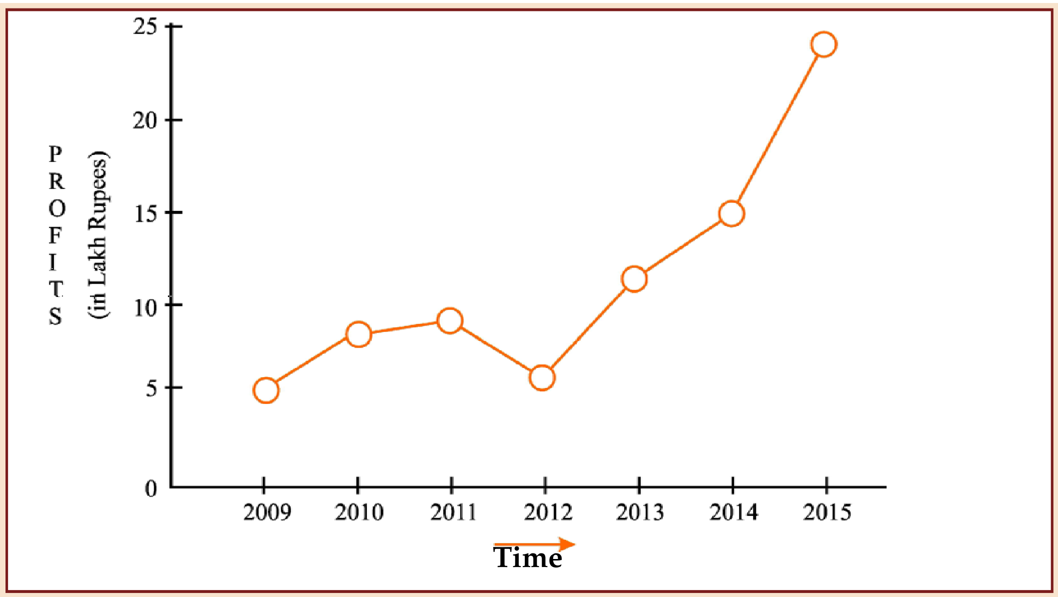


Figure 13.1.1

Showing line chart for the Profit of an Industrial House during 2009 to 2015.

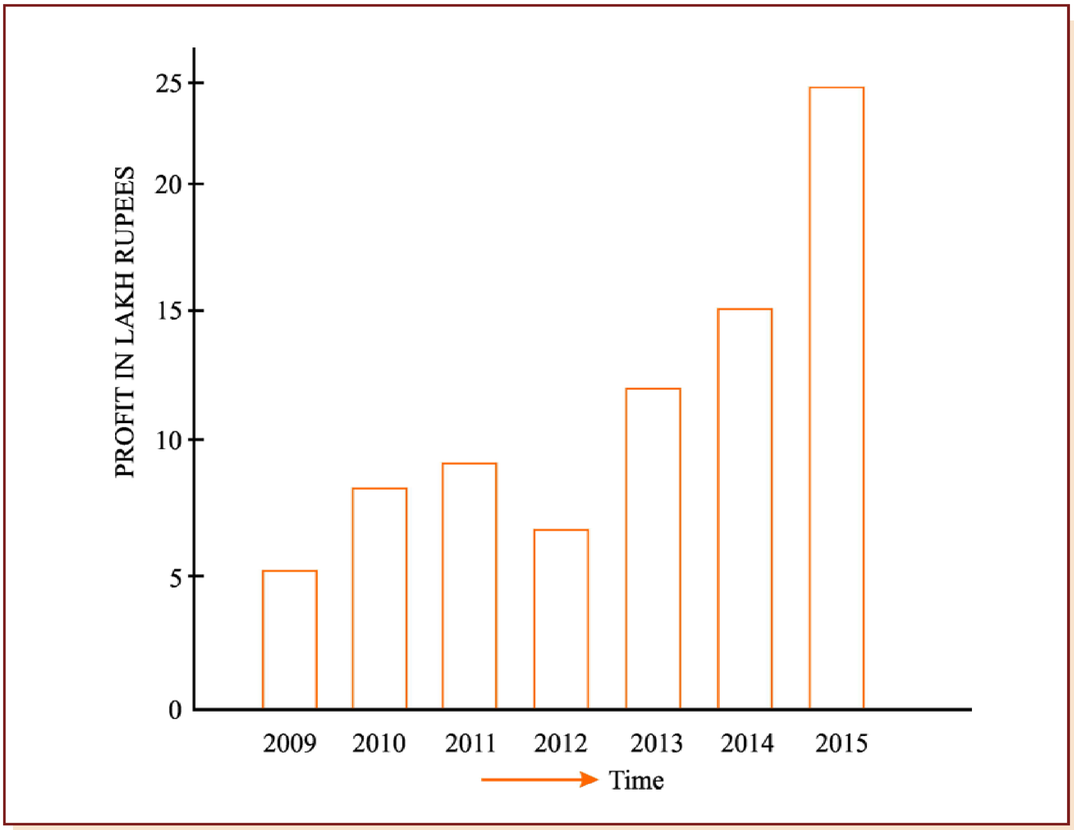


Figure 13.1.2

Showing vertical bar diagram for the Profit of an Industrial house from 2009 to 2015.

Example 13.1.2: The production of wheat and rice of a region are given below :

Year	Production in metric tones	
	Wheat	Rice
2012	12	25
2013	15	30
2014	18	32
2015	19	36

Represent this information using a suitable diagram.

Solution:

We can represent this information by drawing a multiple line chart. Alternately, a multiple bar diagram may be considered. These are depicted in figure 13.1.3 and 13.1.4 respectively.

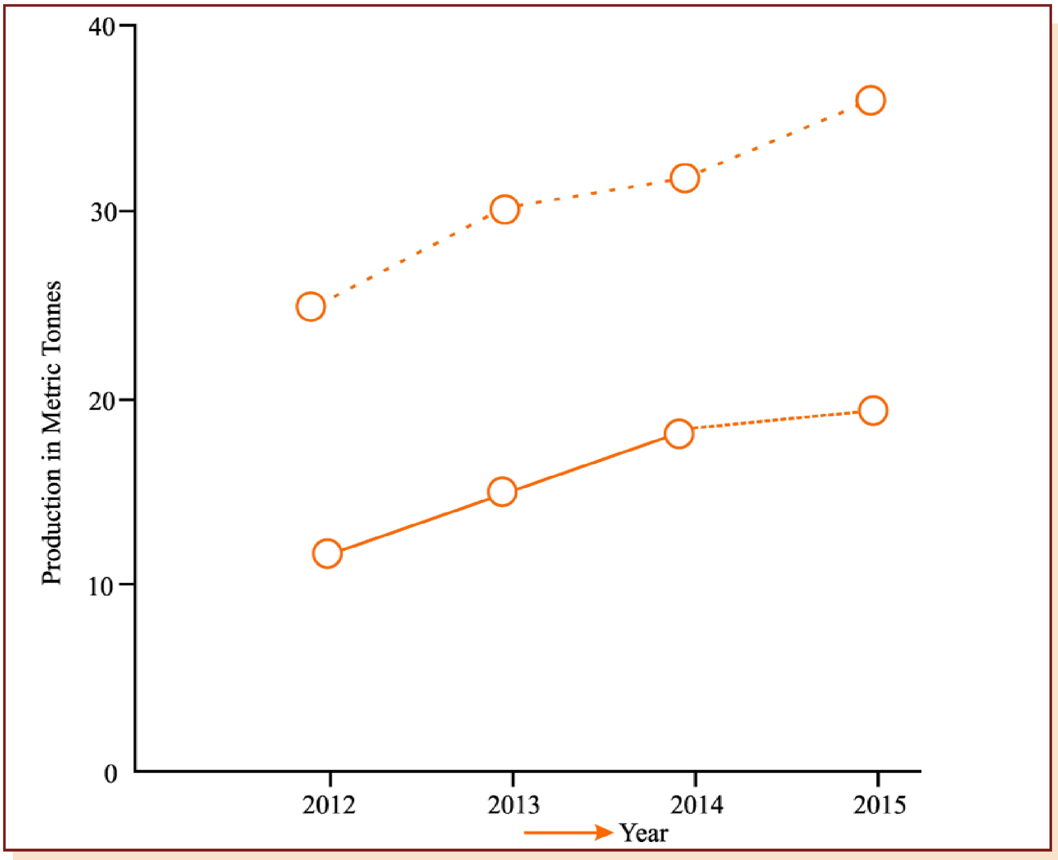


Figure 13.1.3

Multiple line chart showing production of wheat and rice of a region during 2012–2015. (Dotted line represent production of rice and continuous line that of wheat).

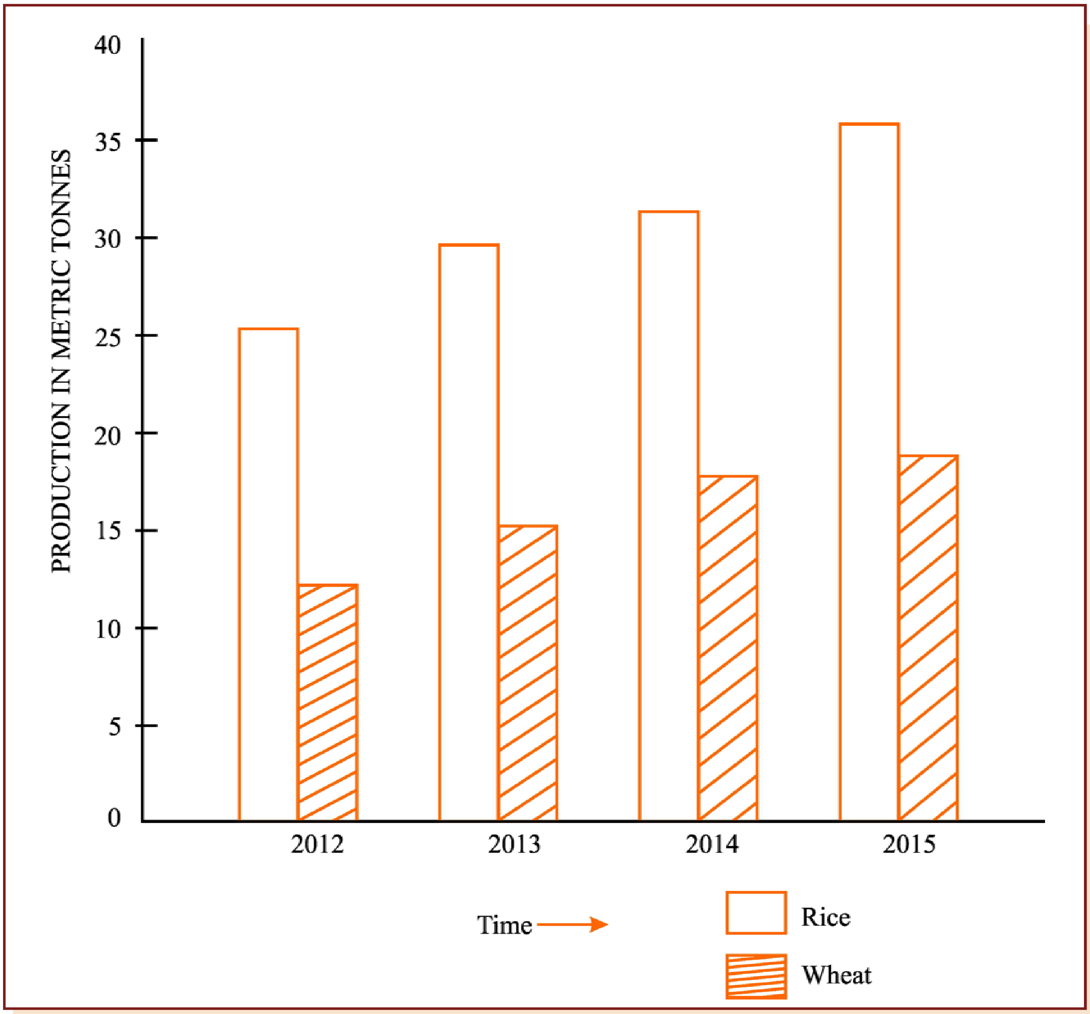


Figure 13.1.4

Multiple bar chart representing production of rice and wheat from 2012 to 2015.

Example 13.1.3: Draw an appropriate diagram with a view to represent the following data :

Source	Revenue in millions of (₹)
Customs	80
Excise	190
Income Tax	160
Corporate Tax	75
Miscellaneous	35

Solution:

Pie chart or divided bar chart would be the ideal diagram to represent this data. We consider Pie chart.

Table 13.1.2

Computation for drawing Pie chart

Source (1)	Revenue in Million rupees (2)	Central angle $= \frac{(2)}{\text{Total of (2)}} \times 360^o$
Customs	80	$\frac{80}{540} \times 360^o = 53^o$ (approx.)
Excise	190	$\frac{190}{540} \times 360^o = 127^o$
Income Tax	160	$\frac{160}{540} \times 360^o = 107^o$
Corporate Tax	75	$\frac{75}{540} \times 360^o = 50^o$
Miscellaneous	35	$\frac{35}{540} \times 360^o = 23^o$
Total	540	360^0

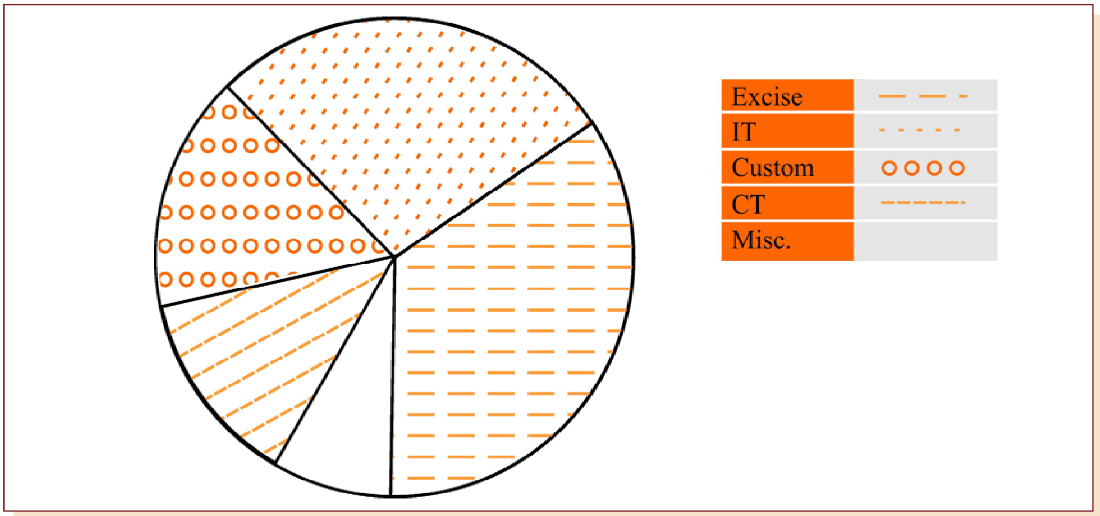


Figure 13.1.5

Pie chart showing the distribution of Revenue

13.1. 4 FREQUENCY DISTRIBUTION

As discussed in the previous section, frequency data occur when we classify statistical data in respect of either a variable or an attribute. A frequency distribution may be defined as a tabular representation of statistical data, usually in an ascending order, relating to a measurable characteristic according to individual value or a group of values of the characteristic under study.

In case, the characteristic under consideration is an attribute, say nationality, then the tabulation is made by allotting numerical figures to the different classes the attribute may belong like, in this illustration, counting the number of Indian, British, French, German and so on. The qualitative characteristic is divided into a number of categories or classes which are mutually exclusive and exhaustive and the figures against all these classes are recorded. The figure corresponding to a particular class, signifying the number of times or how frequently a particular class occurs is known as the frequency of that class. Thus, the number of Indians, as found from the given data, signifies the frequency of the Indians. So frequency distribution is a statistical table that distributes the total frequency to a number of classes.

When tabulation is done in respect of a discrete random variable, it is known as Discrete or Ungrouped or simple Frequency Distribution and in case the characteristic under consideration is a continuous variable, such a classification is termed as Grouped Frequency Distribution. In case of a grouped frequency distribution, tabulation is done not against a single value as in the case of an attribute or a discrete random variable but against a group of values. The distribution of the number of car accidents in Delhi during 12 months of the year 2005 is an example of a ungrouped frequency distribution and the distribution of heights of the students of St. Xavier's College for the year 2004 is an example of a grouped frequency distribution.

Example 13.1.4: Following are the records of babies born in a nursing home in Bangalore during a week (B denoting Boy and G for Girl) :

B	G	G	B	G	G	B	B	G	G
G	G	B	B	B	G	B	B	G	B
B	B	G	B	B	B	G	G	B	G

Construct a frequency distribution according to gender.

Solution:

In order to construct a frequency distribution of babies in accordance with their gender, we count the number of male births and that of female births and present this information in the following table.

Table 13.1.3

Frequency distribution of babies according to Gender

Category	Number of births
Boy (B)	16
Girl (G)	14
Total	30

Frequency Distribution of a Variable

For the construction of a frequency distribution of a variable, we need to go through the following steps :

I Find the largest and smallest observations and obtain the difference between them, known as Range, in case of a continuous variable.

II Form a number of classes depending on the number of isolated values assumed by a discrete variable. In case of a continuous variable, find the number of class intervals using the relation, No. of class Interval × class length ≅ Range.

III Present the class or class interval in a table known as frequency distribution table.

IV Apply 'tally mark' i.e. a stroke against the occurrence of a particulars value in a class or class interval.

V Count the tally marks and present these numbers in the next column, known as frequency column, and finally check whether the total of all these class frequencies tally with the total number of observations.

Example 13.1.5: A review of the first 30 pages of a statistics book reveals the following printing mistakes:

0	1	3	3	2	5	6	0	1	0
4	1	1	0	2	3	2	5	0	4
2	3	2	2	3	3	4	6	1	4

Make a frequency distribution of printing mistakes.

Solution:

Since x, the printing mistakes, is a discrete variable, x can assume seven values 0, 1, 2, 3, 4, 5 and 6. Thus we have 7 classes, each class comprising a single value.

Table 13.1.4

Frequency Distribution of the number of printing mistakes of the first 30 pages of a book

Printing Mistake	Tally marks	Frequency (No. of Pages)
0	~~IIII~~	5
1	~~IIII~~	5
2	~~IIII~~ I	6
3	~~IIII~~ I	6
4	IIII	4
5	II	2
6	II	2
Total	–	30

Example 13.1.6: Following are the weights in kgs. of 36 BBA students of St. Xavier's College.

70	73	49	61	61	47	57	50	59
59	68	45	55	65	68	56	68	55
70	70	57	44	69	73	64	49	63
65	70	65	62	64	73	67	60	50

Construct a frequency distribution of weights, taking class length as 5.

Solution:

We have, Range = Maximum weight – Minimum weight

= 73 kgs. – 44 kgs.

= 29 kgs.

No. of class interval × class lengths ≅ Range

$\Rightarrow$ No. of class interval $\times\ 5 \cong 29$

$\Rightarrow$ No. of class interval $= \frac{29}{5} \cong 6.$

(We always take the next integer as the number of class intervals so as to include both the minimum and maximum values).

Table 13.1.5

Frequency Distribution of weights of 36 BBA Students

Weight in kg (Class Interval)	Tally marks	No. of Students (Frequency)
44-48	III	3
49-53	IIII	4
54-58	~~IIII~~	5
59-63	~~IIII~~ II	7
64-68	~~IIII~~ IIII	9
69-73	~~IIII~~ III	8
Total	–	36

Some important terms associated with a frequency distribution

Class Limit (CL)

Corresponding to a class interval, the class limits may be defined as the minimum value and the maximum value the class interval may contain. The minimum value is known as the lower class limit (LCL) and the maximum value is known as the upper class limit (UCL). For the frequency distribution of weights of BBA Students, the LCL and UCL of the first class interval are 44 kgs. and 48 kgs. respectively.

Class Boundary (CB)

Class boundaries may be defined as the actual class limit of a class interval. For overlapping classification or mutually exclusive classification that excludes the upper class limits like 10–20, 20–30, 30–40, ……… etc. the class boundaries coincide with the class limits. This is usually done for a continuous variable. However, for non-overlapping or mutually inclusive classification that includes both the class limits like 0–9, 10–19, 20–29,…… which is usually applicable for a discrete variable, we have

$$\text{LCB} = \text{LCL} - \frac{D}{2}$$

$$\text{and UCB} = \text{UCL} + \frac{D}{2}$$

where D is the difference between the LCL of the next class interval and the UCL of the given class interval. For the data presented in table 10.5, LCB of the first class interval

$$= 44 \text{ kgs.} - \frac{(49 - 48)}{2} \text{kgs.}$$

= 43.50 kgs.

and the corresponding UCB

$$= 48\text{ kgs.} + \frac{49 - 48}{2}\text{ kgs.}$$

= 48.50 kgs.

Mid-point or Mid-value or class mark

Corresponding to a class interval, this may be defined as the total of the two class limits or class boundaries to be divided by 2. Thus, we have

$$\text{mid-point} = \frac{\text{LCL} + \text{UCL}}{2}$$

$$= \frac{\text{LCB} + \text{UCB}}{2}$$

Referring to the distribution of weight of BBA students, the mid-points for the first two class intervals are

$$\frac{44\text{ kgs.} + 48\text{ kgs.}}{2} \text{ and } \frac{49\text{ kgs.} + 53\text{ kgs.}}{2}$$

i.e. 46 kgs. and 51 kgs. respectively.

Width or size of a class interval

The width of a class interval may be defined as the difference between the UCB and the LCB of that class interval. For the distribution of weights of BBA students, C, the class length or width is 48.50 kgs. – 43.50 kgs. = 5 kgs. for the first class interval. For the other class intervals also, C remains same.

Cumulative Frequency

The cumulative frequency corresponding to a value for a discrete variable and corresponding to a class boundary for a continuous variable may be defined as the number of observations less than the value or less than or equal to the class boundary. This definition refers to the less than cumulative frequency. We can define more than cumulative frequency in a similar manner. Both types of cumulative frequencies are shown in the following table.

Table 13.1.6

Cumulative Frequency Distribution of weights of 36 BBA students

Weight in kg (CB)	Cumulative Frequency	
	Less than	More than
43.50	0	33 + 3 or 36
48.50	0 + 3 or 3	29 + 4 or 33
53.50	3 + 4 or 7	24 + 5 or 29
58.50	7 + 5 or 12	17 + 7 or 24
63.50	12 + 7 or 19	8 + 9 or 17
68.50	19 + 9 or 28	0 + 8 or 8
73.50	28 + 8 or 36	0

Frequency density of a class interval

It may be defined as the ratio of the frequency of that class interval to the corresponding class length. The frequency densities for the first two class intervals of the frequency distribution of weights of BBA students are 3/5 and 4/5 i.e. 0.60 and 0.80 respectively.

Relative frequency and percentage frequency of a class interval

Relative frequency of a class interval may be defined as the ratio of the class frequency to the total frequency. Percentage frequency of a class interval may be defined as the ratio of class frequency to the total frequency, expressed as a percentage. For the last example, the relative frequencies for the first two class intervals are 3/36 and 4/36 respectively and the percentage frequencies are 300/36 and 400/36 respectively. It is quite obvious that whereas the relative frequencies add up to unity, the percentage frequencies add up to one hundred.

13.1.5 GRAPHICAL REPRESENTATION OF A FREQUENCY DISTRIBUTION

We consider the following types of graphical representation of frequency distribution :

(i) Histogram or Area diagram;

(ii) Frequency Polygon;

(iii) Ogives or cumulative Frequency graphs.

(i) **Histogram or Area diagram**

This is a very convenient way to represent a frequency distribution. Histogram helps us to get an idea of the frequency curve of the variable under study. Some statistical measure can be obtained using a histogram. A comparison among the frequencies for different class intervals is possible in this mode of diagrammatic representation.

In order to draw a histogram, the class limits are first converted to the corresponding class boundaries and a series of adjacent rectangles, one against each class interval, with the class interval as base or breadth and the frequency or frequency density usually when the class intervals are not uniform as length or altitude, is erected. The histogram for the distribution of weight of 36 BBA students is shown below. The mode of the weights has also been determined using the histogram.

i.e. Mode = 66.50 kgs.

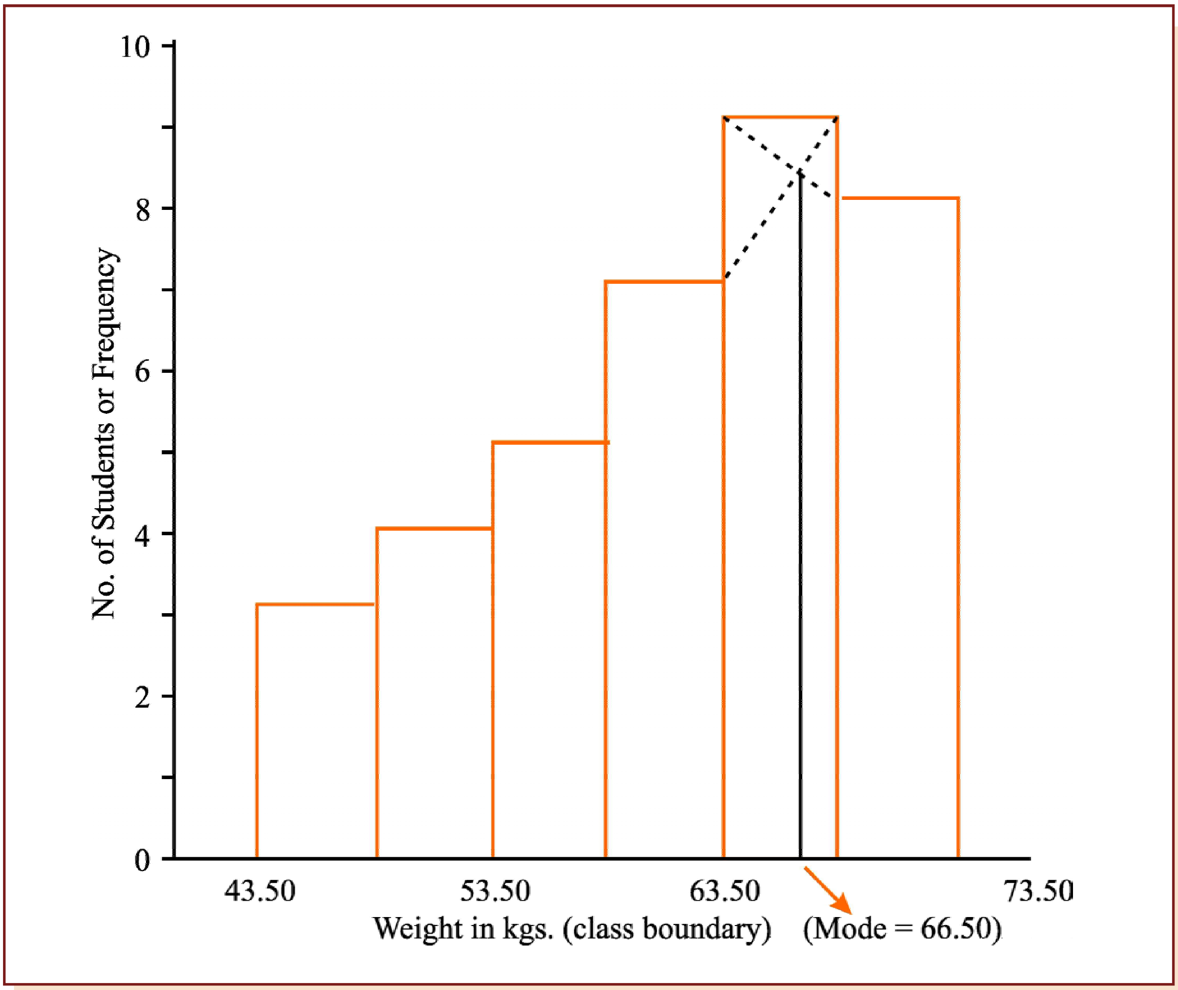


Figure 13.1.6

Showing histogram for the distribution of weight of 36 BBA students

(ii) **Frequency Polygon**

Usually frequency polygon is meant for single frequency distribution. However, we also apply it for grouped frequency distribution provided the width of the class intervals remains the same. A frequency curve can be regarded as a limiting form of frequency polygon. In order to draw a frequency polygon, we plot (x_i, f_i) for i = 1, 2, 3, ……….. n with x_i denoting the mid-point of the its class interval and f_i, the corresponding frequency, n being the number of class intervals. The plotted points are joined successively by line segments and the figure, so drawn, is given the shape of a polygon, a closed figure, by joining the two extreme ends of the drawn figure to two additional points (x_0,0) and (x_{n+1},0).

The frequency polygon for the distribution of weights of BBA students is shown in Figure 13.7. We can also obtain a frequency polygon starting with a histogram by adding the mid-points of the upper sides of the rectangles successively and then completing the figure by joining the two ends as before.

Mid-points	No. of Students (Frequency)
46	3
51	4
56	5
61	7
66	9
71	8

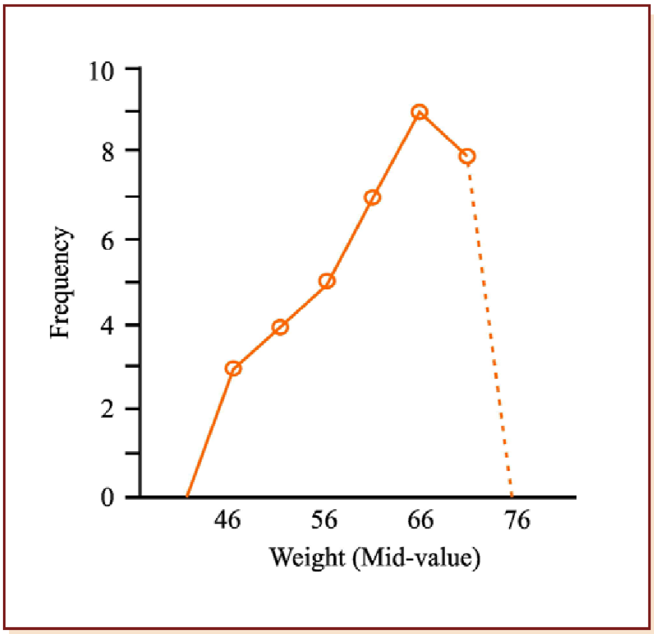


Figure 13.1.7

Showing frequency polygon for the distribution of height of 36 BBA students

(iii) Ogives or Cumulative Frequency Graph

By plotting cumulative frequency against the respective class boundary, we get ogives. As such there are two ogives – less than type ogives, obtained by taking less than cumulative frequency on the vertical axis and more than type ogives by plotting more than type cumulative frequency on the vertical axis and thereafter joining the plotted points successively by line segments. Ogives may be considered for obtaining quartiles graphically. If a perpendicular is drawn from the point of intersection of the two ogives on the horizontal axis, then the x-value of this point gives us the value of median, the second or middle quartile. Ogives further can be put into use for making short term projections.

Figure 13.8 depicts the ogives and the determination of the quartiles. This figure give us the following information.

1st quartile or lower quartile (Q_1) = 55 kgs.

2nd quartile or median (Q_2 or Me) = 62.50 kgs.

3rd quartile or upper quartile (Q_3) = 68 kgs.

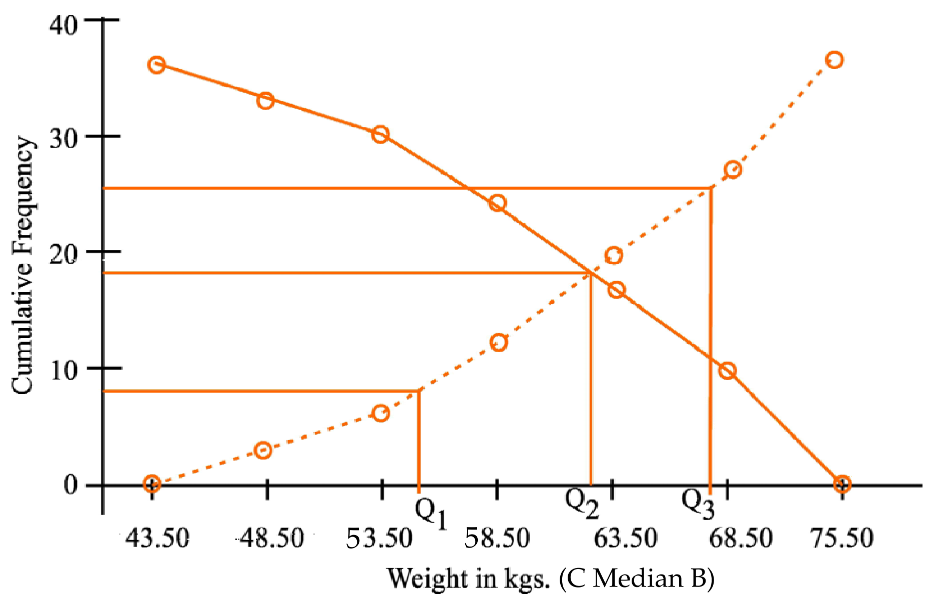


Figure 13.1.8

Showing the ogives for the distribution of weights of 36 BBA students

We find Q_1 = 55 kgs.

Q_2 = Me = 62.50 kgs.

Q_3 = 68 kgs.

Frequency Curve

A frequency curve is a smooth curve for which the total area is taken to be unity. It is a limiting form of a histogram or frequency polygon. The frequency curve for a distribution can be obtained by drawing a smooth and free hand curve through the mid-points of the upper sides of the rectangles forming the histogram.

There exist four types of frequency curves namely

(a) Bell-shaped curve;

(b) U-shaped curve;

(c) J-shaped curve;

(d) Mixed curve.

Most of the commonly used distributions provide bell-shaped curve, which, as suggested by the name, looks almost like a bell. The distribution of height, weight, mark, profit etc. usually belong to this category. On a bell-shaped curve, the frequency, starting from a rather low value, gradually reaches the maximum value, somewhere near the central part and then gradually decreases to reach its lowest value at the other extremity.

For a U-shaped curve, the frequency is minimum near the central part and the frequency slowly but steadily reaches its maximum at the two extremities. The distribution of Kolkata bound commuters belongs to this type of curve as there are maximum number of commuters during the peak hours in the morning and in the evening.

The J-shaped curve starts with a minimum frequency and then gradually reaches its maximum frequency at the other extremity. The distribution of commuters coming to Kolkata from the early morning hour to peak morning hour follows such a distribution. Sometimes, we may also come across an inverted J-shaped frequency curve.

Lastly, we may have a combination of these frequency curves, known as mixed curve. These are exhibited in the following figures.

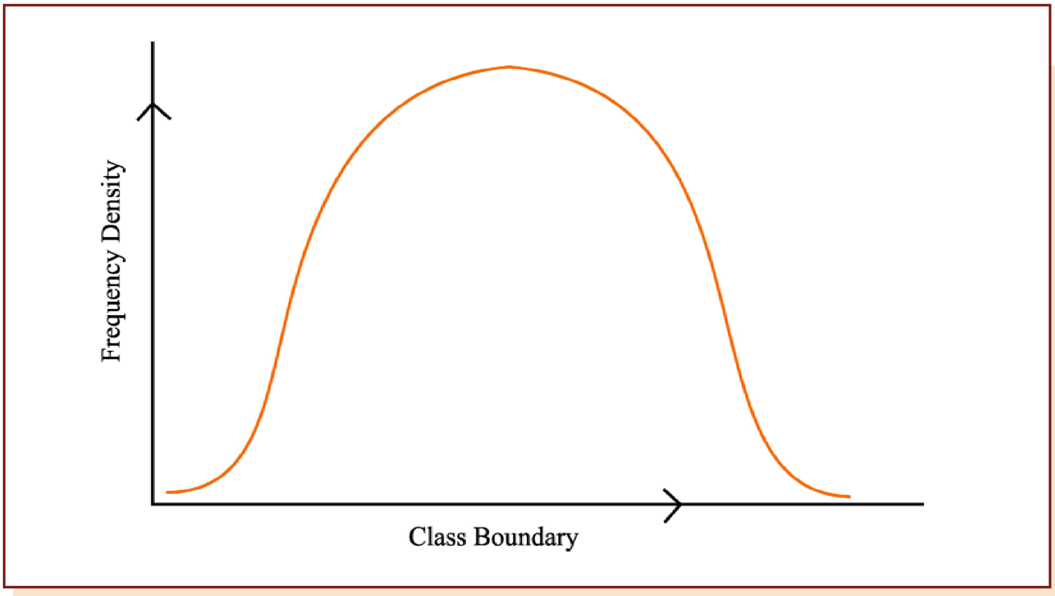


Figure 13.1.9

Bell-shaped curve

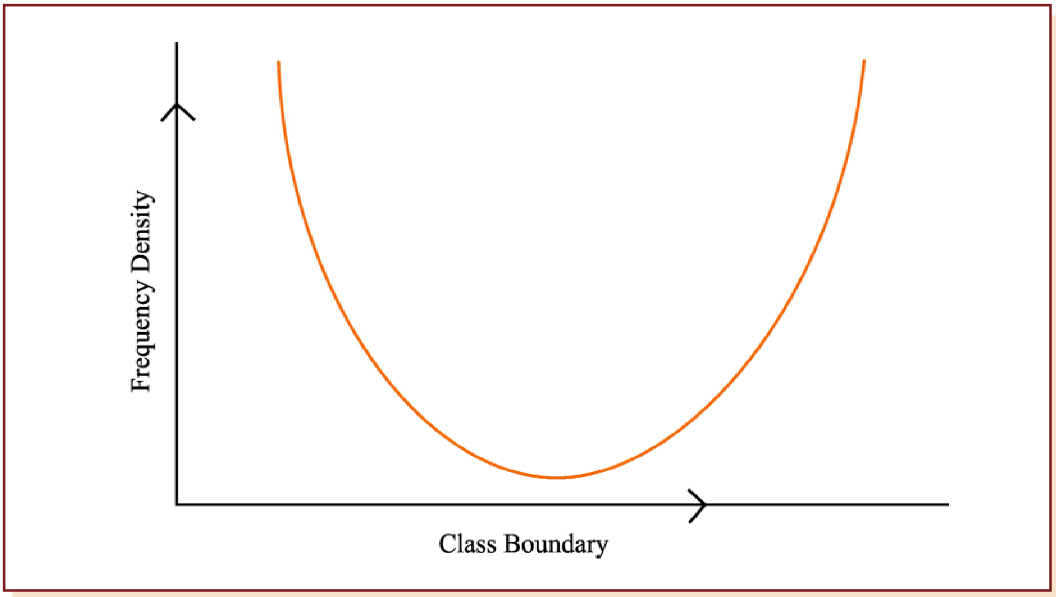


Figure 13.1.10

U-shaped curve

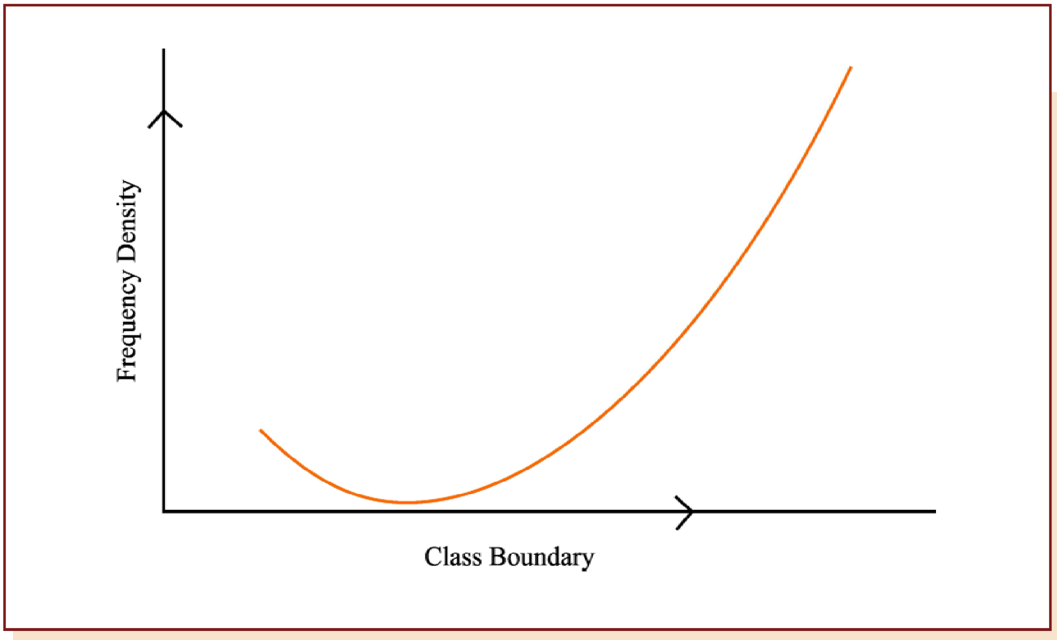


Figure 13.1.11

J-shaped curve

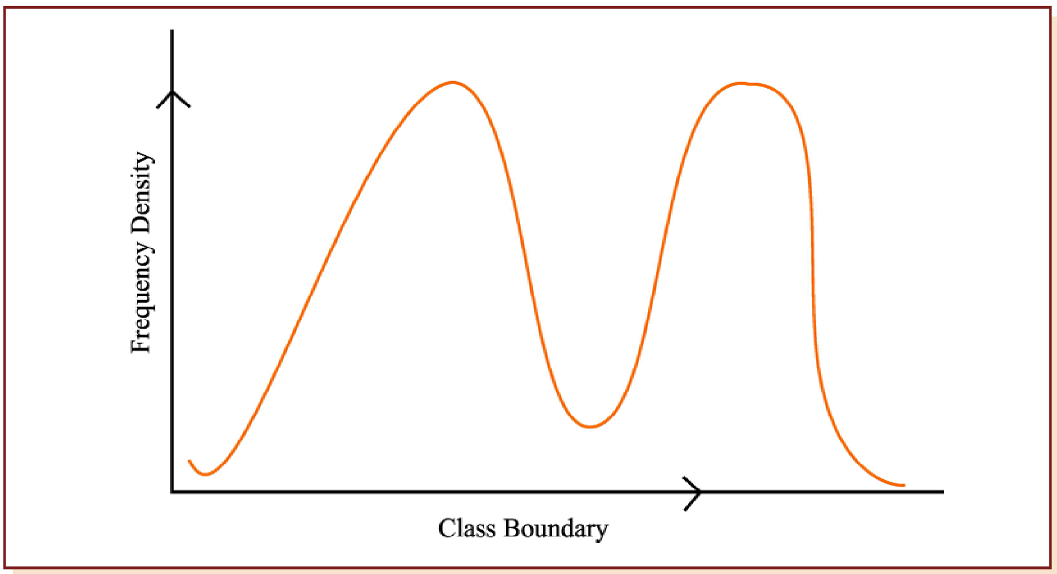


Figure 13.1.12

Mixed curve

SUMMARY

- Statistics deals with the aggregates. An individual, to a statistician has no significance except the fact that it is a part of the aggregate.
- Statistics is concerned with quantitative data. However, qualitative data also can be converted to quantitative data by providing a numerical description to the corresponding qualitative data.
- The theory of statistical inferences is built upon random sampling. If the rules for random sampling are not strictly adhered to, the conclusion drawn on the basis of these unrepresentative samples would be erroneous.
- We can broadly classify data as
  (a) Primary;
  (b) Secondary.
- Mode of Presentation of Data
  (a) Textual presentation;
  (b) Tabular presentation or Tabulation;
  (c) Diagrammatic representation.
- The types of diagrams:
  (a) Line diagram or Historiagram;
  (b) Bar diagram;
  (c) Pie chart.
- Frequency Distribution of a Variable
  (a) Find the largest and smallest observations and obtain the difference between them, known as Range, in case of a continuous variable.
  (b) Form a number of classes depending on the number of isolated values assumed by a discrete variable. In case of a continuous variable, find the number of class intervals using the relation, No. of class Interval × class length $\cong$ Range.
  (c) Present the class or class interval in a table known as frequency distribution table.
  (d) Apply 'tally mark' i.e. a stroke against the occurrence of a particulars value in a class or class interval.
  (e) Count the tally marks and present these numbers in the next column, known as frequency column, and finally check whether the total of all these class frequencies tally with the total number of observations.

UNIT I EXERCISE

Set A

Answer the following questions. Each question carries 1 mark.

1. Which of the following statements is false?
   (a) Statistics is derived from the Latin word 'Status'
   (b) Statistics is derived from the Italian word 'Statista'
   (c) Statistics is derived from the French word 'Statistik'
   (d) None of these.
2. Statistics is defined in terms of numerical data in the
   (a) Singular sense (b) Plural sense
   (c) Either (a) or (b) (d) Both (a) and (b).
3. Statistics is applied in
   (a) Economics (b) Business management
   (c) Commerce and industry (d) All these.
4. Statistics is concerned with
   (a) Qualitative information (b) Quantitative information
   (c) (a) or (b) (d) Both (a) and (b).
5. An attribute is
   (a) A qualitative characteristic (b) A quantitative characteristic
   (c) A measurable characteristic (d) All these.
6. Annual income of a person is
   (a) An attribute (b) A discrete variable
   (c) A continuous variable (d) (b) or (c).
7. Marks of a student is an example of
   (a) An attribute (b) A discrete variable
   (c) A continuous variable (d) None of these.
8. Nationality of a student is
   (a) An attribute (b) A continuous variable
   (c) A discrete variable (d) (a) or (c).
9. Drinking habit of a person is
   (a) An attribute (b) A variable
   (c) A discrete variable (d) A continuous variable.

10. Age of a person is
    (a) An attribute
    (b) A discrete variable
    (c) A continuous variable
    (d) A variable.
11. Data collected on religion from the census reports are
    (a) Primary data
    (b) Secondary data
    (c) Sample data
    (d) (a) or (b).
12. The data collected on the height of a group of students after recording their heights with a measuring tape are
    (a) Primary data
    (b) Secondary data
    (c) Discrete data
    (d) Continuous data.
13. The primary data are collected by
    (a) Interview method
    (b) Observation method
    (c) Questionnaire method
    (d) All these.
14. The quickest method to collect primary data is
    (a) Personal interview
    (b) Indirect interview
    (c) Telephone interview
    (d) By observation.
15. The best method to collect data, in case of a natural calamity, is
    (a) Personal interview
    (b) Indirect interview
    (c) Questionnaire method
    (d) Direct observation method.
16. In case of a rail accident, the appropriate method of data collection is by
    (a) Personal interview
    (b) Direct interview
    (c) Indirect interview
    (d) All these.
17. Which method of data collection covers the widest area?
    (a) Telephone interview method
    (b) Mailed questionnaire method
    (c) Direct interview method
    (d) All these.
18. The amount of non-responses is maximum in
    (a) Mailed questionnaire method
    (b) Interview method
    (c) Observation method
    (d) All these.
19. Some important sources of secondary data are
    (a) International and Government sources
    (b) International and primary sources
    (c) Private and primary sources
    (d) Government sources.

20. Internal consistency of the collected data can be checked when
    (a) Internal data are given
    (b) External data are given
    (c) Two or more series are given
    (d) A number of related series are given.
21. The accuracy and consistency of data can be verified by
    (a) Internal checking
    (b) External checking
    (c) Scrutiny
    (d) Both (a) and (b).
22. The mode of presentation of data are
    (a) Textual, tabulation and diagrammatic
    (b) Tabular, internal and external
    (c) Textual, tabular and internal
    (d) Tabular, textual and external.
23. The best method of presentation of data is
    (a) Textual
    (b) Tabular
    (c) Diagrammatic
    (d) (b) and (c).
24. The most attractive method of data presentation is
    (a) Tabular
    (b) Textual
    (c) Diagrammatic
    (d) (a) or (b).
25. For tabulation, 'caption' is
    (a) The upper part of the table
    (b) The lower part of the table
    (c) The main part of the table
    (d) The upper part of a table that describes the column and sub-column.
26. 'Stub' of a table is the
    (a) Left part of the table describing the columns
    (b) Right part of the table describing the columns
    (c) Right part of the table describing the rows
    (d) Left part of the table describing the rows.
27. The entire upper part of a table is known as
    (a) Caption
    (b) Stub
    (c) Box head
    (d) Body.
28. The unit of measurement in tabulation is shown in
    (a) Box head
    (b) Body
    (c) Caption
    (d) Stub.

29. In tabulation source of the data, if any, is shown in the
(a) Footnote (b) Body
(c) Stub (d) Caption.

30. Which of the following statements is untrue for tabulation?
(a) Statistical analysis of data requires tabulation
(b) It facilitates comparison between rows and not columns
(c) Complicated data can be presented
(d) Diagrammatic representation of data requires tabulation.

31. Hidden trend, if any, in the data can be noticed in
(a) Textual presentation (b) Tabulation
(c) Diagrammatic representation (d) All these.

32. Diagrammatic representation of data is done by
(a) Diagrams (b) Charts
(c) Pictures (d) All these.

33. The most accurate mode of data presentation is
(a) Diagrammatic method (b) Tabulation
(c) Textual presentation (d) None of these.

34. The chart that uses logarithm of the variable is known as
(a) Line chart (b) Ratio chart
(c) Multiple line chart (d) Component line chart.

35. Multiple line chart is applied for
(a) Showing multiple charts
(b) Two or more related time series when the variables are expressed in the same unit
(c) Two or more related time series when the variables are expressed in different unit
(d) Multiple variations in the time series.

36. Multiple axis line chart is considered when
(a) There is more than one time series (b) The units of the variables are different
(c) (a) or (b) (d) (a) and (b).

37. Horizontal bar diagram is used for
(a) Qualitative data (b) Data varying over time
(c) Data varying over space (d) (a) or (c).

38. Vertical bar diagram is applicable when
    (a) The data are qualitative
    (b) The data are quantitative
    (c) When the data vary over time
    (d) (b) or (c).
39. Divided bar chart is considered for
    (a) Comparing different components of a variable
    (b) The relation of different components to the table
    (c) (a) or (b)
    (d) (a) and (b).
40. In order to compare two or more related series, we consider
    (a) Multiple bar chart
    (b) Grouped bar chart
    (c) (a) or (b)
    (d) (a) and (b).
41. Pie-diagram is used for
    (a) Comparing different components and their relation to the total
    (b) Representing qualitative data in a circle
    (c) Representing quantitative data in circle
    (d) (b) or (c).
42. A frequency distribution
    (a) Arranges observations in an increasing order
    (b) Arranges observation in terms of a number of groups
    (c) Relates to a measurable characteristic
    (d) All these.
43. The frequency distribution of a continuous variable is known as
    (a) Grouped frequency distribution
    (b) Simple frequency distribution
    (c) (a) or (b)
    (d) (a) and (b).

44. The distribution of shares is an example of the frequency distribution of
    (a) A discrete variable
    (b) A continuous variable
    (c) An attribute
    (d) (a) or (c).
45. The distribution of profits of a blue-chip company relates to
    (a) Discrete variable
    (b) Continuous variable
    (c) Attributes
    (d) (a) or (b).
46. Mutually exclusive classification
    (a) Excludes both the class limits
    (b) Excludes the upper class limit but includes the lower class limit
    (c) Includes the upper class limit but excludes the upper class limit
    (d) Either (b) or (c).
47. Mutually inclusive classification is usually meant for
    (a) A discrete variable
    (b) A continuous variable
    (c) An attribute
    (d) All these.
48. Mutually exclusive classification is usually meant for
    (a) A discrete variable
    (b) A continuous variable
    (c) An attribute
    (d) Any of these.
49. The LCB is
    (a) An upper limit to LCL
    (b) A lower limit to LCL
    (c) (a) and (b)
    (d) (a) or (b).

50. The UCB is
    (a) An upper limit to UCL
    (b) A lower limit to LCL
    (c) Both (a) and (b)
    (d) (a) or (b).
51. length of a class is
    (a) The difference between the UCB and LCB of that class
    (b) The difference between the UCL and LCL of that class
    (c) (a) or (b)
    (d) Both (a) and (b).
52. For a particular class boundary, the less than cumulative frequency and more than cumulative frequency add up to
    (a) Total frequency
    (b) Fifty per cent of the total frequency
    (c) (a) or (b)
    (d) None of these.
53. Frequency density corresponding to a class interval is the ratio of
    (a) Class frequency to the total frequency
    (b) Class frequency to the class length
    (c) Class length to the class frequency
    (d) Class frequency to the cumulative frequency.
54. Relative frequency for a particular class
    (a) Lies between 0 and 1
    (b) Lies between 0 and 1, both inclusive
    (c) Lies between –1 and 0
    (d) Lies between –1 to 1.
55. Mode of a distribution can be obtained from
    (a) Histogram
    (b) Less than type ogives
    (c) More than type ogives
    (d) Frequency polygon.
56. Median of a distribution can be obtained from
    (a) Frequency polygon
    (b) Histogram
    (c) Less than type ogives
    (d) None of these.
57. A comparison among the class frequencies is possible only in
    (a) Frequency polygon
    (b) Histogram
    (c) Ogives
    (d) (a) or (b).
58. Frequency curve is a limiting form of
    (a) Frequency polygon
    (b) Histogram
    (c) (a) or (b)
    (d) (a) and (b).

59. Most of the commonly used frequency curves are

(a) Mixed (b) Inverted J-shaped

(c) U-shaped (d) Bell-shaped.

60. The distribution of profits of a company follows

(a) J-shaped frequency curve (b) U-shaped frequency curve

(c) Bell-shaped frequency curve (d) Any of these.

Set B

Answer the following questions. Each question carries 2 marks.

1. Out of 1000 persons, 25 per cent were industrial workers and the rest were agricultural workers. 300 persons enjoyed world cup matches on TV. 30 per cent of the people who had not watched world cup matches were industrial workers. What is the number of agricultural workers who had enjoyed world cup matches on TV?

(a) 260 (b) 240 (c) 230 (d) 250

2. A sample study of the people of an area revealed that total number of women were 40% and the percentage of coffee drinkers were 45 as a whole and the percentage of male coffee drinkers was 20. What was the percentage of female non-coffee drinkers?

(a) 10 (b) 15 (c) 18 (d) 20

3. Cost of sugar in a month under the heads raw materials, labour, direct production and others were 12, 20, 35 and 23 units respectively. What is the difference between the central angles for the largest and smallest components of the cost of sugar?

(a) 72^o (b) 48^o (c) 56^o (d) 92^o

4. The number of accidents for seven days in a locality are given below :

No. of accidents :	0	1	2	3	4	5	6
Frequency :	15	19	22	31	9	3	2

What is the number of cases when 3 or less accidents occurred?

(a) 56 (b) 6 (c) 68 (d) 87

5. The following data relate to the incomes of 86 persons :

Income in Rs. :	500–999	1000–1499	1500–1999	2000–2499
No. of persons :	15	28	36	7

What is the percentage of persons earning more than Rs. 1500?

(a) 50 (b) 45 (c) 40 (d) 60

6. The following data relate to the marks of a group of students:

Marks :	Below 10	Below 20	Below 30	Below 40	Below 50
No. of students :	15	38	65	84	100

How many students got marks more than 30?

(a) 65 (b) 50 (c) 35 (d) 43

7. Find the number of observations between 250 and 300 from the following data :

Value :	More than 200	More than 250	More than 300	More than 350
No. of observations :	56	38	15	0

(a) 56 (b) 23 (c) 15 (d) 8

Set C

Answer the following questions. Each question carries 5 marks.

1. In a study about the male and female students of commerce and science departments of a college in 5 years, the following datas were obtained :

1995	2000
70% male students	75% male students
65% read Commerce	40% read Science
20% of female students read Science	50% of male students read Commerce
3000 total No. of students	3600 total No. of students.

After combining 1995 and 2000 if x denotes the ratio of female commerce student to female Science student and y denotes the ratio of male commerce student to male Science student, then

(a) $x = y$ (b) $x > y$ (c) $x < y$ (d) $x \geq y$

2. In a study relating to the labourers of a jute mill in West Bengal, the following information was collected.

'Twenty per cent of the total employees were females and forty per cent of them were married. Thirty female workers were not members of Trade Union. Compared to this, out of 600 male workers 500 were members of Trade Union and fifty per cent of the male workers were married. The unmarried non-member male employees were 60 which formed ten per cent of the total male employees. The unmarried non-members of the employees were 80'. On the basis of this information, the ratio of married male non-members to the married female non-members is

(a) 1 : 3 (b) 3 : 1 (c) 4 : 1 (d) 5 : 1

3. The weight of 50 students in pounds are given below :

82,	95,	120,	174,	179,	176,	159,	91,	85,	175
88,	160,	97,	133,	159,	176,	151,	115,	105,	172
170,	128,	112,	101,	123,	117,	93,	117,	99,	90
113,	119,	129,	134,	178,	105,	147,	107,	155,	157
98,	117,	95,	135,	175,	97,	160,	168,	144,	175

If the data are arranged in the form of a frequency distribution with class intervals as 81-100, 101-120, 121-140, 141-160 and 161-180, then the frequencies for these 5 class intervals are

(a) 6, 9, 10, 11, 14 (b) 12, 8, 7, 11, 12 (c) 10, 12, 8, 11, 9 (d) 12, 12, 6, 9, 11

4. The following data relate to the marks of 48 students in statistics :

56,	10,	54,	38,	21,	43,	12,	22
48,	51,	39,	26,	12,	17,	36,	19
48,	36,	15,	33,	30,	62,	57,	17
5,	17,	45,	46,	43,	55,	57,	38
43,	28,	32,	35,	54,	27,	17,	16
11,	43,	45,	2,	16,	46,	28,	45

What are the frequency densities for the class intervals 30-39, 40-49 and 50-59

(a) 0.20, 0.50, 0.90

(b) 0.70, 0.90, 1.10

(c) 0.1875, 0.1667, 0.2083

(d) 0.90, 1.1, 0.7

5. The following information relates to the age of death of 50 persons in an area :

36,	48,	50,	45,	49,	31,	50,	48,	42,	57
43,	40,	32,	41,	39,	39,	43,	47,	45,	52
47,	48,	53,	37,	48,	50,	41,	49,	50,	53
38,	41,	49,	45,	36,	39,	31,	48,	59,	48
37,	49,	53,	51,	54,	59,	48,	38,	39,	45

If the class intervals are 31-33, 34-36, 37-39, …. Then the percentage frequencies for the last five class intervals are

(a) 18, 18, 10, 2 and 4. (b) 10, 15, 18, 4 and 2. (c) 14, 18, 20, 10 and 2.

(d) 10, 12, 16, 4 and 6.

ANSWERS

Set A

1.	(c)	**2.**	(b)	**3.**	(d)	**4.**	(d)	**5.**	(a)	**6.**	(b)
7.	(b)	**8.**	(a)	**9.**	(a)	**10.**	(c)	**11.**	(b)	**12.**	(a)
13.	(d)	**14.**	(c)	**15.**	(a)	**16.**	(c)	**17.**	(b)	**18.**	(a)
19.	(a)	**20.**	(d)	**21.**	(c)	**22.**	(a)	**23.**	(b)	**24.**	(c)

25.	(d)	**26.**	(d)	**27.**	(c)	**28.**	(a)	**29.**	(a)	**30.**	(b)
31.	(c)	**32.**	(d)	**33.**	(b)	**34.**	(b)	**35.**	(b)	**36.**	(d)
37.	(d)	**38.**	(d)	**39.**	(d)	**40.**	(c)	**41.**	(a)	**42.**	(d)
43.	(a)	**44.**	(a)	**45.**	(b)	**46.**	(b)	**47.**	(a)	**48.**	(d)
49.	(b)	**50.**	(a)	**51.**	(a)	**52.**	(a)	**53.**	(b)	**54.**	(a)
55.	(a)	**56.**	(c)	**57.**	(b)	**58.**	(c)	**59.**	(d)	**60.**	(c)

Set B

1.	(a)	**2.**	(b)	**3.**	(d)	**4.**	(d)	**5.**	(a)	**6.**	(c)
7.	(b)										

Set C

1.	(b)	**2.**	(c)	**3.**	(d)	**4.**	(d)	**5.**	(a)

ADDITIONAL QUESTION BANK

1. Graph is a

   (a) Line diagram (b) Bar diagram (c) Pie diagram (d) Pictogram

2. Details are shown by

   (a) Charts (b) Tabular presentation

   (c) both (d) none

3. The relationship between two variables are shown in

   (a) Pictogram (b) Histogram (c) Bar diagram (d) Line diagram

4. In general the number of types of tabulation are

   (a) two (b) three (c) one (d) four

5. A table has

   (a) four (b) two (c) five (d) none parts.

6. The number of errors in Statistics are

   (a) one (b) two (c) three (d) four

7. The number of “Frequency distribution“ is

   (a) two (b) one (c) five (d) four

8. (Class frequency)/(Width of the class) is defined as

   (a) Frequency density (b) Frequency distribution

   (c) both (d) none

9. Tally marks determines

(a) class width (b) class boundary (c) class limit (d) class frequency

10. Cumulative Frequency Distribution is a

(a) graph (b) frequency (c) Statistical Table (d) distribution

11. To find the number of observations less than any given value

(a) Single frequency distribution (b) Grouped frequency distribution

(c) Cumulative frequency distribution (d) None is used.

12. An area diagram is

(a) Histogram (b) Frequency Polygon

(c) Ogive (d) none

13. When all classes have a common width

(a) Pie Chart (b) Frequency Polygon

(c) both (d) none is used.

14. An approximate idea of the shape of frequency curve is given by

(a) Ogive (b) Frequency Polygon

(c) both (d) none

15. Ogive is a

(a) Line diagram (b) Bar diagram (c) both (d) none

16. Unequal widths of classes in the frequency distribution do not cause any difficulty in the construction of

(a) Ogive (b) Frequency Polygon

(c) Histogram (d) none

17. The graphical representation of a cumulative frequency distribution is called

(a) Histogram (b) Ogive (c) both (d) none.

18. The most common form of diagrammatic representation of a grouped frequency distribution is

(a) Ogive (b) Histogram (c) Frequency Polygon (d) none

19. Vertical bar chart may appear somewhat alike

(a) Histogram (b) Frequency Polygon

(c) both (d) none

20. The number of types of cumulative frequency is

(a) one (b) two (c) three (d) four

21. A representative value of the class interval for the calculation of mean, standard deviation, mean deviation etc. is

    (a) class interval (b) class limit (c) class mark (d) none

22. The number of observations falling within a class is called

    (a) density (b) frequency (c) both (d) none

23. Classes with zero frequencies are called

    (a) nill class (b) empty class (c) class (d) none

24. For determining the class frequencies it is necessary that these classes are

    (a) mutually exclusive (b) not mutually exclusive

    (c) independent (d) none

25. Most extreme values which would ever be included in a class interval are called

    (a) class limits (b) class interval (c) class boundaries (d) none

26. The value exactly at the middle of a class interval is called

    (a) class mark (b) mid value (c) both (d) none

27. Difference between the lower and the upper class boundaries is

    (a) width (b) size (c) both (d) none

28. In the construction of a frequency distribution, it is generally preferable to have classes of

    (a) equal width (b) unequal width (c) maximum (d) none

29. Frequency density is used in the construction of

    (a) Histogram (b) Ogive

    (c) Frequency Polygon (d) none when the classes are of unequal width.

30. "Cumulative Frequency" only refers to the

    (a) less-than type (b) more-than type (c) both (d) none

31. For the construction of a grouped frequency distribution

    (a) class boundaries (b) class limits (c) both (d) none are used.

32. In all Statistical calculations and diagrams involving end points of classes

    (a) class boundaries (b) class value (c) both (d) none are used.

33. Upper limit of any class is ________ from the lower limit of the next class

    (a) same (b) different

    (c) both (d) none

34. Upper boundary of any class coincides with the Lower boundary of the next class.

    (a) true (b) false (c) both (d) none.

35. Excepting the first and the last, all other class boundaries lie midway between the upper limit of a class and the lower limit of the next higher class.

(a) true (b) false (c) both (d) none

36. The lower extreme point of a class is called

(a) lower class limit (b) lower class boundary

(c) both (d) none

37. For the construction of grouped frequency distribution from ungrouped data we use

(a) class limits (b) class boundaries (c) class width (d) none

38. When one end of a class is not specified, the class is called

(a) closed- end class (b) open- end class (c) both (d) none

39. Class boundaries should be considered to be the real limits for the class interval.

(a) true (b) false (c) both (d) none

40. Difference between the maximum & minimum value of a given data is called

(a) width (b) size (c) range (d) none

41. In Histogram if the classes are of unequal width then the heights of the rectangles must be proportional to the frequency densities.

(a) true (b) false (c) both (d) none

42. When all classes have equal width, the heights of the rectangles in Histogram will be numerically equal to the

(a) class frequencies (b) class boundaries (c) both (d) none

43. Consecutive rectangles in a Histogram have no space in between

(a) true (b) false (c) both (d) none

44. Histogram emphasizes the widths of rectangles between the class boundaries.

(a) false (b) true (c) both (d) none

45. To find the mode graphically

(a) Ogive (b) Frequency Polygon

(c) Histogram (d) none may be used.

46. When the width of all classes is same, frequency polygon has not the same area as the Histogram.

(a) True (b) false (c) both (d) none

47. For obtaining frequency polygon we join the successive points whose abscissa represent the corresponding class frequency_____

(a) true (b) false (c) both (d) none

48. In representing simple frequency distributions of a discrete variable

(a) Ogive (b) Histogram (c) Frequency Polygon (d) both is useful.

49. Diagrammatic representation of the cumulative frequency distribution is

(a) Frequency Polygon (b) Ogive (c) Histogram (d) none

50. For the overlapping classes 0–10 , 10–20 , 20–30 etc.the class mark of the class 0–10 is

(a) 5 (b) 0 (c) 10 (d) none

51. For the non-overlapping classes 0–19 , 20–39 , 40–59 the class mark of the class 0–19 is

(a) 0 (b) 19 (c) 9.5 (d) none

52.

Class :	0–10	10–20	20–30	30–40	40–50
Frequency :	5	8	15	6	4

For the class 20–30 , cumulative frequency is

(a) 20 (b) 13 (c) 15 (d) 28

53. An Ogive can be prepared in ____________ different ways.

(a) 2 (b) 3 (c) 4 (d) none

54. The curve obtained by joining the points, whose x- coordinates are the upper limits of the class-intervals and y coordinates are corresponding cumulative frequencies is called

(a) Ogive (b) Histogram (c) Frequency Polygon (d) Frequency Curve

55. The breadth of the rectangle is equal to the length of the class-interval in

(a) Ogive (b) Histogram (c) both (d) none

56. In Histogram, the classes are taken

(a) overlapping (b) non-overlapping (c) both (d) none

57. For overlapping class-intervals the class limit & class boundary are

(a) same (b) not same (c) zero (d) none

58. Data classification is of________kinds

(a) four (b) Three (c) two (d) five

ANSWERS

1.	(a)	**2.**	(b)	**3.**	(d)	**4.**	(a)	**5.**	(c)
6.	(b)	**7.**	(a)	**8.**	(a)	**9.**	(d)	**10.**	(c)
11.	(c)	**12.**	(a)	**13.**	(b)	**14.**	(b)	**15.**	(a)
16.	(c)	**17.**	(b)	**18.**	(b)	**19.**	(a)	**20.**	(b)
21.	(c)	**22.**	(b)	**23.**	(b)	**24.**	(a)	**25.**	(c)

26.	(c)	**27.**	(c)	**28.**	(a)	**29.**	(a)	**30.**	(a)
31.	(b)	**32.**	(a)	**33.**	(b)	**34.**	(a)	**35.**	(a)
36.	(b)	**37.**	(a)	**38.**	(b)	**39.**	(a)	**40.**	(c)
41.	(a)	**42.**	(a)	**43.**	(a)	**44.**	(b)	**45.**	(c)
46.	(b)	**47.**	(b)	**48.**	(c)	**49.**	(b)	**50.**	(a)
51.	(c)	**52.**	(d)	**53.**	(a)	**54.**	(a)	**55.**	(b)
56.	(a)	**57.**	(a)	**58.**	(a)				

UNIT 2 SAMPLING

LEARNING OBJECTIVES

After reading this unit a student will learn -

- Different procedure of sampling which will be the best representative of the population;

13.2.1 INTRODUCTION

There are situations when we would like to know about a vast, infinite universe or population. But some important factors like time, cost, efficiency, vastness of the population make it almost impossible to go for a complete enumeration of all the units constituting the population. Instead, we take recourse to selecting a representative part of the population and infer about the unknown universe on the basis of our knowledge from the known sample. A somewhat clear picture would emerge out if we consider the following cases.

In the first example let us share the problem faced by Mr. Basu. Mr. Basu would like to put a big order for electrical lamps produced by Mr. Ahuja's company "General Electricals". But before putting the order, he must know whether the claim made by Mr. Ahuja that the lamps of General Electricals last for at least 1500 hours is justified.

Miss Manju Bedi is a well-known social activist. Of late, she has noticed that the incidence of a particular disease in her area is on the rise. She claims that twenty per cent of the people in her town have been suffering from the disease.

In both the situations, we are faced with three different types of problems. The first problem is how to draw a representative sample from the population of electrical lamps in the first case and from the population of human beings in her town in the second case. The second problem is to estimate the population parameters i.e., the average life of all the bulbs produced by General Electricals and the proportion of people suffering form the disease in the first and second examples respectively on the basis of sample observations. The third problem relates to decision making i.e., is there enough evidence, once again on the basis of sample observations, to suggest that the claims made by Mr. Ahuja or Miss Bedi are justifiable so that Mr. Basu can take a decision about buying the lamps from General Electricals in the first case and some effective steps can be taken in the second example with a view to reducing the outbreak of the disease. We consider tests of significance or tests of hypothesis before decision making.

13.2.2 BASIC PRINCIPLES OF SAMPLE SURVEY

Sample Survey is the study of the unknown population on the basis of a proper representative sample drawn from it. How can a part of the universe reveal the characteristics of the unknown universe? The answer to this question lies in the basic principles of sample survey comprising the following components:

(a) Law of Statistical regularity

(b) Principle of Inertia

(c) Principle of Optimization

(d) Principle of Validity

(a) According to the law of statistical regularity, if a sample of fairly large size is drawn from the population under discussion at random, then on an average the sample would posses the characteristics of that population.

Thus the sample, to be taken from the population, should be moderately large. In fact larger the sample size, the better in revealing the identity of the population. The reliability of a statistic in estimating a population characteristics varies as the square root of the sample size. However, it is not always possible to increase the sample size as it would put an extra burden on the available resource. We make a compromise on the sample size in accordance with some factors like cost, time, efficiency etc.

Apart from the sample size, the sample should be drawn at random from the population which means that each and every unit of the population should have a pre-assigned probability to belong to the sample.

(b) The results derived from a sample, according to the principle of inertia of large numbers, are likely to be more reliable, accurate and precise as the sample size increases, provided other factors are kept constant. This is a direct consequence of the first principle.

(c) The principle of optimization ensures that an optimum level of efficiency at a minimum cost or the maximum efficiency at a given level of cost can be achieved with the selection of an appropriate sampling design.

(d) The principle of validity states that a sampling design is valid only if it is possible to obtain valid estimates and valid tests about population parameters. Only a probability sampling ensures this validity.

13.2.3 COMPARISON BETWEEN SAMPLE SURVEY AND COMPLETE ENUMERATION

When complete information is collected for all the units belonging to a population, it is defined as complete enumeration or census. In most cases, we prefer sample survey to complete enumeration due to the following factors:

(a) **Speed:** As compared to census, a sample survey could be conducted, usually, much more quickly simply because in sample survey, only a part of the vast population is enumerated.

(b) **Cost:** The cost of collection of data on each unit in case of sample survey is likely to be more as compared to census because better trained personnel are employed for conducting a sample survey. But when it comes to total cost, sample survey is likely to be less expensive as only some selected units are considered in a sample survey.

(c) **Reliability:** The data collected in a sample survey are likely to be more reliable than that in a complete enumeration because of trained enumerators better supervision and application of modern technique.

(d) **Accuracy:** Every sampling is subjected to what is known as sampling fluctuation which is termed as sampling error. It is obvious that complete enumeration is totally free from this sampling error. However, errors due to recording observations, biases on the part of the enumerators, wrong and faulty interpretation of data etc. are prevalent in both sampling and census and this type of error is termed as non-sampling errors. It may be noted that in sample survey, the sampling error can be reduced to a great extent by taking several steps like increasing the sample size, adhering to a probability sampling design strictly and so on. The non-sampling errors also can be contained to a desirable degree by a proper planning which is not possible or feasible in case of complete enumeration.

(e) **Necessity:** Sometimes, sampling becomes necessity. When it comes to destructive sampling where the items get exhausted like testing the length of life of electrical bulbs or sampling from a hypothetical population like coin tossing, there is no alternative to sample survey.

However, when it is necessary to get detailed information about each and every item constituting the population, we go for complete enumeration. If the population size is not large, there is hardly any merit to take recourse to sampling. If the occurrence of just one defect may lead to a complete destruction of the process as in an aircraft, we must go for complete enumeration.

13.2.4 ERRORS IN SAMPLE SURVEY

Errors or biases in a survey may be defined as the deviation between the value of population parameter as obtained from a sample and its observed value. Errors are of two types.

I. Sampling Errors

II. Non-Sampling Errors

Sampling Errors : Since only a part of the population is investigated in a sampling, every sampling design is subjected to this type of errors. The factors contributing to sampling errors are listed below:

(a) Errors arising out due to defective sampling design: Selection of a proper sampling design plays a crucial role in sampling. If a non- probabilistic sampling design is followed, the bias or prejudice of the sampler affects the sampling technique thereby resulting some kind of error.

(b) Errors arising out due to substitution: A very common practice among the enumerators is to replace a sampling unit by a suitable unit in accordance with their convenience when difficulty arises in getting information from the originally selected unit. Since the sampling design is not strictly adhered to, this results in some type of bias.

(c) Errors owing to faulty demarcation of units: It has its origin in faulty demarcation of sampling units. In case of an agricultural survey, the sampler has, usually, a tendency to underestimate or overestimate the character under consideration.

(d) Errors owing to wrong choice of statistic: One must be careful in selecting the proper statistic while estimating a population characteristic.

(e) Variability in the population: Errors may occur due to variability among population units beyond a degree. This could be reduced by following somewhat complicated sampling design like stratified sampling, Multistage sampling etc.

Non-sampling Errors

As discussed earlier, this type of errors happen both in sampling and complete enumeration. Some factors responsible for this particular kind of biases are lapse of memory, preference for certain digits, ignorance, psychological factors like vanity, non- responses on the part of the interviewees wrong measurements of the sampling units, communication gap between the interviewers and the interviewees, incomplete coverage etc. on the part of the enumerators also lead to non-sampling errors.

13.2.5 SOME IMPORTANT TERMS ASSOCIATED WITH SAMPLING

Population or Universe

It may be defined as the aggregate of all the units under consideration. All the lamps produced by "General Electricals" in our first example in the past, present and future constitute the population. In the second example, all the people living in the town of Miss Manju form the population. The number of units belonging to a population is known as population size. If there are one lakh people in her town then the population size, to be denoted by N, is 1 lakh.

A population may be finite or infinite. If a population comprises only a finite number of units, then it is known as a finite population. The population in the second example is obviously, finite. If the population contains an infinite or uncountable number of units, then it is known as an infinite population. The population of electrical lamps of General Electricals is infinite. Similarly, the population of stars, the population of mosquitoes in Kolkata, the population of flowers in Mumbai, the population of insects in Delhi etc. are infinite population.

Population may also be regarded as existent or hypothetical. A population consisting of real objects is known as an existent population. The population of the lamps produced by General Electricals and the population of Miss Manju's town are example of existent populations. A population that exists just hypothetically like the population of heads when a coin is tossed infinitely is known as a hypothetical or an imaginary population.

Sample

A sample may be defined as a part of a population so selected with a view to representing the population in all its characteristics selection of a proper representative sample is pretty important because statistical inferences about the population are drawn only on the basis of the sample observations. If a sample contains n units, then n is known as sample size. If a sample of 500 electrical lamps is taken from the production process of General Electricals, then n = 500. The units forming the sample are known as "Sampling Units". In the first example, the sampling unit is electrical lamp and in the second example, it is a human. A detailed and complete list of all the sampling units is known as a "Sampling Frame". Before drawing sample, it is a must to have a updated sampling frame complete in all respects before the samples are actually drawn.

Parameter

A parameter may be defined as a characteristic of a population based on all the units of the population. Statistical inferences are drawn about population parameters based on the sample observations drawn from that population. In the first example, we are interested about the parameter "Population Mean". If x a denotes the a th member of the population, then population mean m, which represents the average length of life of all the lamps produced by General Electricals is given by

$$\mu = \frac{\sum_{a=1}^{n} X_a}{N} \qquad (13.2.1)$$

Where N denotes the population size i.e. the total number of lamps produced by the company. In the second example, we are concerned about the population proportion P, representing the ratio of the people suffering from the disease to the total number of people in the town. Thus if there are X people possessing this attribute i.e. suffering from the disease, then we have

$$P = \frac{X}{N} \qquad (13.2.2)$$

Another important parameter namely the population variance, to be denoted by s^2 is given by

$$\sigma^2 = \frac{\sum (X_a - \mu)^2}{N} \qquad (13.2.3)$$

Also we have $SD = \sigma = \sqrt{\frac{\sum (X_a - \mu)^2}{N}}$ (13.2.4)

Statistics

A statistic may be defined as a statistical measure of sample observation and as such it is a function of sample observations. If the sample observations are denoted by $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$, then a statistic T may be expressed as $T = f(x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n)$

A statistic is used to estimate a particular population parameter. The estimates of population mean, variance and population proportion are given by

$$\bar{x} = \hat{\mu} = \frac{\sum x_i}{n} \qquad (13.2.5)$$

$$S_2 = \hat{\sigma}^2 = \frac{\sum \left(x_i - \bar{x}\right)^2}{n} \qquad (13.2.6)$$

and $p = p = \hat{P} = \frac{x}{n}$ (13.2.7)

Where x, in the last case, denotes the number of units in the sample in possession of the attribute under discussion.

Sampling Distribution and Standard Error of a Statistic

Starting with a population of N units, we can draw many a sample of a fixed size n. In case of sampling with replacement, the total number of samples that can be drawn is and when it comes to sampling without replacement of the sampling units, the total number of samples that can be drawn is ${}^{N}c_{n}$.

If we compute the value of a statistic, say mean, it is quite natural that the value of the sample mean may vary from sample to sample as the sampling units of one sample may be different from that of another sample. The variation in the values of a statistic is termed as "Sampling Fluctuations".

If it is possible to obtain the values of a statistic (T) from all the possible samples of a fixed sample size along with the corresponding probabilities, then we can arrange the values of the statistic, which is to be treated as a random variable, in the form of a probability distribution. Such a probability distribution is known as the sampling distribution of the statistic. The sampling distribution, just like a theoretical probability distribution possesses different characteristics. The mean of the statistic, as obtained from its sampling distribution, is known as "Expectation" and the standard deviation of the statistic T is known as the "Standard Error (SE)" of T. SE can be regarded as a measure of precision achieved by sampling. SE is inversely proportional to the square root of sample size. It can be shown that

$$SE\left(\bar{x}\right) = \frac{\sigma}{\sqrt{n}} \text{ for SRS WR}$$

$$= \frac{\sigma}{\sqrt{n}} \cdot \sqrt{\frac{N-n}{N-1}} \qquad \text{for SRS WOR} \qquad (13.2.8)$$

Standard Error for Proporation

$$SE\,(p) = \sqrt{\frac{Pq}{n}} \qquad \text{for SRS WR}$$

$$\sqrt{\frac{Pq}{n}}\ \sqrt{\frac{N-n}{N-1}} \qquad \text{for SRS WOR} \qquad (13.2.9)$$

SRSWR and SRSWOR stand for simple random sampling with replacement and simple random sampling without replacement.

The factor $\sqrt{\frac{N-n}{N-1}}$ is known as finite population correction (fpc) or finite population multiplier and may be ignored as it tends to 1 if the sample size (n) is very large or the population under consideration is infinite when the parameters are unknown, they may be replaced by the corresponding statistic.

Illustrations

Example 13.2.1: A population comprises the following units: a, b, c, d, e. Draw all possible samples of size three without replacement.

Solution: Since in this case, sample size (n) = 3 and population size (N) = 5. the total number of possible samples without replacement = $^5c_3 = 10$

These are abc, abd, abe, acd, ace, ade, bcd, bce,bde,cde.

Example 13.2.2: A population comprises 3 member 1, 5, 3. Draw all possible samples of size two

(i) with replacement

(ii) without replacement

Find the sampling distribution of sample mean in both cases.

Solution: (i) With replacement :- Since n = 2 and N = 3, the total number of possible samples of size 2 with replacement = $3^2 = 9$.

These are exhibited along with the corresponding sample mean in table 15.1. Table 15.2 shows the sampling distribution of sample mean i.e., the probability distribution of $\bar{x}$.

Table 13.2.1

All possible samples of size 2 from a population comprising 3 units under WR scheme

Serial No.	Sample of size 2 with replacement	Sample mean ($\bar{x}$)
1	1, 1	1
2	1, 5	3
3	1, 3	2
4	5, 1	3
5	5, 5	5
6	5, 3	4
7	3, 1	2
8	3, 5	4
9	3, 3	3

Table 13.2.2

Sampling distribution of sample mean

$\bar{x}$	1	2	3	4	5	Total
P	1 / 9	2 / 9	3 / 9	2 / 9	1 / 9	1

(ii) without replacement: As N = 3 and n = 2, the total number of possible samples without replacement = $^NC_2 = {}^3C_2 = 3$.

Table 13.2.3

Possible samples of size 2 from a population of 3 units under WOR scheme

Serial No	Sample of size 2 without replacement	Sample mean ($\overline{x}$)
1	1 , 3	2
2	1 , 5	3
3	3 , 5	4

Table 13.2.4

Sampling distribution of mean

:	2	3	4	Total
P:	1 / 3	1/3	1/3	1

Example 13.2.3: Compute the standard deviation of sample mean for the last problem. Obtain the SE of sample mean applying 15.8 and show that they are equal.

Solution: We consider the following cases:

(i) with replacement :

Let U = $\overline{x}$ The sampling distribution of U is given by

U:	1	2	3	4	5
P:	1/9	2/9	3/9	2/9	1/9

$$E(U) = \Sigma P_i U_i$$
$$= 1/9\times1 + 2/9\times2 + 3/9\times3 + 2/9\times4 + 1/9\times5$$
$$= 3$$

$$E(U\ 2) = \Sigma P_i U_i^2$$
$$= 1/9\times1^2 + 2/9\times2^2 + 3/9\times3^2 + 2/9\times4^2 + 1/9\times5^2$$
$$= 31/3$$

$$\therefore v(\overline{x}) = v(U) = E(U^2) - [E(U)]^2$$
$$= 31/3 - 3^2$$
$$= 4/3$$

Hence SE= $\frac{2}{\sqrt{3}}$ (1)

Since the population comprises 3 units, namely 1, 5, and 3 we may take $X_1 = 1, X_2 = 5, X_3 = 3$

The population mean (m) is given by

$$\mu = \frac{\sum X_a}{N}$$

$$= \frac{1+5+3}{3} = 3$$

and the population variance $\sigma^2 = \frac{\sum (X_a - \mu)^2}{N}$

$$\frac{(1-3)^2 + (5-3)^2 + (3-3)^2}{3} = 8/3$$

Applying 15.8 we have, $SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{3}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{3}}$ (2)

Thus comparing (1) and (2), we are able to verify the validity of the formula.

(ii) without replacement :

In this case, the sampling distribution of V = is given by

V:	2	3	4
P:	1/3	1/3	1/3

$E(\bar{x}) = E(V) = 1/3 \times 2 + 1/3 \times 3 + 1/3 \times 4$

$= 3$

$V(\bar{x}) = \text{Var}(V) = E(v^2) - [E(v)]^2$

$= 1/3 \times 2^2 + 1/3 \times 3^2 + 1/3 \times 4^2 - 3^2$

$= 29/3 - 9$

$= 2/3$

$\therefore SE_{\bar{x}} = \frac{2}{\sqrt{3}}$

Applying 13.2.8, we have

$$SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \cdot \sqrt{\frac{N-n}{N-1}}$$

$$= \frac{8}{\sqrt{3}} \times \frac{1}{\sqrt{2}} \times \frac{8}{\sqrt{3}} \times \frac{1}{\sqrt{2}} \times \sqrt{\frac{3-2}{3-1}} = \frac{2}{\sqrt{3}}$$

and thereby, we make the same conclusion as in the previous case.

13.2.6 TYPES OF SAMPLING

There are three different types of sampling which are

I. Probability Sampling

II. Non – Probability Sampling

III. Mixed Sampling

In the first type of sampling there is always a fixed, pre assigned probability for each member of the population to be a part of the sample taken from that population . When each member of the population has an equal chance to belong to the sample, the sampling scheme is known as Simple Random Sampling. Some important probability sampling other than simple random sampling are stratified sampling, Multi Stage sampling, Multi Phase Sampling, Cluster Sampling and so on. In non- probability sampling , no probability attached to the member of the population and as such it is based entirely on the judgement of the sampler. Non-probability sampling is also known as Purposive or Judgement Sampling. Mixed sampling is based partly on some probabilistic law and partly on some pre decided rule. Systematic sampling belongs to this category. Some important and commonly used sampling process are described now.

Simple Random Sampling (SRS)

When the units are selected independent of each other in such a way that each unit belonging to the population has an equal chance of being a part of the sample, the sampling is known as Simple random sampling or just random sampling. If the units are drawn one by one and each unit after selection is returned to the population before the next unit is being drawn so that the composition of the original population remains unchanged at any stage of the sampling then the sampling procedure is known as Simple Random Sampling with replacement. If, however, once the units selected from the population one by one are never returned to the population before the next drawing is made, then the sampling is known as sampling without replacement. The two sampling methods become almost identical if the population is infinite i.e. vary large or a very large sample is taken from the population. The best method of drawing simple random sample is to use random sampling numbers.

Simple random sampling is a very simple and effective method of drawing samples provided (i) the population is not very large (ii) the sample size is not very small and (iii) the population under consideration is not heterogeneous i.e. there is not much variability among the members forming the population. Simple random sampling is completely free from Sampler's biases. All the tests of significance are based on the concept of simple random sampling.

Stratified Sampling

If the population is large and heterogeneous, then we consider a somewhat, complicated sampling design known as stratified sampling which comprises dividing the population into a number of strata or sub-populations in such a way that there should be very little variations among the units comprising a stratum and maximum variation should occur among the different strata. The stratified sample consists of a number of sub samples, one from each stratum. Different sampling scheme may be applied to different strata and , in particular, if simple random sampling is applied for drawing units from all the strata, the sampling procedure is known as stratified random sampling. The purpose of stratified sampling are (i) to make representation of all the sub populations (ii) to provide an estimate of parameter not only for all the strata but also and overall estimate (iii) reduction of variability and thereby an increase in precision.

There are two types of allocation of sample size. When there is prior information that there is not much variation between the strata variances. We consider "Proportional allocation" or "Bowely's allocation where the sample sizes for different strata are taken as proportional to the population sizes. When the strata-variances differ significantly among themselves, we take recourse to "Neyman's allocation" where sample size vary jointly with population size and population standard deviation i.e. $n_i \mu N_i S_i$. Here n_i denotes the sample size for the i^{th} stratum, N_i and S_i being the corresponding population size and population standard deviation. In case of Bowley's allocation, we have $n_i \mu N_i$.

Stratified sampling is not advisable if (i) the population is not large (ii) some prior information is not available and (iii) there is not much heterogeneity among the units of population.

Multi Stage Sampling

In this type of complicated sampling, the population is supposed to compose of first stage sampling units, each of which in its turn is supposed to compose of second stage sampling units, each of which again in its turn is supposed to compose of third stage sampling units and so on till we reach the ultimate sampling unit.

Sampling also, in this type of sampling design, is carried out through stages. Firstly, only a number of first stage units is selected. For each of the selected first stage sampling units, a number of second stage sampling units is selected. The process is carried out until we select the ultimate sampling units. As an example of multi stage sampling, in order to find the extent of unemployment in India, we may take state, district, police station and household as the first stage, second stage, third stage and ultimate sampling units respectively.

The coverage in case of multistage sampling is quite large. It also saves computational labour and is cost-effective. It adds flexibility into the sampling process which is lacking in other sampling schemes. However, compared to stratified sampling, multistage sampling is likely to be less accurate.

Systematic Sampling

It refers to a sampling scheme where the units constituting the sample are selected at regular interval after selecting the very first unit at random i.e., with equal probability. Systematic sampling is partly probability sampling in the sense that the first unit of the systematic sample is selected probabilistically and partly non- probability sampling in the sense that the remaining units of the sample are selected according to a fixed rule which is non-probabilistic in nature.

If the population size N is a multiple of the sample size n i.e. N = nk, for a positive integer k which must be less than n, then the systematic sampling comprises selecting one of the first k units at random, usually by using random sampling number and thereby selecting every kth unit till the complete, adequate and updated sampling frame comprising all the members of the population is exhausted. This type of systematic sampling is known as "linear systematic sampling ". K is known as "sample interval".

However, if N is not a multiple of n, then we may write $N = nk + p$, $p < k$ and as before, we select the first unit from 1 to k by using random sampling number and thereafter selecting every kth unit in a cyclic order till we get the sample of the required size n. This type of systematic sampling is known as "circular systematic sampling."

Systematic sampling is a very convenient method of sampling when a complete and updated sampling frame is available. It is less time consuming, less expensive and simple as compared to the other methods of sampling. However, systematic sampling has a severe drawback. If there is an unknown and undetected periodicity in the sampling frame and the sampling interval is a multiple of that period, then we are going to get a most biased sample, which, by no stretch of imagination, can represent the population under investigation. Furthermore, since it is not a probability sampling, no statistical inference can be drawn about population parameter.

Purposive or Judgement sampling

This type of sampling is dependent solely on the discretion of the sampler and he applies his own judgement based on his belief, prejudice, whims and interest to select the sample. Since this type of sampling is non-probabilistic, it is purely subjective and, as such, varies from person to person. No statistical hypothesis can be tested on the basis of a purposive sampling.

UNIT II EXERCISE

Set A

Answer the following questions. Each question carries one mark.

1. Sampling can be described as a statistical procedure
   (a) To infer about the unknown universe from a knowledge of any sample
   (b) To infer about the known universe from a knowledge of a sample drawn from it
   (c) To infer about the unknown universe from a knowledge of a random sample drawn from it
   (d) Both (a) and (b).
2. The Law of Statistical Regularity says that
   (a) Sample drawn from the population under discussion possesses the characteristics of the population
   (b) A large sample drawn at random from the population would posses the characteristics of the population
   (c) A large sample drawn at random from the population would possess the characteristics of the population on an average
   (d) An optimum level of efficiency can be attained at a minimum cost.
3. A sample survey is prone to
   (a) Sampling errors (b) Non-sampling errors
   (c) Either (a) or (b) (d) Both (a) and (b)

4. The population of roses in Salt Lake City is an example of
   (a) A finite population (b) An infinite population
   (c) A hypothetical population (d) An imaginary population.
5. Statistical decision about an unknown universe is taken on the basis of
   (a) Sample observations (b) A sampling frame
   (c) Sample survey (d) Complete enumeration
6. Random sampling implies
   (a) Haphazard sampling (b) Probability sampling
   (c) Systematic sampling (d) Sampling with the same probability for each unit.
7. A parameter is a characteristic of
   (a) Population (b) Sample
   (c) Both (a) and (b) (d) (a) or (b)
8. A statistic is
   (a) A function of sample observations (b) A function of population units
   (c) A characteristic of a population (d) A part of a population.
9. Sampling Fluctuations may be described as
   (a) The variation in the values of a statistic
   (b) The variation in the values of a sample
   (c) The differences in the values of a parameter
   (d) The variation in the values of observations.
10. The sampling distribution is
   (a) The distribution of sample observations
   (b) The distribution of random samples
   (c) The distribution of a parameter
   (d) The probability distribution of a statistic.
11. Standard error can be described as
   (a) The error committed in sampling
   (b) The error committed in sample survey
   (c) The error committed in estimating a parameter
   (d) Standard deviation of a statistic.

12. A measure of precision obtained by sampling is given by
    (a) Standard error (b) Sampling fluctuation
    (c) Sampling distribution (d) Expectation.
13. As the sample size increases, standard error
    (a) Increases (b) Decreases
    (c) Remains constant (d) Decreases proportionately.
14. If from a population with 25 members, a random sample without replacement of 2 members is taken, the number of all such samples is
    (a) 300 (b) 625
    (c) 50 (d) 600
15. A population comprises 5 members. The number of all possible samples of size 2 that can be drawn from it with replacement is
    (a) 100 (b) 15
    (c) 125 (d) 25
16. Simple random sampling is very effective if
    (a) The population is not very large
    (b) The population is not much heterogeneous
    (c) The population is partitioned into several sections.
    (d) Both (a) and (b)
17. Simple random sampling is
    (a) A probabilistic sampling (b) A non- probabilistic sampling
    (c) A mixed sampling (d) Both (b) and (c).
18. According to Neyman's allocation, in stratified sampling
    (a) Sample size is proportional to the population size
    (b) Sample size is proportional to the sample SD
    (c) Sample size is proportional to the sample variance
    (d) Population size is proportional to the sample variance.
19. Which sampling provides separate estimates for population means for different segments and also an over all estimate?
    (a) Multistage sampling (b) Stratified sampling
    (c) Simple random sampling (d) Systematic sampling

20. Which sampling adds flexibility to the sampling process?
    (a) Simple random sampling (b) Multistage sampling
    (c) Stratified sampling (d) Systematic sampling

21. Which sampling is affected most if the sampling frame contains an undetected periodicity?
    (a) Simple random sampling (b) Stratified sampling
    (c) Multistage sampling (d) Systematic sampling

22. Which sampling is subjected to the discretion of the sampler?
    (a) Systematic sampling (b) Simple random sampling
    (c) Purposive sampling (d) Quota sampling.

23. If a random sample of size 2 with replacement is taken from the population containing the units 3,6 and 1, then the samples would be
    (a) (3,6),(3,1),(6,1)
    (b) (3,3),(6,6),(1,1)
    (c) (3,3),(3,6),(3,1),(6,6),(6,3),(6,1),(1,1),(1,3),(1,6)
    (d) (1,1),(1,3),(1,6),(6,1),(6,2),(6,3),(6,6),(1,6),(1,1)

24. If a random sample of size two is taken without replacement from a population containing the units a,b,c and d then the possible samples are
    (a) (a, b),(a, c),(a, d) (b) (a, b),(b, c), (c, d)
    (c) (a, b), (b, a), (a, c),(c,a), (a, d), (d, a) (d) (a, b), (a, c), (a, d), (b, c), (b, d), (c,d)

ANSWERS

Set A

1.	(c)	2.	(c)	3.	(d)	4.	(b)	5.	(a)	6.	(d)
7.	(a)	8.	(a)	9.	(a)	10.	(d)	11.	(d)	12.	(a)
13.	(b)	14.	(a)	15.	(d)	16.	(b)	17.	(a)	18.	(a)
19.	(b)	20.	(d)	21.	(d)	22.	(c)	23.	(c)	24.	(d)

NOTES

CHAPTER BE

MEASURES OF CENTRAL TENDENCY AND DISPERSION

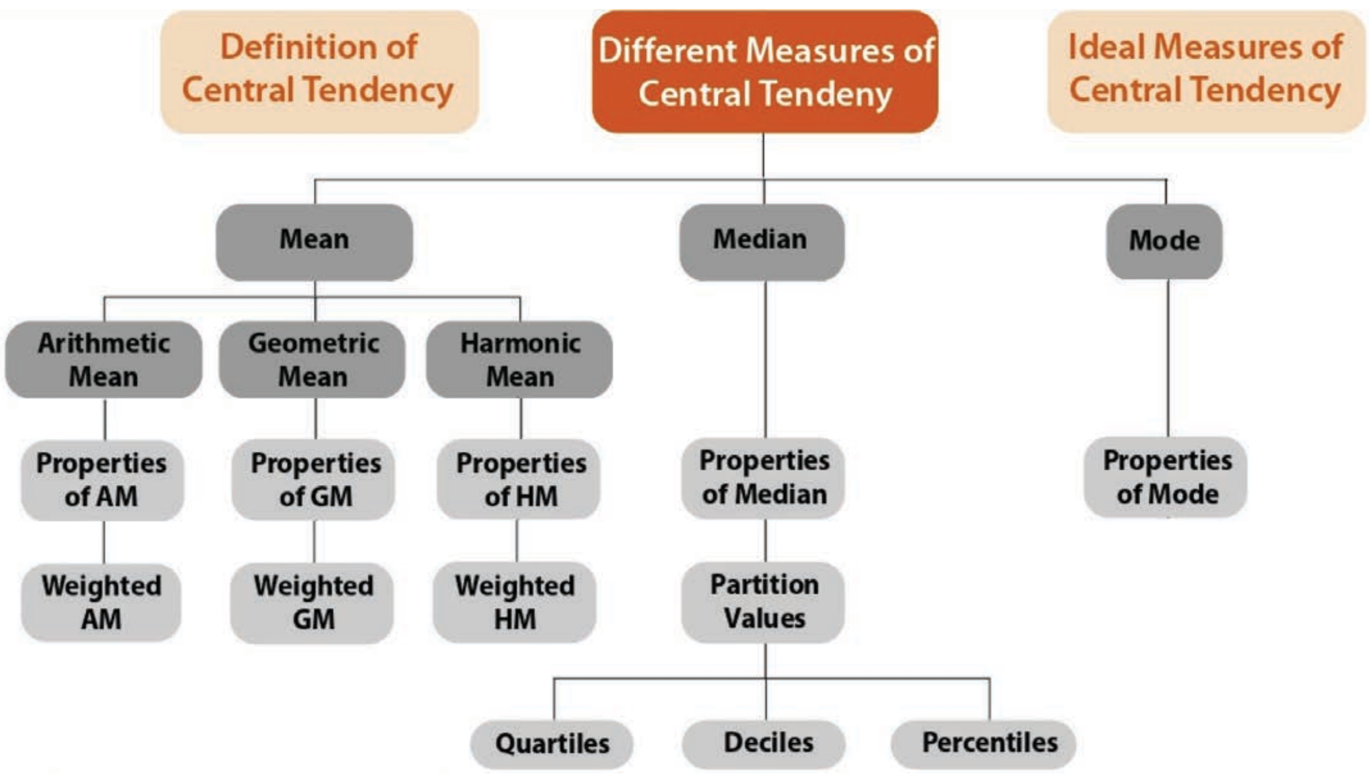
UNIT I: MEASURES OF CENTRAL TENDENCY

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- To understand different measures of central tendency, i.e. Arithmetic Mean, Median, Mode, Geometric Mean and Harmonic Mean, and computational techniques of these measures.
- To learn comparative advantages and disadvantages of these measures and therefore, which measures to use in which circumstance.

UNIT OVERVIEW



14.1.1 DEFINITION OF CENTRAL TENDENCY

In many a case, like the distributions of height, weight, marks, profit, wage and so on, it has been noted that starting with rather low frequency, the class frequency gradually increases till it reaches its maximum somewhere near the central part of the distribution and after which the class frequency steadily falls to its minimum value towards the end. Thus, central tendency may be defined as the tendency of a given set of observations to cluster around a single central or middle value and the single value that represents the given set of observations is described as a measure of central tendency or, location, or average. Hence, it is possible to condense a vast mass of data by a single representative value. The computation of a measure of central tendency plays a very important part in many a sphere. A company is recognized by its high average profit, an educational institution is judged on the basis of average marks obtained by its students and so on. Furthermore, the central tendency also facilitates us in providing a basis for comparison between different distribution. Following are the different measures of central tendency:

(i) Mean
- Arithmetic Mean (AM)
- Geometric Mean (GM)
- Harmonic Mean (HM)

(ii) Median (Me)

(iii) Mode (Mo)

14.1.2 CRITERIA FOR AN IDEAL MEASURE OF CENTRAL TENDENCY

Following are the criteria for an ideal measure of central tendency:

(i) It should be properly and unambiguously defined.

(ii) It should be easy to comprehend.

(iii) It should be simple to compute.

(iv) It should be based on all the observations.

(v) It should have certain desirable mathematical properties.

(vi) It should be least affected by the presence of extreme observations.

14.1.3 ARITHMETIC MEAN

For a given set of observations, the AM may be defined as the sum of all the observations divided by the number of observations. Thus, if a variable x assumes n values $x_1, x_2, x_3, \ldots\ldots\ldots x_n$, then the AM of x, to be denoted by $\overline{X}$, is given by,

$$\overline{X} = \frac{x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots\ldots + x_n}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\overline{X} = \frac{\sum x_i}{n} \qquad \text{……………………..(14.1.1)}$$

In case of a simple frequency distribution relating to an attribute, we have

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + ... + f_nx_n}{f_1 + f_2 + f_3 + ... + f_n}$$

$$= \frac{\sum f_ix_i}{\sum f_i}$$

$$\overline{X} = \frac{\sum f_ix_i}{N} \qquad \text{……………………..(14.1.2)}$$

assuming the observation x_i occurs f_i times, i=1,2,3,……..n and $N=\sum f_i$.
In case of grouped frequency distribution also we may use formula (14.1.2) with x_i as the mid value of the i-th class interval, on the assumption that all the values belonging to the i-th class interval are equal to x_i.
However, in most cases, if the classification is uniform, we consider the following formula for the computation of AM from grouped frequency distribution:

$$\overline{x} = A + \frac{\sum f_id_i}{N} \times C \qquad \text{…………………………..(14.1.3)}$$

Where, $d_i = \dfrac{x_i - A}{C}$

A = Assumed Mean

C = Class Length

ILLUSTRATIONS:

Example 14.1.1: Following are the daily wages in Rupees of a sample of 9 workers: 58, 62, 48, 53, 70, 52, 60, 84, 75. Compute the mean wage.

Solution: Let x denote the daily wage in rupees.

Then as given, x_1=58, x_2=62, x_3=48, x_4=53, x_5=70, x_6=52, x_7=60, x_8=84 and x_9=75.

Applying (14.1.1) the mean wage is given by,

$$\overline{x} = \frac{\sum_{i=1}^{9} x_i}{9}$$

$$= ₹\ \frac{(58+62+48+53+70+52+60+84+75)}{9}$$

$$= ₹\ \frac{562}{9}$$

$$= ₹\ 62.44.$$

Example 14.1.2: Compute the mean weight of a group of BBA students of St. Xavier's College from the following data:

Weight in kgs.	44 – 48	49 – 53	54 – 58	59 – 63	64 – 68	69 – 73
No. of Students	3	4	5	7	9	8

Solution: Computation of mean weight of 36 BBA students

Table 14.1.1

Weight in kgs. (1)	No. of Student (f_i) (2)	Mid-Value (x_i) (3)	$f_i x_i$ (4) = (2) x (3)
44 – 48	3	46	138
49 – 53	4	51	204
54 – 58	5	56	280
59 – 63	7	61	427
64 – 68	9	66	594
69 – 73	8	71	568
Total	36	–	2211

Applying (14.1.2), we get the average weight as

$$\bar{x} = \frac{\sum f_i x_i}{N}$$

$$= \frac{2211}{36} \text{ kgs.}$$

$$= 61.42 \text{ kgs.}$$

Example 14.1.3: Find the AM for the following distribution:

Class Interval	350 – 369	370 – 389	390 – 409	410 – 429	430 – 449	450 – 469	470 – 489
Frequency	23	38	58	82	65	31	11

Solution: We apply formula (14.1.3) since the amount of computation involved in finding the AM is much more compared to **Example 14.1.2**. Any mid value can be taken as A. However, usually A is taken as the middle most mid-value for an odd number of class intervals and any one of the two middle most mid-values for an even number of class intervals. The class length is taken as C.

Table 14.1.2 Computation of AM

Class Interval (1)	Frequency(f_i) (2)	Mid-Value(x_i) (3)	$d_i = \frac{x_i - A}{c} = \frac{x_i - 419.50}{20}$ (4)	$f_i d_i$ (5) = (2)X(4)
350 – 369	23	359.50	– 3	– 69
370 – 389	38	379.50	– 2	– 76
390 – 409	58	399.50	– 1	– 58
410 – 429	82	419.50 (A)	0	0
430 – 449	65	439.50	1	65
450 – 469	31	459.50	2	62
470 – 489	11	479.50	3	33
Total	308	–	–	– 43

The required AM is given by

$$\bar{x} = A + \frac{\sum f_i d_i}{N} \times C$$

$$= 419.50 + \frac{(-43)}{308} \times 20$$

$$= 419.50 - 2.79$$

$$= 416.71$$

Example 14.1.4: Given that the mean height of a group of students is 67.45 inches. Find the missing frequencies for the following incomplete distribution of height of 100 students.

Height in inches	60 – 62	63 – 65	66 – 68	69 – 71	72 – 74
No. of Students	5	18	–	–	8

Solution: Let x denote the height and f_3 and f_4 as the two missing frequencies.

Table 14.1.3

Estimation of missing frequencies

CI (1)	Frequency (f_i) (2)	Mid - Value (x_i) (3)	$d_i = \frac{x_i - A}{c}$ $\frac{x_i - 67}{3}$ (4)	$f_i d_i$ (5) = (2) x (4)
60-62	5	61	-2	-10
63 – 65	18	64	– 1	– 18
66 – 68	f_3	67 (A)	0	0
69 – 71	f_4	70	1	f_4
72 – 74	8	73	2	16
Total	$31 + f_3 + f_4$	–	–	$-12 + f_4$

As given, we have

$$31 + f_3 + f_4 = 100$$

$\Rightarrow$ $$f_3 + f_4 = 69 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$\bar{x} = 67.45$$

$\Rightarrow$ $$A + \frac{\sum f_i d_i}{N} \times C = 67.45$$

$\Rightarrow$ $$67 + \frac{(-12 + f_4)}{100} \times 3 = 67.45$$

$\Rightarrow$ $$(-12 + f_4) \times 3 = (67.45 - 67) \times 100$$

$\Rightarrow$ $$-12 + f_4 = 15$$

$\Rightarrow$ $$f_4 = 27$$

On substituting 27 for f_4 in (1), we get

$f_3 + 27 = 69$, $\Rightarrow$ $f_3 = 42$

Thus, the missing frequencies would be 42 and 27.

Properties of AM

(i) **If all the observations assumed by a variable are constants, say k, then the AM is also k.** For example, if the height of every student in a group of 10 students is 170 cm, then the mean height is, of course, 170 cm.

(ii) the algebraic sum of deviations of a set of observations from their AM is zero

i.e. for unclassified data , $\sum(x_i - \bar{x}) = 0$

and for grouped frequency distribution, $\sum f_i(x_i - \bar{x}) = 0$ } ....(14.1.4)

For example, if a variable x assumes five observations, say 58, 63, 37, 45, 29, then $\bar{x}$ =46.4. Hence, the deviations of the observations from the AM i.e. $(x_i - \bar{x})$ are 11.60, 16.60, –9.40, –1.40 and –17.40, then $\sum(x_i - \bar{x}) = 11.60 + 16.60 + (–9.40) + (–1.40) + (–17.40) = 0$.

(iii) AM is affected due to a change of origin and/or scale which implies that if the original variable x is changed to another variable y by effecting a change of origin, say a, and scale say b, of x i.e. y=a+bx, then the AM of y is given by $\bar{y} = a + b\bar{x}$.

For example, if it is known that two variables x and y are related by 2x+3y+7=0 and $\bar{x} = 15$, then the AM of y is given by $\bar{y} = \frac{-7 - 2\bar{x}}{3}$

$= \frac{-7 - 2 \times 15}{3} = \frac{-37}{3} = -12.33$.

(iv) If there are two groups containing n_1 and n_2 observations and $\bar{x}_1$ and $\bar{x}_2$ as the respective arithmetic means, then the combined AM is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} \quad ....(14.1.5)$$

This property could be extended to k>2 groups and we may write

$$\bar{x} = \frac{\sum n_i\bar{x}_i}{\sum n_i} \quad ....(14.1.6) \quad i = 1, 2,........n.$$

Example 14.1.5: The mean salary for a group of 40 female workers is ₹5,200 per month and that for a group of 60 male workers is ₹ 6800 per month. What is the combined mean salary?

Solution: As given n_1 = 40, n_2 = 60, $\bar{x}_1$ = ₹ 5,200 and $\bar{x}_2$ = ₹ 6,800 hence, the combined mean salary per month is

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$= \frac{40 \times ₹\ 5,200 + 60 \times ₹\ 6,800}{40 + 60} = ₹6,160.$$

14.1.4 MEDIAN – PARTITION VALUES

As compared to AM, median is a positional average which means that the value of the median is dependent upon the position of the given set of observations for which the median is wanted. Median, for a given set of observations, may be defined as the middle-most value when the observations are arranged either in an ascending order or a descending order of magnitude.

As for example, if the marks of the 7 students are 72, 85, 56, 80, 65, 52 and 68, then in order to find the median mark, we arrange these observations in the following ascending order of magnitude: 52, 56, 65, 68, 72, 80, 85.

Since the 4th term i.e. 68 in this new arrangement is the middle most value, the median mark is 68 i.e. Median (Me) = 68.

As a second example, if the wages of 8 workers, expressed in rupees are

56, 82, 96, 120, 110, 82, 106, 100 then arranging the wages as before, in an ascending order of magnitude, we get ₹56, ₹ 82, ₹ 82, ₹ 96, ₹ 100, ₹ 106, ₹ 110, ₹ 120. Since there are two middle-most values, namely, ₹ 96, and ₹ 100 any value between ₹ 96 and ₹ 100 may be, theoretically, regarded as median wage. However, to bring uniqueness, we take the arithmetic mean of the two middle-most values, whenever the number of the observations is an even number. Thus, the median wage in this example, would be

$$M = \frac{₹\,96 + ₹\,100}{2} = ₹\,98$$

In case of a grouped frequency distribution, we find median from the cumulative frequency distribution of the variable under consideration. We may consider the following formula, which can be derived from the basic definition of median.

$$M = l_1 + \left(\frac{\frac{N}{2} - N_l}{N_u - N_l} \right) \times C \quad \text{.......................................} (14.1.7)$$

Where,

l_1 = lower class boundary of the median class i.e. the class containing median.

N = total frequency.

N_l = less than cumulative frequency corresponding to l_1. (Pre median class)

N_u = less than cumulative frequency corresponding to l_2. (Post median class)

l_2 being the upper class boundary of the median class.

C = $l_2 - l_1$ = length of the median class.

Example 14.1.6: Compute the median for the distribution as given in **Example 14.1.3.**

Solution: First, we find the cumulative frequency distribution which is exhibited in **Table 14.1.4.**

Table 14.1.4

Computation of Median

Class boundary	Less than cumulative frequency
349.50	0
369.50	23
389.50	61
409.50 (l_1)	119 (N_l)
429.50 (l_2)	201(N_u)
449.50	266
469.50	297
489.50	308

We find, from the **Table 14.1.4**, $\frac{N}{2} = \frac{308}{2}$ = 154 lies between the two cumulative frequencies 119 and 201 i.e. $119 < 154 < 201$. Thus, we have N_l = 119, N_u = 201 l_1 = 409.50 and l_2 = 429.50. Hence C = 429.50 – 409.50 =20.

Substituting these values in (14.1.7), we get,

$$M = 409.50 + \frac{154-119}{201-119} \times 20$$

$$= 409.50+8.54$$

$$= 418.04.$$

Example 14.1.7: Find the missing frequency from the following data, given that the median mark is 23.

Mark :	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of students :	5	8	?	6	3

Solution: Let us denote the missing frequency by f_3. Table 14.1.5 shows the relevant computation.

Table 14.1.5
(Estimation of missing frequency)

Mark	Less than cumulative frequency
0	0
10	5
$20(l_1)$	$13(N_l)$
$30(l_2)$	$13+f_3(N_u)$
40	$19+f_3$
50	$22+f_3$

Going through the mark column, we find that 20<23<30. Hence l_1=20, l_2=30 and accordingly N_l=13, N_u=13+f_3. Also the total frequency i.e. N is 22+f_3. Thus,

$$M = l_1 + \left(\frac{\frac{N}{2} - N_l}{N_u - N_l}\right) \times C$$

$$\Rightarrow \quad 23 = 20 + \frac{\left(\frac{22+f_3}{2}\right) - 13}{(13+f_3) - 13} \times 10$$

$$\Rightarrow \quad 3 = \frac{22+f_3-26}{f_3} \times 5$$

$$\Rightarrow \quad 3f_3 = 5f_3 - 20$$

$$\Rightarrow \quad 2f_3 = 20$$

$$\Rightarrow \quad f_3 = 10$$

So, the missing frequency is 10.

Properties of median

We cannot treat median mathematically, the way we can do with arithmetic mean. We consider below two important features of median.

(i) If x and y are two variables, to be related by y=a+bx for any two constants a and b, then the median of y is given by
$y_{me} = a + bx_{me}$

For example, if the relationship between x and y is given by 2x – 5y = 10 and if x_{me} i.e. the median of x is known to be 16.
Then 2x – 5y = 10

⇒ $y = -2 + 0.40x$

⇒ $y_{me} = -2 + 0.40\, x_{me}$

⇒ $y_{me} = -2 + 0.40 \times 16$

⇒ $y_{me} = 4.40.$

(ii) For a set of observations, the sum of absolute deviations is minimum when the deviations are taken from the median. This property states that $\sum|x_i - A|$ is minimum if we choose A as the median.

PARTITION VALUES OR QUARTILES OR FRACTILES

These may be defined as values dividing a given set of observations into a number of equal parts. When we want to divide the given set of observations into two equal parts, we consider median. Similarly, quartiles are values dividing a given set of observations into four equal parts. So there are three quartiles – first quartile or lower quartile denoted by Q_1, second quartile or median to be denoted by Q_2 or Me and third quartile or upper quartile denoted by Q_3. First quartile is the value for which one fourth of the observations are less than or equal to Q_1 and the remaining three – fourths observations are more than or equal to Q_1. In a similar manner, we may define Q_2 and Q_3.

Deciles are the values dividing a given set of observation into ten equal parts. Thus, there are nine deciles to be denoted by D_1, D_2, D_3,…..D_9. D_1 is the value for which one-tenth of the given observations are less than or equal to D_1 and the remaining nine-tenth observations are greater than or equal to D_1 when the observations are arranged in an ascending order of magnitude.

Lastly, we talk about the percentiles or centiles that divide a given set of observations into 100 equal parts. The points of sub-divisions being P_1, P_2,………..P_{99}. P_1 is the value for which one hundredth of the observations are less than or equal to P_1 and the remaining ninety-nine hundredths observations are greater than or equal to P_1 once the observations are arranged in an ascending order of magnitude.

For unclassified data, the p^{th} quartile is given by the $(n+1)p^{th}$ value, where n denotes the total number of observations. p = 1/4, 2/4, 3/4 for Q_1, Q_2 and Q_3 respectively. p=1/10, 2/10,………….9/10. For D_1, D_2,……,D_9 respectively and lastly p=1/100, 2/100,….,99/100 for P_1, P_2, P_3….P_{99} respectively.

In case of a grouped frequency distribution, we consider the following formula for the computation of quartiles.

$$Q = l_1 + \left(\frac{Np - N_l}{N_u - N_l}\right) \times C \quad \text{.......} \quad (14.1.8)$$

The symbols, except p, have their usual interpretation which we have already discussed while computing median and just like the unclassified data, we assign different values to p depending on the quartile.

Another way to find quartiles for a grouped frequency distribution is to draw the ogive (less than type) for the given distribution. In order to find a particular quartile, we draw a line parallel to the horizontal axis through the point Np. We draw perpendicular from the point of intersection of this parallel line and the ogive. The x-value of this perpendicular line gives us the value of the quartile under discussion.

Example 14.1.8: Following are the wages of the labourers: ₹ 82, ₹ 56, ₹ 90, ₹ 50, ₹ 120, ₹ 75, ₹ 75, ₹ 80, ₹ 130, ₹ 65. Find Q_1, D_6 and P_{82}.

Solution: Arranging the wages in an ascending order, we get ₹ 50, ₹ 56, ₹ 65, ₹ 75, ₹ 75, ₹ 80, ₹ 82, ₹ 90, ₹ 120, ₹ 130.
Hence, we have

$$Q_1 = \frac{(n+1)}{4}\text{th value}$$

$$= \frac{(10+1)}{4}\text{th value}$$

$$= 2.75^{th}\text{ value}$$

$$= 2^{nd}\text{ value} + 0.75 \times \text{difference between the third and the } 2^{nd}\text{ values.}$$

$$= ₹\,[56 + 0.75 \times (65 - 56)]$$

$$= ₹\,62.75$$

$$D_6 = (15 + 1) \times \frac{6}{10}\text{ th value}$$

$$= 6.60^{th}\text{ value}$$

$$= 6^{th}\text{ value} + 0.60 \times \text{difference between the } 7^{th}\text{ and the } 6^{th}\text{ values.}$$

$$= ₹\,(80 + 0.60 \times 2)$$

$$= ₹\,81.20$$

$$P_{82} = (10+1) \times \frac{82}{100}\text{ th value}$$

$$= 9.02^{th}\text{ value}$$

$$= 9^{th}\text{ value} + 0.02 \times \text{difference between the } 10^{th}\text{ and the } 9^{th}\text{ values}$$

$$= ₹\,(120 + 0.02 \times 10)$$

$$= ₹\,120.20$$

Next, let us consider one problem relating to the grouped frequency distribution.

Example 14.1.9: Following distribution relates to the distribution of monthly wages of 100 workers.

Wages in (₹) :	less than 500	500–699	700–899	900–1099	1100–1499	more than 1500
No. of workers :	5	23	29	27	10	6

Compute Q_3, D_7 and P_{23}.

Solution: This is a typical example of an open end unequal classification as we find the lower class limit of the first class interval and the upper class limit of the last class interval are not stated, and theoretically, they can assume any value between 0 and 500 and 1500 to any number respectively. The ideal measure of the central tendency in such a situation is median as the median or second quartile is based on the fifty percent central values. Denoting the first LCB and the last UCB by the L and U respectively, we construct the following cumulative frequency distribution:

Table 14.1.7
Computation of quartiles

Wages in rupees (CB)	No. of workers (less than cumulative frequency)
L	0
499.50	5
699.50	28
899.50	57
1099.50	84
1499.50	94
U	100

For Q_3, $\frac{3N}{4} = \frac{3 \times 100}{4} = 75$

since, 57<75 <84, we take $N_l = 57$, $N_u = 84$, $l_1 = 899.50$, $l_2 = 1099.50$, $c = l_2 - l_1 = 200$ in the formula (11.8) for computing Q_3.

Therefore, $Q_3 = ₹\left[899.50 + \frac{75-57}{84-57} \times 200\right] = ₹\ 1032.83$

Similarly, for D_7, $\frac{7N}{10} = \frac{7 \times 100}{10} = 70$ which also lies between 57 and 84.

Thus, $D_7 = ₹\left[899.50 + \frac{70-57}{84-57} \times 200\right] = ₹995.80$

Lastly for P_{23}, $\frac{23N}{100} = \frac{23}{100} \times 100 = 23$ and as $5 < 23 < 28$, we have

$P_{23} = ₹\ [499.50 + \frac{23-5}{28-5} \times 200]$

$= ₹\ 656.02$

14.1.5 MODE

For a given set of observations, mode may be defined as the value that occurs the maximum number of times. Thus, mode is that value which has the maximum concentration of the observations around it. This can also be described as the most common value with which, even, a layman may be familiar with.

Thus, if the observations are 5, 3, 8, 9, 5 and 6, then Mode (Mo) = 5 as it occurs twice and all the other observations occur just once. The definition for mode also leaves scope for more than one mode. Thus sometimes we may come across a distribution having more than one mode. Such a distribution is known as a multi-modal distribution. Bi-modal distribution is one having two modes.

Furthermore, it also appears from the definition that mode is not always defined. As an example, if the marks of 5 students are 50, 60, 35, 40, 56, there is no modal mark as all the observations occur once i.e. the same number of times.

We may consider the following formula for computing mode from a grouped frequency distribution:

$$\text{Mode} = l_1 + \left(\frac{f_0 - f_{-1}}{2f_0 - f_{-1} - f_1}\right) \times c \qquad \text{.............................(14.1.9)}$$

where,

l_1 = LCB of the modal class.
i.e. the class containing mode.
f_0 = frequency of the modal class
f_{-1} = frequency of the pre-modal class
f_1 = frequency of the post modal class
C = class length of the modal class

Example 14.1.10: Compute mode for the distribution as described in Example. 14.1.3

Solution: The frequency distribution is shown below:

Table 14.1.8
Computation of mode

Class Interval	Frequency
350 - 369	23
370 - 389	38
390 - 409	58 (f_{-1})
410 - 429	82 (f_0)
430 - 449	65 (f_1)
450 - 469	31
470 - 489	11

Going through the frequency column, we note that the highest frequency i.e. f_0 is 82. Hence, f_{-1} = 58 and f_1 = 65. Also the modal class i.e. the class against the highest frequency is 410 – 429.

Thus l_1 = LCB=409.50 and c=429.50 – 409.50 = 20

Hence, applying formulas (11.9), we get

$$Mo = 409.5 + \frac{82-58}{2\times 82-58-65}\times 20$$

= 421.21 which belongs to the modal class. (410 – 429)

When it is difficult to compute mode from a grouped frequency distribution, we may consider the following empirical relationship between mean, median and mode:

Mean – Mode = 3(Mean – Median) ……………………….(14.1.9A)

or Mode = 3 Median – 2 Mean

(14.1.9A) holds for a moderately skewed distribution. We also note that if y = a+bx, then $y_{mo}=a+bx_{mo}$ …………………………………….(14.1.10)

Example 14.11: For a moderately skewed distribution of marks in statistics for a group of 200 students, the mean mark and median mark were found to be 55.60 and 52.40. What is the modal mark?

Solution: Since in this case, mean = 55.60 and median = 52.40, applying (11.9A), we get the modal mark as

Mode = 3 × Median – 2 × Mean
= 3 × 52.40 – 2 × 55.60
= 46.

Example 14.1.12: If y = 2 + 1.50x and mode of x is 15, what is the mode of y?

Solution:

By virtue of (11.10), we have

y_{mo} = 2 + 1.50 × 15
= 24.50.

14.1.6 GEOMETRIC MEAN AND HARMONIC MEAN

For a given set of n positive observations, the geometric mean is defined as the n-th root of the product of the observations. Thus if a variable x assumes n values $x_1, x_2, x_3, \ldots, x_n$, all the values being positive, then the GM of x is given by

$G= (x_1 \times x_2 \times x_3 \ldots \times x_n)^{1/n}$ .... (14.1.11)

For a grouped frequency distribution, the GM is given by

$G= (x_1^{f_1} \times x_2^{f_2} \times x_3^{f_3} \ldots \times x_n^{f_n})^{1/N}$ .... (14.1.12)

Where $N = \sum f_i$

In connection with GM, we may note the following properties :

(i) Logarithm of G for a set of observations is the AM of the logarithm of the observations; i.e.

$$\log G = \frac{1}{n}\sum \log x \qquad \ldots\ldots\ldots\ldots\ldots\ldots(14.1.13)$$

(ii) if all the observations assumed by a variable are constants, say K > 0, then the GM of the observations is also K.

(iii) GM of the product of two variables is the product of their GM's i.e. if $z = xy$, then

GM of z = (GM of x) × (GM of y) ....(14.1.14)

(iv) GM of the ratio of two variables is the ratio of the GM's of the two variables i.e. if $z = x/y$ then

$$\text{GM of } z = \frac{\text{GM of } x}{\text{GM of } y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(14.1.15)$$

Example 14.1.13: Find the GM of 3, 6 and 12.

Solution: As given $x_1=3$, $x_2=6$, $x_3=12$ and n=3.

Applying (14.1.11), we have $G= (3\times6\times12)^{1/3} = (6^3)^{1/3}=6$.

Example 14.1.14: Find the GM for the following distribution:

x :	2	4	8	16
f :	2	3	3	2

Solution: According to (14.1.12), the GM is given by

$$G = (x_1^{f_1} \times x_2^{f_2} \times x_3^{f_3} \times x_4^{f_4})^{1/N}$$
$$= (2^2 \times 4^3 \times 8^3 \times 16^2)^{1/10}$$
$$= (2)^{2.50}$$
$$= 4\sqrt{2}$$
$$= 5.66$$

Harmonic Mean

For a given set of non-zero observations, harmonic mean is defined as the reciprocal of the AM of the reciprocals of the observation. So, if a variable x assumes n non-zero values x_1, x_2, x_3,...............,x_n, then the HM of x is given by

$$H=\frac{n}{\sum(1/x_i)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(14.1.16)$$

For a grouped frequency distribution, we have

$$H = \frac{N}{\sum\left[\frac{f_i}{x_i}\right]} \qquad \text{..................(14.1.17)}$$

Properties of HM

(i) If all the observations taken by a variable are constants, say k, then the HM of the observations is also k.

(ii) If there are two groups with n_1 and n_2 observations and H_1 and H_2 as respective HM's than the combined HM is given by

$$\frac{n_1 + n_2}{\frac{n_1}{H_1} + \frac{n_2}{H_2}} \qquad \text{..................(14.1.18)}$$

Example 14.15: Find the HM for 4, 6 and 10.

Solution: Applying (14.1.16), we have

$$H = \frac{3}{\frac{1}{4} + \frac{1}{6} + \frac{1}{10}}$$

$$= \frac{3}{0.25 + 0.17 + 0.10}$$

$$= 5.77$$

Example 14.1.16: Find the HM for the following data:

x:	2	4	8	16
f:	2	3	3	2

Solution: Using (14.1.17), we get

$$H = \frac{10}{\frac{2}{2} + \frac{3}{4} + \frac{3}{8} + \frac{2}{16}}$$

$$= 4.44$$

Relation between AM, GM, and HM

For any set of positive observations, we have the following inequality:

$$AM \geq GM \geq HM \qquad \text{…………..} (14.1.19)$$

The equality sign occurs, as we have already seen, when all the observations are equal.

Example 14.1.17: compute AM, GM, and HM for the numbers 6, 8, 12, 36.

Solution: In accordance with the definition, we have

$$AM = \frac{6+8+12+36}{4} = 15.50$$

$$GM = (6 \times 8 \times 12 \times 36)^{1/4}$$
$$= (2^8 \times 3^4)^{1/4} = 12$$

$$HM = \frac{4}{\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{36}} = 9.93$$

The computed values of AM, GM, and HM establish (14.1.19).

Weighted average

When the observations under consideration have a hierarchical order of importance, we take recourse to computing weighted average, which could be either weighted AM or weighted GM or weighted HM.

$$\text{Weighted AM} = \frac{\sum w_i x_i}{\sum w_i} \qquad \text{…………..} (14.1.20)$$

$$\text{Weighted GM} = \text{Anti log}\left(\frac{\sum w_i \log x_i}{\sum w_i}\right) \qquad \text{…………..} (14.1.21)$$

$$\text{Weighted HM} = \frac{\sum w_i}{\sum\left(\frac{w_i}{x_i}\right)} \qquad \text{…………..} (14.1.22)$$

Example 14.1.18: Find the weighted AM and weighted HM of first n natural numbers, the weights being equal to the squares of the corresponding numbers.

Solution: As given,

x	1	2	3	….	n
w	1^2	2^2	3^2	….	n^2

$$\text{Weighted AM} = \frac{\sum w_i x_i}{\sum w_i}$$

$$= \frac{1\times1^2 + 2\times2^2 + 3\times3^2 + \ldots\ldots\ldots\ldots\ldots\ldots n\times n^2}{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + n^2}$$

$$= \frac{1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3}{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2}$$

$$= \frac{\left[\frac{n(n+1)}{2}\right]^2}{\frac{n(n+1)(2n+1)}{6}} = \frac{3n(n+1)}{2(2n+1)}$$

$$\text{Weighted HM} = \frac{\sum w_i}{\sum\left(\frac{w_i}{x_i}\right)}$$

$$= \frac{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots n^2}{\frac{1^2}{1} + \frac{2^2}{2} + \frac{3^2}{3} + \ldots\ldots\ldots \frac{n^2}{n}}$$

$$= \frac{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2}{1 + 2 + 3 + \ldots\ldots\ldots + n}$$

$$= \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}}$$

$$= \frac{2n+1}{3}$$

A General review of the different measures of central tendency

After discussing the different measures of central tendency, now we are in a position to have a review of these measures of central tendency so far as the relative merits and demerits are concerned on the basis of the requisites of an ideal measure of central tendency which we have already mentioned in section 14.1.2. The best measure of central tendency, usually, is the AM. It is rigidly defined, based on all the observations, easy to comprehend, simple to calculate and amenable to mathematical properties. Mean is the stable and highly reliable average. However, AM has one drawback in the sense that it is very much affected hy sampling fluctuations. In case of frequency distribution, mean cannot be advocated for open-end classification.

Like AM, median is also rigidly defined and easy to comprehend and compute. But median is not based on all the observation and does not allow itself to mathematical treatment. However, median is not much affected by sampling fluctuation and it is the most appropriate measure of central tendency for an open-end classification.

The strongest drawback of arthmetic mean is that it is very much affected by extreme observations. By sampling fluctuations, we mean, variation in statistic. It is well known fact, "Mean is least affected by fluctuations of sampling ". In fact, Median is more affected by sampling fluctuations as compare to AM.

Although mode is the most popular measure of central tendency, there are cases when mode remains undefined. Unlike mean, it has no mathematical property. Mode is also affected by sampling fluctuations.

GM and HM, like AM, possess some mathematical properties. They are rigidly defined and based on all the observations. But they are difficult to comprehend and compute and, as such, have limited applications for the computation of average rates and ratios and such like things.

Example 14.1.19: Given two positive numbers a and b, prove that **AH=G²**. Does the result hold for any set of observations?

Solution: For two positive numbers a and b, we have,

$$A = \frac{a+b}{2}$$

$$G = \sqrt{ab}$$

And $$H = \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

$$= \frac{2ab}{a+b}$$

Thus $$AH = \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$$= ab = G^2$$

This result holds for only two positive observations and not for any set of observations.

Example 14.1.20: The AM and GM for two observations are 5 and 4 respectively. Find the two observations.

Solution: If a and b are two positive observations then as given

$$\frac{a+b}{2} = 5$$

$$\Rightarrow \quad a+b = 10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$\sqrt{ab} = 4$$

$$\Rightarrow \quad ab = 16 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\therefore (a-b)^2 = (a+b)^2 - 4ab$$

$$= 10^2 - 4 \times 16$$

$$= 36$$

$\Rightarrow \quad a - b = 6$ (ignoring the negative sign)……………………….(3)

Adding (1) and (3) We get,

$$2a = 16$$

$$\Rightarrow \quad a = 8$$

From (1), we get $b = 10 - a = 2$

Thus, the two observations are 8 and 2.

Example 14.1.21: Find the mean and median from the following data:

Marks	:	less than 10	less than 20	less than 30
No. of Students	:	5	13	23
Marks	:	less than 40	less than 50	
No. of Students	:	27	30	

Also compute the mode using the approximate relationship between mean, median and mode.

Solution: What we are given in this problem is less than cumulative frequency distribution. We need to convert this cumulative frequency distribution to the corresponding frequency distribution and thereby compute the mean and median.

Table 14.1.19

Computation of Mean Marks for 30 students

Marks Class Interval (1)	No. of Students (f_i) (2)	Mid - Value (x_i) (3)	$f_i x_i$ (4)= (2)×(3)
0 – 10	5	5	25
10 – 20	13 – 5 = 8	15	120
20 – 30	23 – 13 = 10	25	250
30 – 40	27 – 23 = 4	35	140
40 – 50	30 – 27 = 3	45	135
Total	30	–	670

Hence the mean mark is given by

$$\bar{x} = \frac{\sum f_i x_i}{N}$$

$$= \frac{670}{30}$$

$$= 22.33$$

Table 14.1.10

Computation of Median Marks

Marks (Class Boundary)	No.of Students (Less than cumulative Frequency)
0	0
10	5
20	13
30	23
40	27
50	30

Since $\frac{N}{2} = \frac{30}{2} = 15$ lies between 13 and 23,

we have $l_1 = 20$, $N_l = 13$, $N_u = 23$

and $C = l_2 - l_1 = 30 - 20 = 10$

Thus,

$$\text{Median} = 20 + \frac{15-13}{23-13} \times 10$$

$$= 22$$

Since Mode = 3 Median – 2 Mean (approximately), we find that

Mode = 3x22 – 2x22.33

= 21.34

Example 14.1.22: Following are the salaries of 20 workers of a firm expressed in thousand rupees: 5, 17, 12, 23, 7, 15, 4, 18, 10, 6, 15, 9, 8, 13, 12, 2, 12, 3, 15, 14. The firm gave bonus amounting to ₹ 2,000, ₹ 3,000, ₹ 4,000, ₹ 5,000 and ₹ 6,000 to the workers belonging to the salary groups 1,000 – 5,000, 6,000 – 10,000 and so on and lastly 21,000 – 25,000. Find the average bonus paid per employee.

Solution: We first construct frequency distribution of salaries paid to the 20 employees. The average bonus paid per employee is given by $\frac{\sum f_i x_i}{N}$ Where x_i represents the amount of bonus paid to the i^{th} salary group and f_i, the number of employees belonging to that group which would be obtained on the basis of frequency distribution of salaries.

Table 14.1.11
Computation of Average bonus

Salary in thousand ₹ (Class Interval) (1)	Tally Mark (2)	No of workers (f_i) (3)	Bonus in Rupees x_i (4)	$f_i x_i$ (5) = (3) × (4)
1-5	||||	4	2000	8000
6-10	~~||||~~	5	3000	15000
11-15	~~||||~~ |||	8	4000	32000
16-20	||	2	5000	10000
21-25	|	1	6000	6000
TOTAL	–	20	–	71000

Hence, the average bonus paid per employee

$$= (₹)\frac{71000}{20}$$

$$= (₹)\ 3550$$

SUMMARY

- The best measure of central tendency, usually, is the AM. It is rigidly defined, based on all the observations, easy to comprehend, simple to calculate and amenable to mathematical properties. However, AM has one drawback in the sense that it is very much affected by sampling fluctuations. In case of frequency distribution, mean cannot be advocated for open-end classification.
- Median is also rigidly defined and easy to comprehend and compute. But median is not based on all the observation and does not allow itself to mathematical treatment. However, median is not much affected by sampling fluctuation and it is the most appropriate measure of central tendency for an open-end classification.
- Mode is the most popular measure of central tendency, there are cases when mode remains undefined. Unlike mean, it has no mathematical property. Mode is also affected by sampling fluctuations.
- Relationship between Mean, Median and Mode

  Mean – Mode = 3(Mean – Median)

  Mode = 3 Median – 2 Mean

- Relation between AM, GM, and HM

  AM ≥ GM ≥ HM

- GM and HM, like AM, possess some mathematical properties. They are rigidly defined and based on all the observations. But they are difficult to comprehend and compute and, as such, have limited applications for the computation of average rates and ratios and such like things.

EXERCISE — UNIT-I

Set A

Write down the correct answers. Each question carries 1 mark.

1. Measures of central tendency for a given set of observations measures

   (a) The scatterness of the observations (b) The central location of the observations

   (c) Both (a) and (b) (d) None of these.

2. While computing the AM from a grouped frequency distribution, we assume that

   (a) The classes are of equal length

   (b) The classes have equal frequency

   (c) All the values of a class are equal to the mid-value of that class

   (d) None of these.

3. Which of the following statements is wrong?

   (a) Mean is rigidly defined

   (b) Mean is not affected due to sampling fluctuations

   (c) Mean has some mathematical properties

   (d) All these

4. Which of the following statements is true?

   (a) Usually mean is the best measure of central tendency

   (b) Usually median is the best measure of central tendency

   (c) Usually mode is the best measure of central tendency

   (d) Normally GM is the best measure of central tendency

5. For open-end classification, which of the following is the best measure of central tendency?

   (a) AM (b) GM (c) Median (d) Mode

6. The presence of extreme observations does not affect

   (a) AM (b) Median (c) Mode (d) Any of these.

7. In case of an even number of observations which of the following is median?

   (a) Any of the two middle-most value

(b) The simple average of these two middle values

(c) The weighted average of these two middle values

(d) Any of these

8. The most commonly used measure of central tendency is

(a) AM (b) Median (c) Mode (d) Both GM and HM.

9. Which one of the following is not uniquely defined?

(a) Mean (b) Median (c) Mode (d) All of these measures

10. Which of the following measure of the central tendency is difficult to compute?

(a) Mean (b) Median (c) Mode (d) GM

11. Which measure(s) of central tendency is(are) considered for finding the average rates?

(a) AM (b) GM (c) HM (d) Both (b) and (c)

12. For a moderately skewed distribution, which of he following relationship holds?

(a) Mean – Mode = 3 (Mean – Median) (b) Median – Mode = 3 (Mean – Median)

(c) Mean – Median = 3 (Mean – Mode) (d) Mean – Median = 3 (Median – Mode)

13. Weighted averages are considered when

(a) The data are not classified

(b) The data are put in the form of grouped frequency distribution

(c) All the observations are not of equal importance

(d) Both (a) and (c).

14. Which of the following results hold for a set of distinct positive observations?

(a) AM ≥ GM ≥ HM (b) HM ≥ GM ≥ AM

(c) AM > GM > HM (d) GM > AM > HM

15. When a firm registers both profits and losses, which of the following measure of central tendency cannot be considered?

(a) AM (b) GM (c) Median (d) Mode

16. Quartiles are the values dividing a given set of observations into

(a) Two equal parts (b) Four equal parts (c) Five equal parts (d) None of these

17. Quartiles can be determined graphically using

(a) Histogram (b) Frequency Polygon (c) Ogive (d) Pie chart.

18. Which of the following measure(s) possesses (possess) mathematical properties?

(a) AM (b) GM (c) HM (d) All of these

19. Which of the following measure(s) satisfies (satisfy) a linear relationship between two variables?

(a) Mean (b) Median (c) Mode (d) All of these

20. Which of he following measures of central tendency is based on only fifty percent of the central values?

(a) Mean (b) Median (c) Mode (d) Both (a) and (b)

Set B

Write down the correct answers. Each question carries 2 marks.

1. If there are 3 observations 15, 20, 25 then the sum of deviation of the observations from their AM is

(a) 0 (b) 5 (c) –5 (d) None of these.

2. What is the median for the following observations?

5, 8, 6, 9, 11, 4.

(a) 6 (b) 7 (c) 8 (d) None of these

3. What is the modal value for the numbers 5, 8, 6, 4, 10, 15, 18, 10?

(a) 18 (b) 10 (c) 14 (d) None of these

4. What is the GM for the numbers 8, 24 and 40?

(a) 24 (b) 12 (c) $8 \cdot \sqrt[3]{15}$ (d) 10

5. The harmonic mean for the numbers 2, 3, 5 is

(a) 2.00 (b) 3.33 (c) 2.90 (d) $-\sqrt[3]{30}$.

6. If the AM and GM for two numbers are 6.50 and 6 respectively then the two numbers are

(a) 6 and 7 (b) 9 and 4 (c) 10 and 3 (d) 8 and 5.

7. If the AM and HM for two numbers are 5 and 3.2 respectively then the GM will be

(a) 16.00 (b) 4.10 (c) 4.05 (d) 4.00.

8. What is the value of the first quartile for observations 15, 18, 10, 20, 23, 28, 12, 16?

(a) 17 (b) 16 (c) 12.75 (d) 12

9. The third decile for the numbers 15, 10, 20, 25, 18, 11, 9, 12 is

(a) 13 (b) 10.70 (c) 11 (d) 11.50

10. If there are two groups containing 30 and 20 observations and having 50 and 60 as arithmetic means, then the combined arithmetic mean is

(a) 55 (b) 56 (c) 54 (d) 52.

11. The average salary of a group of unskilled workers is ₹ 10,000 and that of a group of skilled workers is ₹ 15,000. If the combined salary is ₹ 12,000, then what is the percentage of skilled workers?

(a) 40% (b) 50% (c) 60% (d) none of these

12. If there are two groups with 75 and 65 as harmonic means and containing 15 and 13 observation then the combined HM is given by

(a) 65 (b) 70.36 (c) 70 (d) 71.

13. What is the HM of 1,1/2, 1/3,...............1/n?

(a) n (b) 2n (c) $\frac{2}{(n+1)}$ (d) $\frac{n(n+1)}{2}$

14. An aeroplane flies from A to B at the rate of 500 km/hour and comes back from B to A at the rate of 700 km/hour. The average speed of the aeroplane is

(a) 600 km. per hour (b) 583.33 km. per hour

(c) $100\sqrt{35}$ km. per hour (d) 620 km. per hour.

15. If a variable assumes the values 1, 2, 3…5 with frequencies as 1, 2, 3…5, then what is the AM?

(a) $\frac{11}{3}$ (b) 5 (c) 4 (d) 4.50

16. Two variables x and y are given by y= 2x – 3. If the median of x is 20, what is the median of y?

(a) 20 (b) 40 (c) 37 (d) 35

17. If the relationship between two variables u and v are given by 2u + v + 7 = 0 and if the AM of u is 10, then the AM of v is

(a) 17 (b) –17 (c) –27 (d) 27.

18. If x and y are related by x–y–10 = 0 and mode of x is known to be 23, then the mode of y is

(a) 20 (b) 13 (c) 3 (d) 23.

19. If GM of x is 10 and GM of y is 15, then the GM of xy is

(a) 150 (b) log 10 × log 15 (c) log 150 (d) None of these.

20. If the AM and GM for 10 observations are both 15, then the value of HM is

(a) Less than 15 (b) More than 15

(c) 15 (d) Can not be determined.

Set C

Write down the correct answers. Each question carries 5 marks.

1. What is the value of mean and median for the following data:

Marks:	5–14	15–24	25–34	35–44	45–54	55–64
No. of Students:	10	18	32	26	14	10

(a) 30 and 28 (b) 29 and 30 (c) 33.68 and 32.94 (d) 34.21 and 33.18

2. The mean and mode for the following frequency distribution

Class interval:	350–369	370–389	390–409	410–429	430–449	450–469
Frequency:	15	27	31	19	13	6

are

(a) 400 and 390 (b) 400.58 and 390 (c) 400.58 and 394.50 (d) 400 and 394.

3. The median and modal profits for the following data

Profit in '000 ₹:	below 5	below 10	below 15	below 20	below 25	below 30
No. of firms:	10	25	45	55	62	65

are

(a) 11.60 and 11.50 (b) ₹ 11556 and ₹ 11267

(c) ₹11875 and ₹ 11667 (d) 11.50 and 11.67.

4. Following is an incomplete distribution having modal mark as 44

Marks:	0–20	20–40	40–60	60–80	80–100
No. of Students:	5	18	?	12	5

What would be the mean marks?

(a) 45 (b) 46 (c) 47 (d) 48

5. The data relating to the daily wage of 20 workers are shown below:

₹ 50, ₹ 55, ₹ 60, ₹ 58, ₹ 59, ₹ 72, ₹ 65, ₹ 68, ₹ 53, ₹ 50, ₹ 67, ₹ 58,
₹ 63, ₹ 69, ₹ 74, ₹ 63, ₹ 61, ₹ 57, ₹ 62, ₹ 64.

The employer pays bonus amounting to ₹ 100, ₹ 200, ₹ 300, ₹ 400 and ₹ 500 to the wage earners in the wage groups 50-55, 55-60 and so on, and lastly 70-75, during the festive month of October.

What is the average bonus paid per wage earner?

(a) ₹ 200 (b) ₹ 250 (c) ₹ 285 (d) ₹270

6. The third quartile and 65th percentile for the following data are

Profits in '000 ₹:	les than 10	10–19	20–29	30–39	40–49	50–59
No. of firms:	5	18	38	20	9	2

(a) ₹ 33,500 and ₹ 29,184 (b) ₹ 33,000 and ₹ 28,680

(c) ₹ 33,600 and ₹ 29,000 (d) ₹ 33,250 and ₹ 29,250.

7. For the following incomplete distribution of marks of 100 pupils, median mark is known to be 32.

Marks:	0–10	10–20	20–30	30–40	40–50	50–60
No. of Students:	10	–	25	30	–	10

What is the mean mark?

(a) 32 (b) 31 (c) 31.30 (d) 31.50

8. The mode of the following distribution is ₹ 66. What would be the median wage?

Daily wages (₹):	30–40	40–50	50–60	60–70	70–80	80–90
No of workers:	8	16	22	28	–	12

(a) ₹ 64.00 (b) ₹ 64.56 (c) ₹ 63.21 (d) ₹ 64.25

ANSWERS

Set A

1.	(b)	**2.**	(c)	**3.**	(b)	**4.**	(a)	**5.**	(c)	**6.**	(b)
7.	(b)	**8.**	(a)	**9.**	(c)	**10.**	(d)	**11.**	(d)	**12.**	(a)
13.	(b)	**14.**	(c)	**15.**	(b)	**16.**	(b)	**17.**	(c)	**18.**	(d)
19.	(d)	**20.**	(b)								

Set B

1.	(a)	**2.**	(b)	**3.**	(b)	**4.**	(c)	**5.**	(c)	**6.**	(b)
7.	(d)	**8.**	(c)	**9.**	(b)	**10.**	(c)	**11.**	(a)	**12.**	(c)
13.	(c)	**14.**	(b)	**15.**	(a)	**16.**	(c)	**17.**	(c)	**18.**	(b)
19.	(a)	**20.**	(c)								

Set C

1.	(c)	**2.**	(c)	**3.**	(c)	**4.**	(d)	**5.**	(c)	**6.**	(a)
7.	(c)	**8.**	(c)								

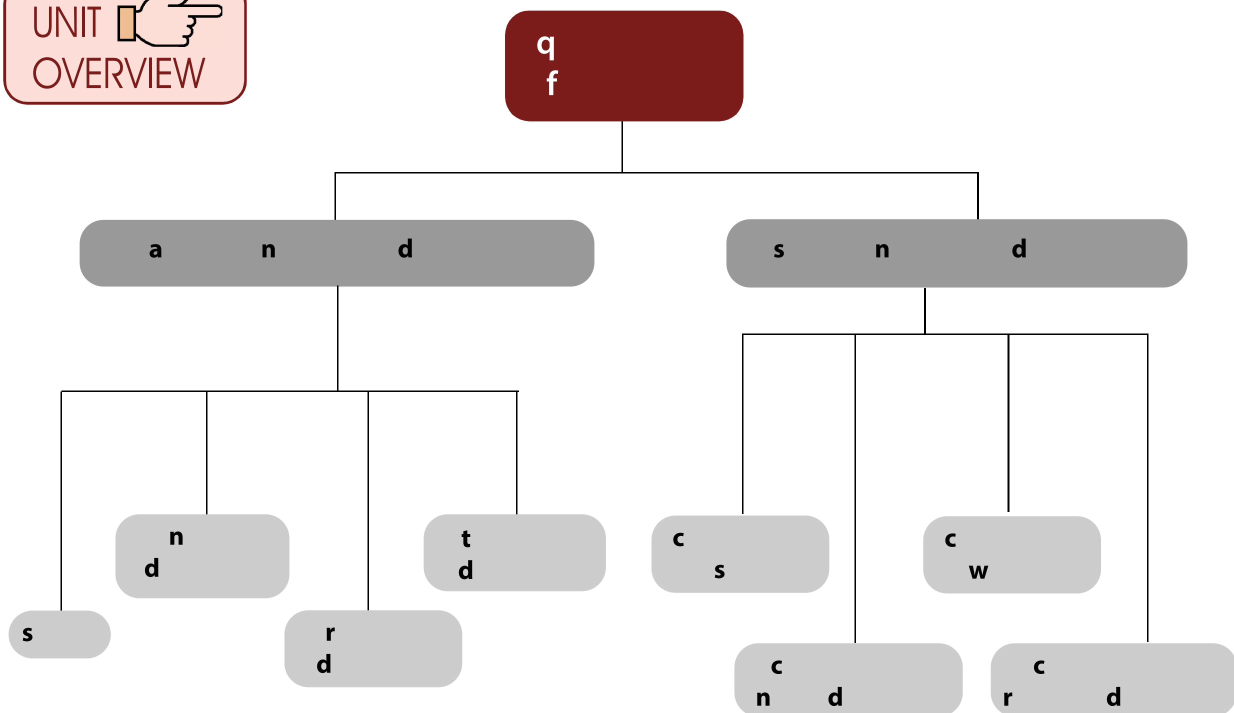
UNIT II: DISPERSION

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- To understand different measures of Dispersion i.e Range, Quartile Deviation, Mean Deviation and Standard Deviation and computational techniques of these measures.
- To learn comparative advantages and disadvantages of these measures and therefore, which measures to use in which circumstance.
- To understand a set of observation, it is equally important to have knowledge of dispersion which indicates the volatility. In advanced stage of chartered accountancy course, volatility measures will be useful in understanding risk involved in financial decision making.

UNIT OVERVIEW



14.2.1 DEFINITION OF DISPERSION

The second important characteristic of a distribution is given by dispersion. Two distributions may be identical in respect of its first important characteristic i.e. central tendency and yet they may differ on account of scatterness. The following figure shows a number of distributions having identical measure of central tendency and yet varying measure of scatterness. Obviously, distribution is having the maximum amount of dispersion.

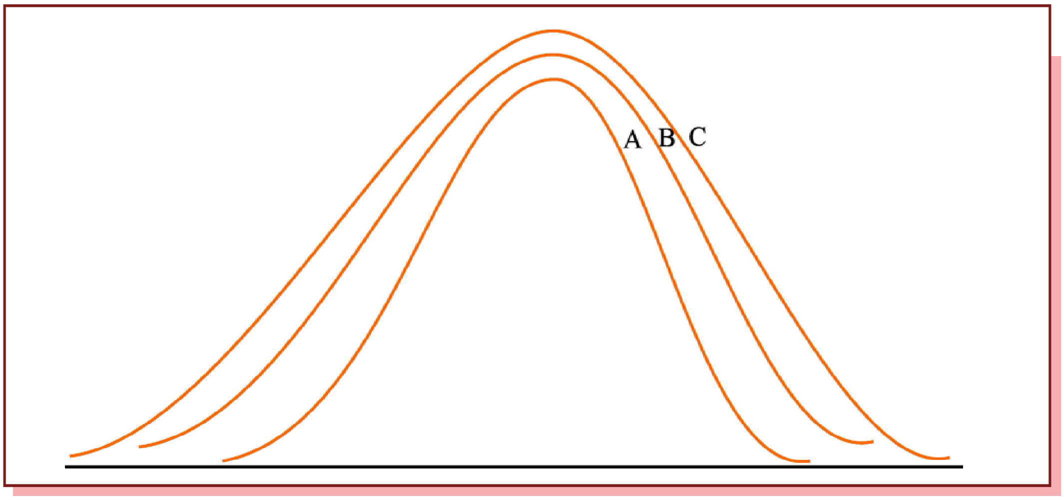


Figure 14.2.1

Showing distributions with identical measure of central tendency
and varying amount of dispersion.

Dispersion for a given set of observations may be defined as the amount of deviation of the observations, usually, from an appropriate measure of central tendency. Measures of dispersion may be broadly classified into

1. Absolute measures of dispersion.

2. Relative measures of dispersion.

Absolute measures of dispersion are classified into

(i) Range

(ii) Mean Deviation

(iii) Standard Deviation

(iv) Quartile Deviation

Likewise, we have the following relative measures of dispersion :

(i) Coefficient of Range.

(ii) Coefficient of Mean Deviation

(iii) Coefficient of Variation

(iv) Coefficient of Quartile Deviation.

We may note the following points of distinction between the absolute and relative measures of dispersion :

I Absolute measures are dependent on the unit of the variable under consideration whereas the relative measures of dispersion are unit free.

II For comparing two or more distributions, relative measures and not absolute measures of dispersion are considered.

III Compared to absolute measures of dispersion, relative measures of dispersion are difficult to compute and comprehend.

Characteristics for an ideal measure of dispersion

As discussed in section 14.2.1 an ideal measure of dispersion should be properly defined, easy to comprehend, simple to compute, based on all the observations, unaffected by sampling fluctuations and amenable to some desirable mathematical treatment.

14.2.2 RANGE

For a given set of observations, range may be defined as the difference between the largest and smallest of observations. Thus if L and S denote the largest and smallest observations respectively then we have

Range = L – S

The corresponding relative measure of dispersion, known as coefficient of range, is given by

$$\text{Coefficient of range} = \frac{L-S}{L+S} \times 100$$

For a grouped frequency distribution, range is defined as the difference between the two extreme class boundaries. The corresponding relative measure of dispersion is given by the ratio of the difference between the two extreme class boundaries to the total of these class boundaries, expressed as a percentage.

We may note the following important result in connection with range:

Result:

Range remains unaffected due to a change of origin but affected in the same ratio due to a change in scale i.e., if for any two constants a and b, two variables x and y are related by y = a + bx,

Then the range of y is given by

$$R_y = |\ b| \times R_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.2.1)$$

Example 14.2.1: Following are the wages of 8 workers expressed in Rupees.
82, 96, 52, 75, 70, 65, 50, 70. Find the range and also its coefficient.

Solution: The largest and the smallest wages are L = ₹ 96 and S= ₹ 50
Thus range = ₹ 96 – ₹ 50 = ₹ 46

$$\text{Coefficient of range} = \frac{96-50}{96+50} \times 100$$
$$= 31.51$$

Example 14.2.2: What is the range and its coefficient for the following distribution of weights?

Weights in kgs. :	50 – 54	55 – 59	60 – 64	65 – 69	70 – 74
No. of Students :	12	18	23	10	3

Solution: The lowest class boundary is 49.50 kgs. and the highest class boundary is 74.50 kgs. Thus we have

Range = 74.50 kgs. – 49.50 kgs.

= 25 kgs.

Also, coefficient of range $= \frac{74.50 - 49.50}{74.50 + 49.50} \times 100$

$= \frac{25}{124} \times 100$

$= 20.16$

Example 14.2.3 : If the relationship between x and y is given by 2x+3y=10 and the range of x is ₹ 15, what would be the range of y?

Solution: Since 2x+3y=10

Therefore, $y = \frac{10}{3} - \frac{2}{3}x$

Applying (14.2.1) , the range of y is given by

$R_y = |b| \times R_x = 2/3 \times ₹\ 15$

$= ₹\ 10.$

14.2.3 MEAN DEVIATION

Since range is based on only two observations, it is not regarded as an ideal measure of dispersion. A better measure of dispersion is provided by mean deviation which, unlike range, is based on all the observations. For a given set of observation, mean deviation is defined as the arithmetic mean of the absolute deviations of the observations from an appropriate measure of central tendency. Hence if a variable x assumes n values $x_1, x_2, x_3 \ldots x_n$, then the mean deviation of x about an average A is given by

$$MD_A = \frac{1}{n}\sum|x_i - A| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14.2.2)$$

For a grouped frequency distribution, mean deviation about A is given by

$$MD_A = \frac{1}{n}\sum|x_i - A|f_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(14.2.2)$$

Where x_i and f_i denote the mid value and frequency of the i-th class interval and

$N = \sum f_i$

In most cases we take A as mean or median and accordingly, we get mean deviation about mean or mean deviation about median.

A relative measure of dispersion applying mean deviation is given by

$$\text{Coefficient of mean deviation} = \frac{\text{Mean deviation about A}}{\text{A}} \times 100 \quad \text{................(14.2.3)}$$

Mean deviation takes its minimum value when the deviations are taken from the median. Also mean deviation remains unchanged due to a change of origin but changes in the same ratio due to a change in scale i.e. if y = a + bx, a and b being constants,

then MD of y = |b| × MD of x ....(14.2.4)

Example 14.2.4: What is the mean deviation about mean for the following numbers?

5, 8, 10, 10, 12, 9.

Solution:

The mean is given by

$$\overline{X} = \frac{5+8+10+10+12+9}{6} = 9$$

Table 14.2.1

Computation of MD about AM

x_i	$\lvert x_i - \overline{x} \rvert$
5	4
8	1
10	1
10	1
12	3
9	0
Total	10

Thus mean deviation about mean is given by

$$\frac{\sum \lvert x_i - \overline{x} \rvert}{n} = \frac{10}{6} = 1.67$$

Example. 14.2.5: Find mean deviations about median and also the corresponding coefficient for the following profits ('000 ₹) of a firm during a week.

82, 56, 75, 70, 52, 80, 68.

Solution:

The profits in thousand rupees is denoted by x. Arranging the values of x in an ascending order, we get

52, 56, 68, 70, 75, 80, 82.

Therefore, Me = 70. Thus, Median profit = ₹ 70,000.

Table 14.2.2

Computation of Mean deviation about median

x_i	$\|x_i - Me\|$
52	18
56	14
68	2
70	0
75	5
80	10
82	12
Total	61

Thus mean deviation about median = $\frac{\sum |x_i - \text{Median}|}{n}$

$$= (₹)\,\frac{61}{7}$$

$$= ₹\ 8714.28$$

Coefficient of mean deviation = $\frac{\text{MD about median}}{\text{Median}} \times 100$

$$= \frac{8714.28}{70000} \times 100$$

$$= 12.45$$

Example 14.2.6 : Compute the mean deviation about the arithmetic mean for the following data:

x :	1	3	5	7	9
f :	5	8	9	2	1

Also find the coefficient of the mean deviation about the AM.

Solution: We are to apply formula (14.1.2) as these data refer to a grouped frequency distribution the AM is given by

$$\bar{x} = \frac{\sum f_i x_i}{N}$$

$$= \frac{5\times1+8\times3+9\times5+2\times7+1\times9}{5+8+9+2+1} = 3.88$$

Table 14.2.3

Computation of MD about the AM

x (1)	f (2)	$\lvert x-\bar{x}\rvert$ (3)	$f\lvert x-\bar{x}\rvert$ (4) = (2) × (3)
1	5	2.88	14.40
3	8	0.88	7.04
5	9	1.12	10.08
7	2	3.12	6.24
9	1	5.12	5.12
Total	25	–	42.88

Thus, MD about AM is given by

$$\frac{\sum f\lvert x-\bar{x}\rvert}{N}$$

$$= \frac{42.88}{25}$$

$$= 1.72$$

Coefficient of MD about its AM $= \frac{\text{MD about AM}}{\text{AM}} \times 100$

$$= \frac{1.72}{3.88}\times100$$

$$= 44.33$$

Example 14.2.7: Compute the coefficient of mean deviation about median for the following distribution:

Weight in kgs.	:	40-50	50-60	60-70	70-80
No. of persons	:	8	12	20	10

Solution: We need to compute the median weight in the first stage

Table 14.2.4

Computation of median weight

Weight in kg (CB)	No. of Persons (Cumulative Frequency)
40	0
50	8
60	20
70	40
80	50

Hence, $$M = l_1 + \left(\frac{\frac{N}{2} - N_l}{N_u - N_l}\right) \times C$$

$$= \left[60 + \frac{25 - 20}{40 - 20} \times 10\right] \text{kg.} = 62.50\text{kg.}$$

Table 14.2.5

Computation of mean deviation of weight about median

weight (kgs.) (1)	mid-value (x_i) kgs. (2)	No. of persons (f_i) (3)	$\lvert x_i - Me \rvert$ (kgs.) (4)	$f_i \lvert x_i - Me \rvert$ (kgs.) (5)=(3)×(4)
40–50	45	8	17.50	140
50–60	55	12	7.50	90
60–70	65	20	2.50	50
70–80	75	10	12.50	125
Total	–	50	–	405

$$\text{Mean deviation about median} = \frac{\sum f_i |x_i - \text{Median}|}{N}$$

$$= \frac{405}{50}\text{kg.}$$

$$= 8.10 \text{ kg.}$$

$$\text{Coefficient of mean deviation about median} = \frac{\text{Mean deviation about median}}{\text{Median}} \times 100$$

$$= \frac{8.10}{62.50} \times 100$$

$$= 12.96$$

Example 14.2.8: If x and y are related as $4x+3y+11 = 0$ and mean deviation of x is 5.40, what is the mean deviation of y?

Solution: Since $4x + 3y + 11 = 0$

Therefore, $y = \left(\frac{-11}{3}\right) + \left(\frac{-4}{3}\right)x$

Hence MD of y= $|b| \times$ MD of x

$$= \frac{4}{3} \times 5.40$$

$$= 7.20$$

14.2.4 STANDARD DEVIATION

Although mean deviation is an improvement over range so far as a measure of dispersion is concerned, mean deviation is difficult to compute and further more, it cannot be treated mathematically. The best measure of dispersion is, usually, standard deviation which does not possess the demerits of range and mean deviation.

Standard deviation for a given set of observations is defined as the root mean square deviation when the deviations are taken from the AM of the observations. If a variable x assumes n values x_1, x_2, x_3 ………..x_n then its standard deviation(s) is given by

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} \quad \text{…………………………(14.2.5)}$$

For a grouped frequency distribution, the standard deviation is given by

$$s = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{N}} \quad \text{……………………… (14.2.6)}$$

(14.2.5) and (14.2.6) can be simplified to the following forms

$$s = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2} \quad \text{for unclassified data}$$

$$= \sqrt{\frac{\sum f_i x_i^2}{N} - \bar{x}^2} \quad \text{for a grouped frequency distribution.} \qquad \ldots (14.2.7)$$

Sometimes the square of standard deviation, known as variance, is regarded as a measure of dispersion. We have, then,

$$\text{Variance} = s^2 = \frac{\sum (x_i - \bar{x})^2}{n} \quad \text{for unclassified data}$$

$$= \frac{\sum f_i (x_i - \bar{x})^2}{N} \quad \text{for a grouped frequency distribution} \quad \ldots\ldots\ldots\ldots\ldots..(14.2.8)$$

A relative measure of dispersion using standard deviation is given by coefficient of variation (cv) which is defined as the ratio of standard deviation to the corresponding arithmetic mean, expressed as a percentage.

$$\text{Coefficient of Variation (CV)} = \frac{SD}{AM} \times 100 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14..2.9)$$

ILLUSTRATIONS:

Example 14.2.9: Find the standard deviation and the coefficient of variation for the following numbers: 5, 8, 9, 2, 6

Solution: We present the computation in the following table.

Table 14.2.6
Computation of standard deviation

x_i	x_i^2
5	25
8	64
9	81
2	4
6	36
30	$\sum x_i^2 = 210$

Applying (14.2.7), we get the standard deviation as

$$s = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$$

$$= \sqrt{\frac{210}{5} - \left(\frac{30}{5}\right)^2} \qquad \left(\text{since } \bar{x} = \frac{\Sigma x_i}{n}\right)$$

$$= \sqrt{42 - 36}$$

$$= \sqrt{6}$$

$$= 2.45$$

The coefficient of variation is

$$CV = 100 \times \frac{SD}{AM}$$

$$= 100 \times \frac{2.45}{6}$$

$$= 40.83$$

Example 14.2.10: Show that for any two numbers a and b, standard deviation is given by $\frac{|a-b|}{2}$.

Solution: For two numbers a and b, AM is given by $\bar{x} = \frac{a+b}{2}$

The variance is

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{2}$$

$$= \frac{\left(a - \frac{a+b}{2}\right)^2 + \left(b - \frac{a+b}{2}\right)^2}{2}$$

$$= \frac{\frac{(a-b)^2}{4} + \frac{(a-b)^2}{4}}{2}$$

$$= \frac{(a-b)^2}{4}$$

$$\Rightarrow \quad s = \frac{|a-b|}{2}$$

(The absolute sign is taken, as SD cannot be negative).

Example 14.2.11: Prove that for the first n natural numbers, SD is $\sqrt{\frac{n^2-1}{12}}$.

Solution: for the first n natural numbers AM is given by

$$\bar{x} = \frac{1+2+3+....................+n}{n}$$

$$= \frac{n(n+1)}{2n}$$

$$= \frac{n+1}{2}$$

$$\therefore SD = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$$

$$= \sqrt{\frac{1^2+2^2+3^2..................+n^2}{n} - \left(\frac{n+1}{2}\right)^2}$$

$$= \sqrt{\frac{n(n+1)(2n+1)}{6n} - \frac{(n+1)^2}{4}}$$

$$= \sqrt{\frac{(n+1)(4n+2-3n-3)}{12}} = \sqrt{\frac{n^2-1}{12}}$$

Thus, SD of first n natural numbers is SD = $\sqrt{\frac{n^2-1}{12}}$

We consider the following formula for computing standard deviation from grouped frequency distribution with a view to saving time and computational labour:

$$S = \sqrt{\frac{\sum f_i d_i^{\ 2}}{N} - \left(\frac{\sum f_i d_i}{N}\right)^2} \qquad ....(14.2.10)$$

Where $d_i = \frac{x_i - A}{C}$

Example 14.2.12: Find the SD of the following distribution:

Weight (kgs.)	:	50-52	52-54	54-56	56-58	58-60
No. of Students	:	17	35	28	15	5

Solution:

Table 14.2.7
Computation of SD

Weight (kgs.) (1)	No. of Students (f_i) (2)	Mid-value (x_i) (3)	$d_i = \frac{x_i - 55}{2}$ (4)	$f_i d_i$ (5)=(2)×(4)	$f_i d_i^2$ (6)=(5)×(4)
50-52	17	51	–2	–34	68
52-54	35	53	–1	–35	35
54-56	28	55	0	0	0
56-58	15	57	1	15	15
58-60	5	59	2	10	20
Total	100	–	–	– 44	138

Applying (14.2.7), we get the SD of weight as

$$= \sqrt{\frac{\sum f_i d_i^2}{N} - \left(\frac{\sum f_i d_i}{N}\right)^2} \times C$$

$$= \sqrt{\frac{138}{100} - \frac{(-44)^2}{100}} \times 2\text{kgs.}$$

$$= \sqrt{1.38 - 0.1936} \times 2 \text{ kgs.}$$

$$= 2.18 \text{ kgs.}$$

Properties of standard deviation

I. If all the observations assumed by a variable are constant i.e. equal, then the SD is zero. This means that if all the values taken by a variable x is k, say, then s = 0. This result applies to range as well as mean deviation.

II. SD remains unaffected due to a change of origin but is affected in the same ratio due to a change of scale i.e., if there are two variables x and y related as y = a+bx for any two constants a and b, then SD of y is given by

$$s_y = |b| s_x \qquad \text{...........................(14.2.11)}$$

III. If there are two groups containing n_1 and n_2 observations, $\bar{x}_1$ and $\bar{x}_2$ as respective AM's, s_1 and s_2 as respective SD's , then the combined SD is given by

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}} \qquad \text{...........................(14.2.12)}$$

where, $d_1 = \bar{x}_1 - \bar{x}$

$d_2 = \bar{x}_2 - \bar{x}$

and $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$ = combined AM

This result can be extended to more than 2 groups. For $x \geq 2$ groups, we have

$$s = \sqrt{\frac{\sum n_i s_i^2 + \sum n_i d_i^2}{\sum n_i}} \quad \text{………………………….. (14.2.13)}$$

With $d_i = x_i - \bar{x}$

and $\bar{x} = \frac{\sum n_i \bar{x}_i}{\sum n_i}$

Where $\bar{x}_1 = \bar{x}_2$ (14.2.13) is reduced to

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}}$$

Example 14.2.13: If AM and coefficient of variation of x are 10 and 40 respectively, what is the variance of (15–2x)?

Solution: let y = 15 – 2x

Then applying (14.2.4), we get,

$s_y = 2 \times s_x$ ………………………………… (1)

As given cv_x = coefficient of variation of x = 40 and $\bar{x} = 10$

This $cv_x = \frac{s_x}{x} \times 100$

$\Rightarrow \quad 40 = \frac{S_x}{10} \times 100$

$\Rightarrow \quad S_x = 4$

From (1), $S_y = 2 \times 4 = 8$

Therefore, variance of $(15 - 2x) = S_y^2 = 64$

Example 14.2.14: Compute the SD of 9, 5, 8, 6, 2.

Without any more computation, obtain the SD of

Sample I	–1,	–5,	–2,	–4,	–8,
Sample II	90,	50,	80,	60,	20,
Sample III	23,	15,	21,	17,	9.

Solution:

Table 14.2.7
Computation of SD

x_i	x_i^2
9	81
5	25
8	64
6	36
2	4
30	210

The SD of the original set of observations is given by

$$s = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$= \sqrt{\frac{210}{5} - \left(\frac{30}{5}\right)^2}$$

$$= \sqrt{42 - 36}$$

$$= \sqrt{6}$$

$$= 2.45$$

If we denote the original observations by x and the observations of sample I by y, then we have

$$y = -10 + x$$

$$y = (-10) + (1)\,x$$

$$\therefore S_y = |1| \times S_x$$

$$= 1 \times 2.45$$

$$= 2.45$$

In case of sample II, x and y are related as

$$Y = 10x$$

$$= 0 + (15)x$$

$\therefore s_y = |10| \times s_x$

$= 10 \times 2.45$

$= 24.50$

And lastly, y= (5)+(2)x

$\Rightarrow s_y = 2 \times 2.45$

$= 4.90$

Example 14.2.15: For a group of 60 boy students, the mean and SD of stats. marks are 45 and 2 respectively. The same figures for a group of 40 girl students are 55 and 3 respectively. What is the mean and SD of marks if the two groups are pooled together?

Solution: As given $n_1 = 60$, $\bar{x}_1 = 45$, $s_1 = 2$ $n_2 = 40$, $\bar{x}_2 = 55$, $s_2 = 3$

Thus the combined mean is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$= \frac{60 \times 45 + 40 \times 55}{60 + 40}$$

$$= 49$$

Thus $d_1 = \bar{x}_1 - \bar{x} = 45 - 49 = -4$

$d_2 = \bar{x}_2 - \bar{x} = 55 - 49 = 6$

Applying (14.2.13), we get the combined SD as

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$$

$$s = \sqrt{\frac{60 \times 2^2 + 40 \times 3^2 + 60 \times (-4)^2 + 40 \times 6^2}{60 + 40}}$$

$$= \sqrt{30}$$

$$= 5.48$$

Example 14.2.16: The mean and standard deviation of the salaries of the two factories are provided below :

Factory	No. of Employees	Mean Salary	SD of Salary
A	30	₹ 4800	₹ 10
B	20	₹ 5000	₹ 12

i) Find the combined mean salary and standard deviation of salary.

ii) Examine which factory has more consistent structure so far as satisfying its employees are concerned.

Solution: Here we are given

$n_1 = 30,\ \bar{x}_1 = ₹\ 4800,\ s_1 = ₹\ 10,$

$n_2 = 20,\ \bar{x}_2 = ₹\ 5000,\ s_2 = ₹\ 12$

i) $\dfrac{30 \times ₹\ 4800 + 20 \times ₹\ 5000}{30 + 20} = ₹\ 4800$

$d_1 = \bar{x}_1 - \bar{x} = ₹\ 4{,}800 - ₹\ 4880 = -₹\ 80$

$d_2 = \bar{x}_2 - \bar{x} = ₹\ 5{,}000 - ₹\ 4880 = ₹\ 120$

hence, the combined SD in rupees is given by

$$s = \sqrt{\frac{30\times10^2 + 20\times12^2 + 30\times(-80)^2 + 20\times120^2}{30+20}}$$

$= \sqrt{9717.60}$

$= 98.58$

thus the combined mean salary and the combined standard deviation of salary are ₹ 4880 and ₹ 98.58 respectively.

ii) In order to find the more consistent structure, we compare the coefficients of variation of the two factories. Letting $CV_A = 100\times\frac{S_A}{\bar{x}_A}$ and $CV_B = 100\times\frac{S_B}{\bar{x}_B}$

We would say factory A is more consistent

if $CV_A < CV_B$. Otherwise factory B would be more consistent.

Now $CV_A = 100\times\frac{S_A}{\bar{x}_A} = 100\times\frac{S_1}{\bar{x}_1} = \frac{100\times10}{4800} = 0.21$

and $CV_B = 100\times\frac{S_B}{\bar{x}_B} = 100\times\frac{S_2}{\bar{x}_2} = \frac{100\times12}{5000} = 0.24$

Thus we conclude that factory A has more consistent structure.

Example 14.2.17: A student computes the AM and SD for a set of 100 observations as 50 and 5 respectively. Later on, she discovers that she has made a mistake in taking one observation as 60 instead of 50. What would be the correct mean and SD if

i) The wrong observation is left out?

ii) The wrong observation is replaced by the correct observation?

Solution: As given, n = 100, $\bar{x} = 50$, S = 5

Wrong observation = 60, correct observation = 50

$$\bar{x} = \frac{\sum x_i}{n}$$

$$\Rightarrow \quad \sum x_i = n\bar{x} = 100 \times 50 = 5000$$

and $\quad s^2 = \frac{\sum x_i^2}{n} - \bar{x}^2$

$$\Rightarrow \sum x_i^2 = n(\bar{x}^2 + s^2) = 100(50^2 + 5^2) = 252500$$

i) Sum of the 99 observations = 5000 – 60 = 4940

AM after leaving the wrong observation = 4940/99 = 49.90

Sum of squares of the observation after leaving the wrong observation

$= 252500 - 60^2 = 248900$

Variance of the 99 observations $= 248900/99 - (49.90)^2$

= 2514.14 – 2490.01

= 24.13

∴ SD of 99 observations = 4.91

ii) Sum of the 100 observations after replacing the wrong observation by the correct observation = 5000 – 60 + 50 = 4990

$$AM = \frac{4990}{100} = 49.90$$

Corrected sum of squares $= 252500 + 50^2 - 60^2 = 251400$

Corrected SD $= \sqrt{\frac{251400}{100} - (49.90)^2}$

$= \sqrt{23.94} = 4.90$

14.2.5 QUARTILE DEVIATION

Another measure of dispersion is provided by **quartile deviation** or **semi-inter–quartile** range which is given by

$$Q_d = \frac{Q_3 - Q_1}{2} \quad \text{..................................(14.2.14)}$$

A relative measure of dispersion using quartiles is given by coefficient of quartile deviation which is

$$\text{Coefficient of quartile deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1} \times 100 \quad \text{..................................(14.2.15)}$$

Quartile deviation provides the best measure of dispersion for open-end classification. It is also less affected due to extreme observations or sampling fluctuations. Like other measures of dispersion, quartile deviation remains unaffected due to a change of origin but is affected in the same ratio due to change in scale.

Example 14.2.18 : Following are the marks of the 10 students : 56, 48, 65, 35, 42, 75, 82, 60, 55, 50. Find quartile deviation and also its coefficient.

Solution:

After arranging the marks in an ascending order of magnitude, we get 35, 42, 48, 50, 55, 56, 60, 65, 75, 82

$$\text{First quartile } (Q_1) = \frac{(n+1)}{4}\text{th observation}$$

$$= \frac{(10+1)}{4}\text{th observation}$$

$= 2.75^{th}$ observation

$= 2^{nd}$ observation + 0.75 × difference between the third and the 2^{nd} observation.

$= 42 + 0.75 \times (48 - 42)$

$= 46.50$

$$\text{Third quartile } (Q_3) = \frac{3(n+1)}{4}\text{th observation}$$

$= 8.25^{th}$ observation

$= 65 + 0.25 \times 10$

$= 67.50$

Thus applying (14.2.14), we get the quartile deviation as

$$\frac{Q_3 - Q_1}{2} = \frac{67.50 - 46.50}{2} = 10.50$$

Also, using (14.2.15), the coefficient of quartile deviation

$$= \frac{Q_3 - Q_1}{Q_3 + Q_1} \times 100$$

$$= \frac{67.50 - 46.50}{67.50 + 46.50} \times 100$$

$= 18.42$

Example 14.2.19 : If the quartile deviation of x is 6 and 3x + 6y = 20, what is the quartile deviation of y?

Solution: $3x + 6y = 20$

$$\Rightarrow \quad y = \left(\frac{20}{6}\right) + \left(\frac{-3}{6}\right)x$$

Therefore, quartile deviation of $y = \frac{|-3|}{6} \times$ quartile deviation of x

$$= \frac{1}{2} \times 6$$

$= 3.$

Example 14.2.20: Find an appropriate measures of dispersion from the following data:

Daily wages (₹)	:	upto 20	20-40	40-60	60-80	80-100
No. of workers (₹)	:	5	11	14	7	3

Solution: Since this is an open-end classification, the appropriate measure of dispersion would be quartile deviation as quartile deviation does not taken into account the first twenty five percent and the last twenty five per cent of the observations.

Table 14.2.8
Computation of Quartile

Daily wages in (₹) (Class boundary)	No. of workers (less than cumulative frequency)
a	0
20	5
40	16
60	30
80	37
100	40

Here a denotes the first Class Boundary

$$Q_1 = ₹ \left[20 + \frac{10-5}{16-5} \times 20\right] = ₹\ 29.09$$

$$Q_3 = ₹ \left[40 + \frac{30-16}{30-16} \times 20\right] = ₹\ 60$$

$Q_3 = ₹\ 60$

Thus quartile deviation of wages is given by

$$\frac{Q_3 - Q_1}{2}$$

$$= \frac{₹\ 60 - ₹\ 29.09}{2}$$

$= ₹\ 15.46$

Example 14.2.21: The mean and variance of 5 observations are 4.80 and 6.16 respectively. If three of the observations are 2, 3 and 6, what are the remaining observations?

Solution: Let the remaining two observations be a and b, then as given

$$\frac{2+3+6+a+b}{5} = 4.80$$

$\Rightarrow \quad 11+a+b = 24$

$\Rightarrow \quad a+b = 13 \quad ....(1)$

and $\quad \frac{2^2+a^2+b^2+3^2+6^2}{5}-(4.80)^2$

$\Rightarrow \quad \frac{49+a^2+b^2}{5}-23.04 = 6.16$

$\Rightarrow \quad 49 + a^2 + b^2 = 146$

$\Rightarrow \quad a^2 + b^2 = 97 \quad ....(2)$

From (1), we get $a = 13 - b$ ....(3)

Eliminating a from (2) and (3), we get

$(13 - b)^2 + b^2 = 97$

$\Rightarrow \quad 169 - 26b + 2b^2 = 97$

$\Rightarrow \quad b^2 - 13\,b + 36 = 0$

$\Rightarrow \quad (b-4)(b-9) = 0$

$\Rightarrow \quad b = 4$ or 9

From (3), $a = 9$ or 4

Thus the remaining observations are 4 and 9.

Example 14.2.22: After shift of origin and change of scale, a frequency distribution of a continuous variable with equal class length takes the following form of the changed variable (d):

d	:	–2	–1	0	1	2
Frequency	:	17	35	28	15	5

If the mean and standard deviation of the original frequency distribution are 54.12 and 2.1784 respectively, find the original frequency distribution.

Solution: We need find out the origin A and scale C from the given conditions.

Since $d_i = \frac{x_i - A}{C}$

$\Rightarrow \quad x_i = A + Cd_i$

Once A and C are known, the mid- values x_i's would be known. Finally, we convert the mid-values to the corresponding class boundaries by using the formula:

$$LCB = x_i - C/2$$

and $$UCB = x_i + C/2$$

On the basis of the given data, we find that

$$\sum f_i d_i = -44, \sum f_i d_i^2 = 138 \text{ and } N = 100$$

$$\text{Hence } s = \sqrt{\frac{\sum f_i d_i^2}{N} - \left(\frac{\sum f_i d_i}{N}\right)^2} \times C$$

$$\Rightarrow \quad 2.1784 = \sqrt{\frac{138}{100} - \left(\frac{-44}{100}\right)^2} \times C$$

$$\Rightarrow \quad 2.1784 = \sqrt{1.38 - 0.1936} \times C$$

$$\Rightarrow \quad 2.1784 = 1.0892 \times C$$

$$\Rightarrow \quad C = 2$$

$$\text{Further, } \bar{x} = A + \frac{\sum f_i d_i}{N} \times C$$

$$\Rightarrow \quad 54.12 = A + \frac{-44}{100} \times 2$$

$$\Rightarrow \quad 54.12 = A - 0.88$$

$$\Rightarrow \quad A = 55$$

$$\text{Thus } x_i = A + Cd_i$$

$$\Rightarrow \quad x_i = 55 + 2d_i$$

Table 14.2.9

Computation of the Original Frequency Distribution

d_i	f_i	$x_i = 55 + 2d_i$	Class interval $x_i \pm \frac{C}{2}$
–2	17	51	50-52
–1	35	53	52-54
0	28	55	54-56
1	15	57	56-58
2	5	59	58-60

Example 14.2.23: Compute coefficient of variation from the following data:

Age	:	under 10	under 20	under 30	under 40	under 50	under 60
No. of persons							
Dying	:	10	18	30	45	60	80

Solution: What is given in this problem is less than cumulative frequency distribution. We need first convert it to a frequency distribution and then compute the coefficient of variation.

Table 14.2.10

Computation of coefficient of variation

Age in years class Interval	No. of persons dying (f_i)	Mid-value (x_i)	$d_i = \frac{x_i - 25}{10}$	$f_i d_i$	$f_i d_i^2$
0-10	10	5	–2	–20	40
10-20	18–10= 8	15	–1	–8	8
20-30	30–18=12	25	0	0	0
30-40	45–30=15	35	1	15	15
40-50	60–45=15	45	2	30	60
50-60	80–60=20	55	3	60	180
Total	80	–	–	77	303

The AM is given by:

$$\bar{x} = A + \frac{\sum f_i d_i}{N} \times C$$

$$= \left(25 + \frac{77}{80} \times 10\right) \text{ years}$$

= 34.63 years

The standard deviation is

$$s = \sqrt{\frac{\sum f_i d_i^2}{N} - \left(\frac{\sum f_i d_i}{N}\right)^2} \times C$$

$$= \sqrt{\frac{303}{80} - \left(\frac{77}{80}\right)^2} \times 10 \text{ years}$$

$$= \sqrt{3.79 - 0.93} \times 10 \text{ years}$$
$$= 16.91 \text{ years}$$

Thus the coefficient of variation is given by

$$CV = \frac{S}{X} \times 100$$

$$= \frac{16.91}{34.63} \times 100$$

$$= 48.83$$

Example 14.2.24: You are given the distribution of wages in two factors A and B

Wages in ₹	:	100-200	200-300	300-400	400-500	500-600	600-700
No. of workers in A	:	8	12	17	10	2	1
No. of workers in B	:	6	18	25	12	2	2

State in which factory, the wages are more variable.

Solution:

As explained in example 14.2.3, we need compare the coefficient of variation of A(i.e. v_A) and of B (i.e v_B).

If $v_A > v_B$, then the wages of factory A would be more variable. Otherwise, the wages of factory B would be more variable where

$$V_A = 100 \times \frac{s_A}{\bar{x}_A} \quad \text{and} \quad V_B = 100 \times \frac{s_B}{\bar{x}_B}$$

Table 14.2.11

Computation of coefficient of variation of wages of Two Factories A and B

Wages in rupees (1)	Mid-value x (2)	d= (3)	No. of workers of A f_A (4)	No. of workers of B f_B (5)	$f_A d$ (6)=(3)×(4)	$f_A d^2$ (7)=(3)×(6)	$f_B d$ (8)=(3)×(5)	$f_B d^2$ (9)=(3)×(8)
100-200	150	–2	8	6	–16	32	–12	24
200-300	250	–1	12	18	–12	12	–18	18
300-400	350	0	17	25	0	0	0	0
400-500	450	1	10	12	10	10	12	12
500-600	550	2	2	2	4	8	4	8
600-700	650	3	1	2	3	9	6	18
Total	–	–	50	65	–11	71	– 8	80

For Factory A

$$\bar{x}_A = ₹\left(350 + \frac{-11}{50} \times 100\right) = ₹\ 328$$

$$S_A = ₹\sqrt{\frac{71}{50} - \left(\frac{-11}{50}\right)^2} \times 100 = ₹\ 117.12$$

$$\therefore V_A = \frac{S_A}{\bar{x}_A} \times 100 = 35.71$$

For Factory B

$$\bar{x}_B = ₹\left(350 + \frac{-8}{65} \times 100\right) = ₹\ 337.69$$

$$S_B = ₹\sqrt{\frac{80}{65} - \left(\frac{-8}{65}\right)^2} \times 100$$

$$= ₹\ 110.25$$

$$\therefore V_B = \frac{110.25}{337.69} \times 100 = 32.65$$

As $V_A > V_B$, the wages for factory A is more variable.

SUMMARY

- Standard deviation is the most widely and commonly used measure of dispersion
- Range is the quickest to compute and as such, has its application in statistical quality control. However, range is based on only two observations and affected too much by the presence of extreme observation(s).
- Mean deviation is rigidly defined, based on all the observations and not much affected by sampling fluctuations. However, mean deviation is difficult to comprehend and its computation is also time consuming and laborious. Furthermore, unlike SD, mean deviation does not possess mathematical properties.
- Quartile deviation is also rigidly defined, easy to compute and not much affected by sampling fluctuations. The presence of extreme observations has no impact on quartile deviation since quartile deviation is based on the central fifty-percent of the observations. However, quartile deviation is not based on all the observations and it has no desirable mathematical properties. Nevertheless, quartile deviation is the best measure of dispersion for open-end classifications.

EXERCISE — UNIT-II

Set A

Write down the correct answers. Each question carries one mark.

1. Which of the following statements is correct?
   (a) Two distributions may have identical measures of central tendency and dispersion.
   (b) Two distributions may have the identical measures of central tendency but different measures of dispersion.
   (c) Two distributions may have the different measures of central tendency but identical measures of dispersion.
   (d) All the statements (a), (b) and (c).
2. Dispersion measures
   (a) The scatterness of a set of observations
   (b) The concentration of a set of observations
   (c) Both (a) and (b)
   (d) Neither (a) and (b).
3. When it comes to comparing two or more distributions we consider
   (a) Absolute measures of dispersion (b) Relative measures of dispersion
   (c) Both (a) and (b) (d) Either (a) or (b).
4. Which one is easiest to compute?
   (a) Relative measures of dispersion (b) Absolute measures of dispersion
   (c) Both (a) and (b) (d) Range
5. Which one is an absolute measure of dispersion?
   (a) Range (b) Mean Deviation
   (c) Standard Deviation (d) All these measures
6. Which measure of dispersion is most usefull?
   (a) Standard deviation (b) Quartile deviation
   (c) Mean deviation (d) Range
7. Which measures of dispersions is not affected by the presence of extreme observations?
   (a) Range (b) Mean deviation
   (c) Standard deviation (d) Quartile deviation
8. Which measure of dispersion is based on the absolute deviations only?
   (a) Standard deviation (b) Mean deviation
   (c) Quartile deviation (d) Range

9. Which measure is based on only the central fifty percent of the observations?
   (a) Standard deviation (b) Quartile deviation
   (c) Mean deviation (d) All these measures
10. Which measure of dispersion is based on all the observations?
   (a) Mean deviation (b) Standard deviation
   (c) Quartile deviation (d) (a) and (b) but not (c)
11. The appropriate measure of dispersion for open-end classification is
   (a) Standard deviation (b) Mean deviation
   (c) Quartile deviation (d) All these measures.
12. The most commonly used measure of dispersion is
   (a) Range (b) Standard deviation
   (c) Coefficient of variation (d) Quartile deviation.
13. Which measure of dispersion has some desirable mathematical properties?
   (a) Standard deviation (b) Mean deviation
   (c) Quartile deviation (d) All these measures
14. If the profits of a company remains the same for the last ten months, then the standard deviation of profits for these ten months would be ?
   (a) Positive (b) Negative (c) Zero (d) (a) or (c)
15. Which measure of dispersion is considered for finding a pooled measure of dispersion after combining several groups?
   (a) Mean deviation (b) Standard deviation
   (c) Quartile deviation (d) Any of these
16. A shift of origin has no impact on
   (a) Range (b) Mean deviation
   (c) Standard deviation (d) All these and quartile deviation.
17. The range of 15, 12, 10, 9, 17, 20 is
   (a) 5 (b) 12 (c) 13 (d) 11.
18. The standard deviation of 10, 16, 10, 16, 10, 10, 16, 16 is
   (a) 4 (b) 6 (c) 3 (d) 0.
19. For any two numbers SD is always
   (a) Twice the range (b) Half of the range
   (c) Square of the range (d) None of these.

20. If all the observations are increased by 10, then
    (a) SD would be increased by 10
    (b) Mean deviation would be increased by 10
    (c) Quartile deviation would be increased by 10
    (d) All these three remain unchanged.
21. If all the observations are multiplied by 2, then
    (a) New SD would be also multiplied by 2
    (b) New SD would be half of the previous SD
    (c) New SD would be increased by 2
    (d) New SD would be decreased by 2.

Set B

Write down the correct answers. Each question carries two marks.

1. What is the coefficient of range for the following wages of 8 workers?
   ₹ 80, ₹ 65, ₹ 90, ₹ 60, ₹ 75, ₹ 70, ₹ 72, ₹ 85.
   (a) ₹ 30 (b) ₹ 20 (c) 30 (d) 20
2. If R_x and R_y denote ranges of x and y respectively where x and y are related by 3x+2y+10=0, what would be the relation between x and y?
   (a) $R_x = R_y$ (b) $2 R_x = 3 R_y$ (c) $3 R_x = 2 R_y$ (d) $R_x = 2 R_y$
3. What is the coefficient of range for the following distribution?

Class Interval :	10-19	20-29	30-39	40-49	50-59
Frequency:	11	25	16	7	3

   (a) 22 (b) 50 (c) 72.46 (d) 75.82
4. If the range of x is 2, what would be the range of –3x +50 ?
   (a) 2 (b) 6 (c) –6 (d) 44
5. What is the value of mean deviation about mean for the following numbers?
   5, 8, 6, 3, 4.
   (a) 5.20 (b) 7.20 (c) 1.44 (d) 2.23
6. What is the value of mean deviation about mean for the following observations?
   50, 60, 50, 50, 60, 60, 60, 50, 50, 50, 60, 60, 60, 50.
   (a) 5 (b) 7 (c) 35 (d) 10
7. The coefficient of mean deviation about mean for the first 9 natural numbers is
   (a) 200/9 (b) 80 (c) 400/9 (d) 50.

8. If the relation between x and y is 5y–3x = 10 and the mean deviation about mean for x is 12, then the mean deviation of y about mean is

(a) 7.20 (b) 6.80 (c) 20 (d) 18.80.

9. If two variables x and y are related by 2x + 3y –7 =0 and the mean and mean deviation about mean of x are 1 and 0.3 respectively, then the coefficient of mean deviation of y about its mean is

(a) –5 (b) 12 (c) 50 (d) 4.

10. The mean deviation about mode for the numbers 4/11, 6/11, 8/11, 9/11, 12/11, 8/11 is

(a) 1/6 (b) 1/11 (c) 6/11 (d) 5/11.

11. What is the standard deviation of 5, 5, 9, 9, 9, 10, 5, 10, 10?

(a) $\sqrt{14}$ (b) $\frac{\sqrt{42}}{3}$ (c) 4.50 (d) 8

12. If the mean and SD of x are a and b respectively, then the SD of $\frac{x-a}{b}$ is

(a) –1 (b) 1 (c) ab (d) a/b.

13. What is the coefficient of variation of the following numbers?
53, 52, 61, 60, 64.

(a) 8.09 (b) 18.08 (c) 20.23 (d) 20.45

14. If the SD of x is 3, what us the variance of (5–2x)?

(a) 36 (b) 6 (c) 1 (d) 9

15. If x and y are related by 2x+3y+4 = 0 and SD of x is 6, then SD of y is

(a) 22 (b) 4 (c) $\sqrt{5}$ (d) 9.

16. The quartiles of a variable are 45, 52 and 65 respectively. Its quartile deviation is

(a) 10 (b) 20 (c) 25 (d) 8.30.

17. If x and y are related as 3x+4y = 20 and the quartile deviation of x is 12, then the quartile deviation of y is

(a) 16 (b) 14 (c) 10 (d) 9.

18. If the SD of the 1st n natural numbers is 2, then the value of n must be

(a) 2 (b) 7 (c) 6 (d) 5.

19. If x and y are related by y = 2x+ 5 and the SD and AM of x are known to be 5 and 10 respectively, then the coefficient of variation of y is

(a) 25 (b) 30 (c) 40 (d) 20.

20. The mean and SD for a, b and 2 are 3 and $\frac{2}{\sqrt{3}}$ respectively, The value of ab would be

(a) 5 (b) 6 (c) 11 (d) 3.

Set C

Write down the correct answer. Each question carries 5 marks.

1. What is the mean deviation about mean for the following distribution?

Variable:	5	10	15	20	25	30
Frequency:	3	4	6	5	3	2

(a) 6.00 (b) 5.93 (c) 6.07 (d) 7.20

2. What is the mean deviation about median for the following data?

X :	3	5	7	9	11	13	15
F :	2	8	9	16	14	7	4

(a) 2.50 (b) 2.46 (c) 2.43 (d) 2.37

3. What is the coefficient of mean deviation for the following distribution of heights? Take deviation from AM.

Height in inches:	60-62	63-65	66-68	69-71	72-74
No. of students:	5	22	28	17	3

(a) 2.31 inches (b) 3.45 inches (c) 3.82 inches (d) 2.48 inches

4. The mean deviation of weights about median for the following data:

Weight (lb) :	131-140	141-150	151-160	161-170	171-180	181-190
No. of persons :	3	8	13	15	6	5

Is given by

(a) 10.97 (b) 8.23 (c) 9.63 (d) 11.45.

5. What is the standard deviation from the following data relating to the age distribution of 200 persons?

Age (year) :	20	30	40	50	60	70	80
No. of people:	13	28	31	46	39	23	20

(a) 15.29 (b) 16.87 (c) 18.00 (d) 17.52

6. What is the coefficient of variation for the following distribution of wages?

Daily Wages (₹):	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
No. of workers	17	28	21	15	13	6

(a) ₹ 14.73 (b) 14.73 (c) 26.93 (d) 20.82

7. Which of the following companies A and B is more consistent so far as the payment of dividend is concerned ?

Dividend paid by A :	5	9	6	12	15	10	8	10
Dividend paid by B :	4	8	7	15	18	9	6	6

(a) A (b) B (c) Both (a) and (b) (d) Neither (a) nor (b)

8. The mean and SD for a group of 100 observations are 65 and 7.03 respectively. If 60 of these observations have mean and SD as 70 and 3 respectively, what is the SD for the group comprising 40 observations?

   (a) 16 (b) 25 (c) 4 (d) 2

9. If two samples of sizes 30 and 20 have means as 55 and 60 and variances as 16 and 25 respectively, then what would be the SD of the combined sample of size 50?

   (a) 5.00 (b) 5.06 (c) 5.23 (d) 5.35

10. The mean and SD of a sample of 100 observations were calculated as 40 and 5.1 respectively by a CA student who took one of the observations as 50 instead of 40 by mistake. The current value of SD would be

    (a) 4.90 (b) 5.00 (c) 4.88 (d) 4.85.

11. The value of appropriate measure of dispersion for the following distribution of daily wages

Wages (₹):	Below 30	30-39	40-49	50-59	60-79	Above 80
No. of workers	5	7	18	32	28	10

is given by

(a) ₹ 11.03 (b) ₹ 10.50 (c) 11.68 (d) ₹ 11.68.

UNIT-II: ANSWERS

Set A

1.	(d)	**2.**	(a)	**3.**	(b)	**4.**	(d)	**5.**	(d)	**6.**	(a)
7.	(d)	**8.**	(b)	**9.**	(b)	**10.**	(d)	**11.**	(c)	**12.**	(b)
13.	(a)	**14.**	(c)	**15.**	(b)	**16.**	(d)	**17.**	(d)	**18.**	(c)
19.	(b)	**20.**	(d)	**21.**	(a)						

Set B

1.	(d)	**2.**	(c)	**3.**	(c)	**4.**	(b)	**5.**	(c)	**6.**	(a)
7.	(c)	**8.**	(a)	**9.**	(b)	**10.**	(a)	**11.**	(b)	**12.**	(b)
13.	(a)	**14.**	(a)	**15.**	(b)	**16.**	(a)	**17.**	(d)	**18.**	(b)
19.	(c)	**20.**	(c)								

Set C

1.	(c)	**2.**	(d)	**3.**	(a)	**4.**	(a)	**5.**	(b)	**6.**	(c)
7.	(a)	**8.**	(c)	**9.**	(b)	**10.**	(b)	**11.**	(a)		

ADDITIONAL QUESTION BANK

1. The number of measures of central tendency is

   (a) two (b) three (c) four (d) five

2. The words "mean" or "average" only refer to

   (a) A.M (b) G.M (c) H.M (d) none

3. ———————— is the most stable of all the measures of central tendency.

   (a) G.M (b) H.M (c) A.M (d) none.

4. Mean is of ———— types.

   (a) 3 (b) 4 (c) 8 (d) 5

5. Weighted A.M is related to

   (a) G.M (b) frequency (c) H.M (d) none.

6. Frequencies are also called as weights.

   (a) True (b) false (c) both (d) none

7. The algebraic sum of deviations of observations from their A.M is

   (a) 2 (b) -1 (c) 1 (d) 0

8. G.M of a set of n observations is the ———— root of their product.

   (a) n/2 th (b) (n+1)th (c) nth (d) (n -1)th

9. The algebraic sum of deviations of 8, 1, 6 from the A.M viz.5 is

   (a) -1 (b) 0 (c) 1 (d) none

10. G.M of 8, 4,2 is

    (a) 4 (b) 2 (c) 8 (d) none

11. ———————— is the reciprocal of the A.M of reciprocal of observations.

    (a) H.M (b) G.M (c) both (d) none

12. A.M is never less than G.M

    (a) True (b) false (c) both (d) none

13. G.M is less than H.M

    (a) true (b) false (c) both (d) none

14. The value of the middlemost item when they are arranged in order of magnitude is called

    (a) standard deviation (b) mean (c) mode (d) median

15. Median is unaffected by extreme values.

    (a) true (b) false (c) both (d) none

16. Median of 2, 5, 8, 4, 9, 6, 71 is

(a) 9 (b) 8 (c) 5 (d) 6

17. The value which occurs with the maximum frequency is called

(a) median (b) mode (c) mean (d) none

18. In the formula Mode = $L_1 + (d_1 X c)/ (d_1 + d_2)$

d_1 is the difference of frequencies in the modal class & the ———— class.

(a) preceding (b) following (c) both (d) none

19. In the formula Mode = $L_1 + (d_1 X c)/ (d_1 + d_2)$

d_2 is the difference of frequencies in the modal class & the ———— class.

(a) preceding (b) succeeding (c) both (d) none

20. In formula of median for grouped frequency distribution N is

(a) total frequency (b) frequency density
(c) frequency (d) cumulative frequency

21. When all observations occur with equal frequency ———— does not exit.

(a) median (b) mode (c) mean (d) none

22. Mode of the observations 2, 5, 8, 4, 3, 4, 4, 5, 2, 4, 4 is

(a) 3 (b) 2 (c) 5 (d) 4

23. For the observations 5, 3, 6, 3, 5, 10, 7, 2 there are ———— modes.

(a) 2 (b) 3 (c) 4 (d) 5

24. ———— of a set of observations is defined to be their sum, divided by the no. of observations.

(a) H.M (b) G.M (c) A.M (d) none

25. Simple average is sometimes called

(a) weighted average (b) unweighted average
(c) relative average (d) none

26. When a frequency distribution is given, the frequencies themselves treated as weights.

(a) True (b) false (c) both (d) none

27. Each value is considered only once for

(a) simple average (b) weighted average
(c) both (d) none

28. Each value is considered as many times as it occurs for

(a) simple average (b) weighted average
(c) both (d) none

29. Multiplying the values of the variable by the corresponding weights and then dividing the sum of products by the sum of weights is

    (a) simple average (b) weighted average
    (c) both (d) none

30. Simple & weighted average are equal only when all the weights are equal.

    (a) True (b) false (c) both (d) none

31. The word "average " used in "simple average" and "weighted average" generally refers to

    (a) median (b) mode (c) A.M , G.M or H.M (d) none

32. ———— average is obtained on dividing the total of a set of observations by their number

    (a) simple (b) weighted (c) both (d) none

33. Frequencies are generally used as

    (a) range (b) weights (c) mean (d) none

34. The total of a set of observations is equal to the product of their number of observations and the

    (a) A.M (b) G.M (c) H.M (d) none

35. The total of the deviations of a set of observations from their A.M is always

    (a) 0 (b) 1 (c) -1 (d) none

36. Deviation may be positive or negative or zero

    (a) true (b) false (c) both (d) none

37. The sum of the squares of deviations of a set of observations has the smallest value, when the deviations are taken from their

    (a) A.M (b) H.M (c) G.M (d) none

38. For a given set of positive observations H.M is less than G.M

    (a) true (b) false (c) both (d) none

39. For a given set of positive observations A.M is greater than G.M

    (a) true (b) false (c) both (d) none

40. Calculation of G.M is more difficult than

    (a) A.M (b) H.M (c) median (d) none

41. ———— has a limited use

    (a) A.M (b) G.M (c) H.M (d) (b) and (c)

42. A.M of 8, 1, 6 is

    (a) 5 (b) 6 (c) 4 (d) none

43. ———— can be calculated from a frequency distribution with open end intervals

(a) Median (b) Mean (c) Mode (d) none

44. The values of all items are taken into consideration in the calculation of

(a) median (b) mean (c) mode (d) none

45. The values of extreme items do not influence the average in case of

(a) median (b) mean (c) mode (d) none

46. In a distribution with a single peak and moderate skewness to the right, it is closer to the concentration of the distribution in case of

(a) mean (b) median (c) both (d) none

47. If the variables x & z are so related that z = ax + b for each $x = x_i$ where a & b are constants, then $\overline{z} = a\,\overline{x} + b$

(a) true (b) false (c) both (d) none

48. G.M is defined only when

(a) all observations have the same sign and none is zero

(b) all observations have the different sign and none is zero

(c) all observations have the same sign and one is zero

(d) all observations have the different sign and one is zero

49. ———— is useful in averaging ratios, rates and percentages.

(a) A.M (b) G.M (c) H.M (d) Both (b) and (c)

50. G.M is useful in construction of index number.

(a) true (b) false (c) both (d) none

51. More laborious numerical calculations involves in G.M than A.M

(a) True (b) false (c) both (d) none

52. H.M is defined when no observation is

(a) 3 (b) 2 (c) 1 (d) 0

53. When all values occur with equal frequency, there is no

(a) mode (b) mean (c) median (d) none

54. ———— cannot be treated algebraically

(a) mode (b) mean (c) median (d) Both (a) and (c)

55. For the calculation of ———— , the data must be arranged in the form of a frequency distribution.

(a) median (b) mode (c) mean (d) none

56. ———— is equal to the value corresponding to cumulative frequency

(a) mode (b) mean (c) median (d) none

57. ————— is the value of the variable corresponding to the highest frequency

(a) mode (b) mean (c) median (d) none

58. The class in which mode belongs is known as

(a) median class (b) mean class (c) modal class (d) none

59. The formula of mode is applicable if classes are of ————— width.

(a) equal (b) unequal (c) both (d) none

60. For calculation of ——— we have to construct cumulative frequency distribution

(a) mode (b) median (c) mean (d) none

61. For calculation of ——— we have to construct a grouped frequency distribution

(a) median (b) mode (c) mean (d) none

62. Relation between mean, median & mode is

(a) mean - mode = 2 (mean - median) (b) mean - median = 3 (mean - mode)
(c) mean - median = 2 (mean - mode) (d) mean - mode = 3 (mean - median)

63. When the distribution is symmetrical, mean, median and mode

(a) coincide (b) do not coincide (c) both (d) none

64. Mean, median & mode are equal for the

(a) Binomial distribution (b) Normal distribution
(c) both (d) none

65. In most frequency distributions, it has been observed that the three measures of central tendency viz. mean, median & mode, obey the approximate relation, provided the distribution is

(a) very skew (b) not very skew (c) both (d) none

66. —————— divides the total number of observations into two equal parts.

(a) mode (b) mean (c) median (d) none

67. Measures which are used to divide or partition the observations into a fixed number of parts are collectively known as

(a) partition values (b) quartiles (c) both (d) none

68. The middle most value of a set of observations is

(a) median (b) mode (c) mean (d) none

69. The number of observations smaller than ——— is the same as the number larger than it.

(a) median (b) mode (c) mean (d) none

* Question no. 64 is based on theoretical distribution.

70. ———— is the value of the variable corresponding to cumulative frequency N /2

(a) mode (b) mean (c) median (d) none

71. ———————— divide the total no. observations into 4 equal parts.

(a) median (b) deciles (c) quartiles (d) percentiles

72. ———————— quartile is known as Upper quartile

(a) First (b) Second (c) Third (d) none

73. Lower quartile is

(a) first quartile (b) second quartile (c) upper quartile (d) none

74. The number of observations smaller than lower quartile is the same as the no. lying between lower and middle quartile.

(a) true (b) false (c) both (d) none

75. ———— are used for measuring central tendency, dispersion & skewness.

(a) Median (b) Deciles (c) Percentiles (d) Quartiles.

76. The second quartile is known as

(a) median (b) lower quartile (c) upper quartile (d) none

77. The lower & upper quartiles are used to define

(a) standard deviation (b) quartile deviation
(c) both (d) none

78. Three quartiles are used in

(a) Pearson's formula (b) Bowley's formula
(c) both (d) none

79. Less than First quartile, the frequency is equal to

(a) N /4 (b) 3N /4 (c) N /2 (d) none

80. Between first & second quartile, the frequency is equal to

(a) 3N/4 (b) N /2 (c) N /4 (d) none

81. Between second & upper quartile, the frequency is equal to

(a) 3N/4 (b) N /4 (c) N /2 (d) none

82. Above upper quartile, the frequency is equal to

(a) N /4 (b) N /2 (c) 3N /4 (d) none

83. Corresponding to first quartile, the cumulative frequency is

(a) N /2 (b) N / 4 (c) 3N /4 (d) none

* Question no. 78 is based on skewness, which is not in syllabus.

84. Corresponding to second quartile, the cumulative frequency is

(a) N/4 (b) 2 N/4 (c) 3N/4 (d) none

85. Corresponding to upper quartile, the cumulative frequency is

(a) 3N/4 (b) N/4 (c) 2N/4 (d) none

86. The values which divide the total number of observations into 10 equal parts are

(a) quartiles (b) percentiles (c) deciles (d) none

87. There are ———— deciles.

(a) 7 (b) 8 (c) 9 (d) 10

88. Corresponding to first decile, the cumulative frequency is

(a) N/10 (b) 2N/10 (c) 9N/10 (d) none

89. Fifth decile is equal to

(a) mode (b) median (c) mean (d) none

90. The values which divide the total number of observations into 100 equal parts is

(a) percentiles (b) quartiles (c) deciles (d) none

91. Corresponding to second decile, the cumulative frequency is

(a) N/10 (b) 2N/10 (c) 5N/10 (d) none

92. There are ——— percentiles.

(a) 100 (b) 98 (c) 97 (d) 99

93. 10^{th} percentile is equal to

(a) 1^{st} decile (b) 10^{th} decile (c) 9^{th} decile (d) none

94. 50^{th} percentile is known as

(a) 50^{th} decile (b) 50^{th} quartile (c) mode (d) median

95. 20^{th} percentile is equal to

(a) 19^{th} decile (b) 20^{th} decile (c) 2^{nd} decile (d) none

96. (3^{rd} quartile —— 1^{st} quartile)/2 is

(a) skewness (b) median (c) quartile deviation (d) none

97. 1^{st} percentile is less than 2^{nd} percentile.

(a) true (b) false (c) both (d) none

98. 25^{th} percentile is equal to

(a) 1^{st} quartile (b) 25^{th}quartile (c) 24^{th} quartile (d) none

99. 90^{th} percentile is equal to

(a) 9^{th} quartile (b) 90^{th} decile (c) 9^{th} decile (d) none

100. 1^{st} decile is greater than 2^{nd} decile

(a) True (b) false (c) both (d) none

101. Quartile deviation is a measure of dispersion.

(a) true (b) false (c) both (d) none

102. To define quartile deviation we use

(a) lower & middle quartiles (b) lower & upper quartiles
(c) upper & middle quartiles (d) none

103. Calculation of quartiles, deciles ,percentiles may be obtained graphically from

(a) Frequency Polygon (b) Histogram (c) Ogive (d) none

104. 7^{th} decile is the abscissa of that point on the Ogive whose ordinate is

(a) 7N/10 (b) 8N /10 (c) 6N /10 (d) none

105. Rank of median is

(a) (n+ 1)/2 (b) (n+ 1)/4 (c) 3(n + 1)/4 (d) none

106. Rank of 1^{st} quartile is

(a) (n+ 1)/2 (b) (n+ 1)/4 (c) 3(n + 1)/4 (d) none

107. Rank of 3rd quartile is

(a) 3(n+ 1)/4 (b) (n+ 1)/4 (c) (n + 1)/2 (d) none

108. Rank of k th decile is

(a) (n+ 1)/2 (b) (n+ 1)/4 (c) (n + 1)/10 (d) k(n +1)/10

109. Rank of k th percentile is

(a) (n+ 1)/100 (b) k(n+ 1)/10 (c) k(n + 1)/100 (d) none

110. ———————— is equal to value corresponding to cumulative frequency (N + 1)/2 from simple frequency distribution

(a) Median (b) 1^{st} quartile (c) 3^{rd} quartile (d) 4^{th} quartile

111. ———— is equal to the value corresponding to cumulative frequency (N + 1)/4 from simple frequency distribution

(a) Median (b) 1^{st} quartile (c) 3^{rd} quartile (d) 1^{st} decile

112. ———— is equal to the value corresponding to cumulative frequency 3 (N + 1)/4 from simple frequency distribution

(a) Median (b) 1^{st} quartile (c) 3^{rd} quartile (d) 1^{st} decile

113. ———— is equal to the value corresponding to cumulative frequency k (N + 1)/10 from simple frequency distribution

(a) Median (b) k^{th} decile (c) k^{th} percentile (d) none

114. ———— is equal to the value corresponding to cumulative frequency k(N + 1)/100 from simple frequency distribution

(a) k^{th} decile (b) k^{th} percentile (c) both (d) none

115. For grouped frequency distribution ———— is equal to the value corresponding to cumulative frequency N /2

(a) median (b) 1^{st} quartile (c) 3^{rd} quartile (d) none

116. For grouped frequency distribution ———— is equal to the value corresponding to cumulative frequency N /4

(a) median (b) 1^{st} quartile (c) 3^{rd} quartile (d) none

117. For grouped frequency distribution ———— is equal to the value corresponding to cumulative frequency 3N /4

(a) median (b) 1^{st} quartile (c) 3^{rd} quartile (d) none

118. For grouped frequency distribution ———— is equal to the value corresponding to cumulative frequency kN/10

(a) median (b) kth percentile (c) kth decile (d) none

119. For grouped frequency distribution ———— is equal to the value corresponding to cumulative frequency kN /100

(a) k^{th} quartile (b) k^{th} percentile (c) k^{th} decile (d) none

120. In Ogive, abscissa corresponding to ordinate N/2 is

(a) median (b) 1^{st} quartile (c) 3^{rd} quartile (d) none

121. In Ogive, abscissa corresponding to ordinate N/4 is

(a) median (b) 1^{st} quartile (c) 3^{rd} quartile (d) none

122. In Ogive, abscissa corresponding to ordinate 3N/4 is

(a) median (b) 3^{rd} quartile (c) 1^{st} quartile (d) none

123. In Ogive, abscissa corresponding to ordinate ———— is kth decile.

(a) kN/10 (b) kN/100 (c) kN/50 (d) none

124. In Ogive , abscissa corresponding to ordinate ———— is kth percentile.

(a) kN/10 (b) kN/100 (c) kN/50 (d) none

125. For 899, 999, 391, 384, 590, 480, 485, 760, 111, 240
Rank of median is

(a) 2.75 (b) 5.5 (c) 8.25 (d) none

126. For 333, 999, 888, 777, 666, 555, 444
Rank of 1^{st} quartile is

(a) 3 (b) 1 (c) 2 (d) 7

127. For 333, 999, 888, 777, 1000, 321, 133
Rank of 3rd quartile is
(a) 7 (b) 4 (c) 5 (d) 6

128. Price per kg.(₹) : 45 50 35; Kgs.Purchased : 100 40 60 Total frequency is
(a) 300 (b) 100 (c) 150 (d) 200

129. The length of a rod is measured by a tape 10 times. You are to estimate the length of the rod by averaging these 10 determinations.
What is the suitable form of average in this case?
(a) A.M (b) G.M (c) H.M (d) none

130. A person purchases 5 rupees worth of eggs from 10 different markets.You are to find the average no. of eggs per rupee for all the markets taken together. What is the suitable form of average in this case?
(a) A.M (b) G.M (c) H.M (d) none

131. You are given the population of India for the courses of 1981 & 1991. You are to find the population of India at the middle of the period by averaging these population figures, assuming a constant rate of increase of population.
What is the suitable form of average in this case?
(a) A.M (b) G.M (c) H.M (d) none

132. ———————— is least affected by sampling fluctions.
(a) Standard deviation (b) Quartile deviation
(c) both (d) none

133. "Root –Mean Square Deviation from Mean" is
(a) Standard deviation (b) Quartile deviation
(c) both (d) none

134. Standard Deviation is
(a) absolute measure (b) relative measure (c) both (d) none

135. Coefficient of variation is
(a) absolute measure (b) relative measure (c) both (d) none

136. ———————— deviation is called semi-interquartile range.
(a) Percentile (b) Standard (c) Quartile (d) none

137. —————————— Deviation is defined as half the difference between the lower & upper quartiles.
(a) Quartile (b) Standard (c) both (d) none

138. Quartile Deviation for the data 1, 3, 4, 5, 6, 6, 10 is

(a) 3 (b) 1 (c) 6 (d) 1.5

139. Coefficient of Quartile Deviation is

(a) (Quartile Deviation x 100)/Median (b) (Quartile Deviation x 100)/Mean
(c) (Quartile Deviation x 100) /Mode (d) none

140. Mean for the data 6, 4, 1, 6, 5, 10, 3 is

(a) 7 (b) 5 (c) 6 (d) none

141. Coefficient of variation = (Standard Deviation x 100)/Mean

(a) true (b) false (c) both (d) none

142. If mean = 5, Standard deviation = 2.6 then the coefficient of variation is

(a) 49 (b) 51 (c) 50 (d) 52

143. If median = 5, Quartile deviation = 1. 5 then the coefficient of quartile deviation is

(a) 33 (b) 35 (c) 30 (d) 20

144. A.M of 2, 6, 4, 1, 8, 5, 2 is

(a) 4 (b) 3 (c) 4 (d) none

145. Most useful among all measures of dispersion is

(a) S.D (b) Q.D (c) Mean deviation (d) none

146. For the observations 6, 4, 1, 6, 5, 10, 4, 8 Range is

(a) 10 (b) 9 (c) 8 (d) none

147. A measure of central tendency tries to estimate the

(a) central value (b) lower value (c) upper value (d) none

148. Measures of central tendency are known as

(a) differences (b) averages (c) both (d) none

149. Mean is influenced by extreme values.

(a) true (b) false (c) both (d) none

150. Mean of 6, 7, 11, 8 is

(a) 11 (b) 6 (c) 7 (d) 8

151. The sum of differences between the actual values and the arithmetic mean is

(a) 2 (b) -1 (c) 0 (d) 1

152. When the algebraic sum of deviations from the arithmetic mean is not equal to zero, the figure of arithmetic mean ———————— correct.

(a) is (b) is not (c) both (d) none

153. In the problem

No. of shirts:	30–32	33–35	36–38	39–41	42–44
No. of persons:	15	14	42	27	18

The assumed mean is

(a) 34 (b) 37 (c) 40 (d) 43

154. In the problem

Size of items:	1–3	3–8	8–15	15–26
Frequency:	5	10	16	15

The assumed mean is

(a) 20.5 (b) 2 (c) 11.5 (d) 5.5

155. The average of a series of overlapping averages, each of which is based on a certain number of item within a series is known as

(a) moving average (b) weighted average
(c) simple average (d) none

156. ———— averages is used for smoothening a time series.

(a) moving average (b) weighted average
(c) simple average (d) none

157. Pooled Mean is also called

(a) Mean (b) Geometric Mean (c) Grouped Mean (d) none

158. Half of the numbers in an ordered set have values less than the ———— and half will have values greater than the ———— .

(a) mean, median (b)median, median (c) mode, mean (d) none.

159. The median of 27, 30, 26, 44, 42, 51, 37 is

(a) 30 (b) 42 (c) 44 (d) 37

160. For an even number of values the median is the

(a) average of two middle values (b) middle value
(c) both (d) none

161. In the case of a continuous frequency distribution, the size of the ———— item indicates class interval in which the median lies.

(a) $(n-1)/2^{th}$ (b) $(n+1)/2^{th}$ (c) $n/2^{th}$ (d) none

162. The deviations from median are ———— if negative signs are ignored as compared to other measures of central tendency.

(a) minimum (b) maximum (c) same (d) none

* Question no. 155 and 156 is based on moving averages, which is not in foundation syllabus.

163. Ninth Decile lies in the class interval of the item

(a) n/9 (b) 9n/10 (c) 9n/20 (d) none item.

164. Ninety Ninth Percentile lies in the class interval of the item

(a) 99n/100 (b) 99n/10 (c) 99n/200 (d) none item.

165. ———— is the value of the variable at which the concentration of observation is the densest.

(a) mean (b) median (c) mode (d) none

166.

Height in cms:	60–62	63–65	66–68	69–71	72–74
No. of students:	15	118	142	127	18

Modal group is

(a) 66–68 (b) 69–71 (c) 63–65 (d) none

167. A distribution is said to be symmetrical when the frequency rises & falls from the highest value in the ———— proportion.

(a) unequal (b) equal (c) both (d) none

168. ———— always lies in between the arithmetic mean & mode.

(a) G.M (b) H.M (c) Median (d) none

169. Logarithm of G.M is the ———— of logarithms of the different values.

(a) weighted mean (b) simple mean (c) both (d) none

170. ———— is not much affected by fluctuations of sampling.

(a) A.M (b) G.M (c) H.M (d) none

171. The data 1, 2, 4, 8, 16 are in

(a) Arithmetic progression (b) Geometric progression

(c) Harmonic progression (d) none

172. ———— & ———— can not be calculated if any observation is zero.

(a) G.M & A.M (b) H.M & A.M (c) H.M & G. M (d) None.

173. ———— & ———— are called ratio averages.

(a) H.M & G.M (b) H. M & A.M (c) A.M & G.M (d) none

174. ———— is a good substitute to a weighted average.

(a) A.M (b) G.M (c) H.M (d) none

175. For ordering shoes of various sizes for resale, a ———— size will be more appropriate.

(a) median (b) modal (c) mean (d) none

176. ———— is called a positional measure.

(a) mean (b) mode (c) median (d) none

* Question no. 174 is not in foundation syllabus.

177. 50% of actual values will be below & 50% of will be above ————

(a) mode (b) median (c) mean (d) none

178. Extreme values have ———— effect on mode.

(a) high (b) low (c) no (d) none

179. Extreme values have ———— effect on median.

(a) high (b) low (c) no (d) none

180. Extreme values have ———— effect on A.M.

(a) greatest (b) least (c) some (d) none

181. Extreme values have ———— effect on H.M.

(a) least (b) greatest (c) medium (d) none

182. ———— is used when representation value is required & distribution is asymmetric.

(a) mode (b) mean (c) median (d) none

183. ———— is used when most frequently occurring value is required (discrete variables).

(a) mode (b) mean (c) median (d) none

184. ———— is used when rate of growth or decline required.

(a) mode (b) A.M (c) G.M (d) none

185. In finding ————, the distribution has open-end classes.

(a) median (b) mean (c) standard deviation (d) none

186. The cumulative frequency distribution is used for

(a) median (b) mode (c) mean (d) none

187. In ——— the quantities are in ratios.

(a) A.M (b) G.M (c) H.M (d) none

188. ———— is used when variability has also to be calculated.

(a) A.M (b) G.M (c) H.M (d) none

189. ———— is used when the sum of absolute deviations from the average should be least.

(a) Mean (b) Mode (c) Median (d) None

190. ———— is used when sampling variability should be least.

(a) Mode (b) Median (c) Mean (d) none

191. ———— is used when distribution pattern has to be studied at varying levels.

(a) A.M (b) Median (c) G.M (d) none

192. The average discovers

(a) uniformity in variability (b) variability in uniformity of distribution
(c) both (d) none

193. The average has relevance for

(a) homogeneous population (b) heterogeneous population
(c) both (d) none

194. The correction factor is applied in

(a) inclusive type of distribution (b) exclusive type of distribution
(c) both (d) none

195. "Mean has the least sampling variability" prove the mathematical property of mean

(a) True (b) false (c) both (d) none

196. "The sum of deviations from the mean is zero" —— is the mathematical property of mean

(a) True (b) false (c) both (d) none

197. "The mean of the two samples can be combined" — is the mathematical property of mean

(a) True (b) false (c) both (d) none

198. "Choices of assumed mean does not affect the actual mean"— prove the mathematical property of mean

(a) True (b) false (c) both (d) none

199. "In a moderately asymmetric distribution mean can be found out from the given values of median & mode"— is the mathematical property of mean

(a) True (b) false (c) both (d) none

200. The mean wages of two companies are equal. It signifies that the workers of both the companies are equally well-off.

(a) True (b) false (c) both (d) none

201. The mean wage in factory A is ₹ 6,000 whereas in factory B it is ₹ 5,500. It signifies that factory A pays more to all its workers than factory B.

(a) True (b) false (c) both (d) none

202. Mean of 0, 3, 5, 6, 7, 9, 12, 0, 2 is

(a) 4.9 (b) 5.7 (c) 5.6 (d) none

203. Median of 15, 12, 6, 13, 12, 15, 8, 9 is

(a) 13 (b) 8 (c) 12 (d) 9

204. Median of 0.3, 5, 6, 7, 9, 12, 0, 2 is

(a) 7 (b) 6 (c) 3 (d) 5

205. Mode of 0, 3, 5, 6, 7, 9, 12, 0, 2 is

(a) 6 (b) 0 (c) 3 (d) 5

206. Mode of 15, 12, 5, 13, 12, 15, 8, 8, 9, 9, 10, 15 is

(a)15 (b) 12 (c) 8 (d) 9

207. Median of 40, 50, 30, 20, 25, 35, 30, 30, 20, 30 is

(a) 25 (b) 30 (c) 35 (d) none

208. Mode of 40, 50, 30, 20, 25, 35, 30, 30, 20, 30 is

(a) 25 (b) 30 (c) 35 (d) none

209. ———————— in particular helps in finding out the variability of the data.

(a) Dispersion (b) Median (c) Mode (d) None

210. Measures of central tendency are called averages of the ———order.

(a) 1^{st} (b) 2^{nd} (c) 3^{rd} (d) none

211. Measures of dispersion are called averages of the ———order.

(a) 1^{st} (b) 2^{nd} (c) 3^{rd} (d) none

212. In measuring dispersion, it is necessary to know the amount of ———— & the degree of ————.

(a) variation, variation (b) variation, median
(c) median, variation (d) none

213. The amount of variation is designated as ———————— measure of dispersion.

(a) relative (b) absolute (c) both (d) none

214. The degree of variation is designated as ———————— measure of dispersion.

(a) relative (b) absolute (c) both (d) none

215. For purposes of comparison between two or more series with varying size or no. of items, varying central values or units of calculation, only ———————— measures can be used.

(a) absolute (b) relative (c) both (d) none

216. The relation Relative range = Absolute range/Sum of the two extremes. is

(a) True (b) false (c) both (d) none

217. The relation Absolute range = Relative range/Sum of the two extremes is

(a) True (b) false (c) both (d) none

218. In quality control ———— is used as a substitute for standard deviation.

(a) mean deviation (b) median (c) range (d) none

219. ———————— factor helps to know the value of standard deviation.

(a) Correction (b) Range (c) both (d) none

220. ———————— is extremely sensitive to the size of the sample

(a) Range (b) Mean (c) Median (d) Mode

221. As the sample size increases, ———— also tends to increase.

(a) Range (b) Mean (c) Median (d) Mode

222. As the sample size increases, range also tends to increase though not proportionately.

(a) true (b) false (c) both (d) none.

223. As the sample size increases, range also tends to

(a) decrease (b) increase (c) same (d) none

224. The dependence of range on extreme items can be avoided by adopting

(a) standard deviation (b) mean deviation (c) quartile deviation (d) none

225. Quartile deviation is called

(a) semi inter quartile range (b) quartile range (c) both (d) none

226. When 1st quartile = 20, 3rd quartile = 30, the value of quartile deviation is

(a) 7 (b) 4 (c) -5 (d) 5

227. $(Q_3 - Q_1)/(Q_3 + Q_1)$ is

(a) coefficient of Quartile Deviation (b) coefficient of Mean Deviation
(c) coefficient of Standard deviation (d) none

228. Standard deviation is denoted by

(a) σ^2 (b) σ (c) $\sqrt{\sigma}$ (d) none

229. The square of standard deviation is known as

(a) variance (b) standard deviation
(c) mean deviation (d) none

230. Mean of 25, 32, 43, 53, 62, 59, 48, 31, 24, 33 is

(a) 44 (b) 43 (c) 42 (d) 41

231. For the following frequency distribution

Class interval:	10–20	20–30	30–40	40–50	50–60	60–70
Frequency:	20	9	31	18	10	9

assumed mean can be taken as

(a) 55 (b) 45 (c) 35 (d) none

232. The value of the standard deviation does not depend upon the choice of the origin.

(a) True (b) false (c) both (d) none

233. Coefficient of standard deviation is

(a) S.D/Median (b) S.D/Mean (c) S.D/Mode (d) none

234. The value of the standard deviation will change if any one of the observations is changed.

(a). True (b) false (c) both (d) none

235. When all the values are equal then variance & standard deviation would be

(a) 2 (b) -1 (c) 1 (d) 0

236. For values lie close to the mean, the standard deviations are

(a) big (b) small (c) moderate (d) none

237. If the same amount is added to or subtracted from all the values, variance & standard deviation shall

(a) changed (b) unchanged (c) both (d) none

238. If the same amount is added to or subtracted from all the values, the mean shall increase or decrease by the ———— amount

(a) big (b) small (c) same (d) none

239. If all the values are multiplied by the same quantity, the ————— & ———— also would be multiple of the same quantity.

(a) mean, standard deviation (b) mean , median
(c) mean, mode (d) median , deviations

240. For a moderately non-symmetrical distribution, Mean deviation = 4/5 of standard deviation

(a) true (b) false (c) both (d) none

241. For a moderately non-symmetrical distribution, Quartile deviation = Standard deviation/3

(a) true (b) false (c) both (d) none

242. For a moderately non-symmetrical distribution, probable error of standard deviation = Standard deviation/3

(a) true (b) false (c) both (d) none

243. Quartile deviation = Probable error of Standard deviation.

(a) true (b) false (c) both (d) none

244. Coefficient of Mean Deviation is

(a) Mean deviation x 100/Mean or mode (b) Standard deviation x 100/Mean or median

(c) Mean deviation x 100/Mean or median (d) none

245. Coefficient of Quartile Deviation = Quartile Deviation x 100/Median

(a) true (b) false (c) both (d) none

246. Karl Pearson's measure gives

(a) coefficient of Mean Variation (b) coefficient of Standard deviation
(c) coefficient of variation (d) none

247. In ——— range has the greatest use.

(a) Time series (b) quality control (c) both (d) none

248. Mean is an absolute measure & standard deviation is based upon it. Therefore standard deviation is a relative measure.

(a) true (b) false (c) both (d) none

249. Semi-quartile range is one-fourth of the range in a normal symmetrical distribution.

(a) Yes (b) No (c) both (d) none

250. Whole frequency table is needed for the calculation of

(a) range (b) variance (c) both (d) none

251. Relative measures of dispersion make deviations in similar units comparable.

(a) true (b) false (c) both (d) none

252. Quartile deviation is based on the

(a) highest 50% (b) lowest 25%
(c) highest 25% (d) middle 50% of the item.

253. S.D is less than Mean deviation

(a) true (b) false (c) both (d) none

254. Coefficient of variation is independent of the unit of measurement.

(a) true (b) false (c) both (d) none

255. Coefficient of variation is a relative measure of

(a) mean (b) deviation (c) range (d) dispersion.

256. Coefficient of variation is equal to

(a) Standard deviation x 100 / median (b) Standard deviation x 100 / mode
(c) Standard deviation x 100 / mean (d) none

257. Coefficient of Quartile Deviation is equal to

(a) Quartile deviation x 100 / median (b) Quartile deviation x 100 / mean
(c) Quartile deviation x 100 / mode (d) none

258. If each item is reduced by 15 A.M is

(a) reduced by 15 (b) increased by 15 (c) reduced by 10 (d) none

259. If each item is reduced by 10, the range is

(a) increased by 10 (b) decreased by 10 (c) unchanged (d) none

260. If each item is reduced by 20, the standard deviation

(a) increased (b) decreased (c) unchanged (d) none

261. If the variables are increased or decreased by the same amount the standard deviation is

(a) decreased (b) increased (c) unchanged (d) none

262. If the variables are increased or decreased by the same proportion, the standard deviation changes by

(a) same proportion (b) different proportion (c) both (d) none

263. The mean of the 1st n natural no. is

(a) n/2 (b) (n-1)/2 (c) (n+1)/2 (d) none

264. If the class interval is open-end then it is difficult to find

(a) frequency (b) A.M (c) both (d) none

265. Which one is true—

(a) A.M = assumed mean + arithmetic mean of deviations of terms

(b) G.M = assumed mean + arithmetic mean of deviations of terms

(c) Both (d) none

266. If the A.M of any distribution be 25 & one term is 18. Then the deviation of 18 from A.M is

(a) 7 (b) -7 (c) 43 (d) none

267. For finding A.M in Step-deviation method, the class intervals should be of

(a) equal lengths (b) unequal lengths (c) maximum lengths (d) none

268. The sum of the squares of the deviations of the variable is ———— when taken about A.M

(a) maximum (b) zero (c) minimum (d) none

269. The A.M of 1, 3, 5, 6, x, 10 is 6 . The value of x is

(a) 10 (b) 11 (c) 12 (d) none

270. The G.M of 2 & 8 is

(a) 2 (b) 4 (c) 8 (d) none

271. (n+1)/2 th term is median if n is

(a) odd (b) even (c) both (d) none

272. For the values of a variable 5, 2, 8, 3, 7, 4, the median is

(a) 4 (b) 4.5 (c) 5 (d) none

273. The abscissa of the maximum frequency in the frequency curve is the

(a) mean (b) median (c) mode (d) none

274.

Variable:	2	3	4	5	6	7
No. of men:	5	6	8	13	7	4

Mode is

(a) 6 (b) 4 (c) 5 (d) none

275. The class having maximum frequency is called

(a) modal class (b) median class (c) mean class (d) none

276. For determination of mode, the class intervals should be

(a) overlapping (b) maximum (c) minimum (d) none

277. First Quartile lies in the class interval of the

(a) $n/2^{th}$ item (b) $n/4^{th}$ item (c) $3n/4^{th}$ item (d) $n/10^{th}$ item

278. The value of a variate that occur most often is called

(a) median (b) mean (c) mode (d) none

279. For the values of a variable 3, 1, 5, 2, 6, 8, 4 the median is

(a) 3 (b) 5 (c) 4 (d) none

280. If $y = 5x - 20$ & $\bar{x} = 30$ then the value of $\bar{y}$ is

(a) 130 (b) 140 (c) 30 (d) none

281. If $y = 3x - 100$ and $\bar{x} = 50$ then the value of $\bar{y}$ is

(a) 60 (b) 30 (c) 100 (d) 50

282. The median of the numbers 11, 10, 12, 13, 9 is

(a) 12.5 (b) 12 (c) 10.5 (d) 11

283. The mode of the numbers 7, 7, 7, 9, 10, 11, 11, 11, 12 is

(a) 11 (b) 12 (c) 7 (d) 7 & 11

284. In a symmetrical distribution when the 3^{rd} quartile plus 1^{st} quartile is halved, the value would give

(a) mean (b) mode (c) median (d) none

285. In Zoology ———————— is used.

(a) median (b) mean (c) mode (d) none

286. For calculation of Speed & Velocity

(a) G.M (b) A.M (c) H.M (d) none is used.

287. The S.D is always taken from

(a) median (b) mode (c) mean (d) none

288. Coefficient of Standard deviation is equal to

(a) S.D/A.M (b) A.M/S.D (c) S.D/GM (d) none

289. The distribution, for which the coefficient of variation is less, is ——— consistent.

(a) less (b) more (c) moderate (d) none

ANSWERS

1. (b)	2. (a)	3. (c)	4. (a)	5. (b)
6. (a)	7. (d)	8. (c)	9. (b)	10. (a)
11. (a)	12. (a)	13. (b)	14. (d)	15. (a)
16. (d)	17. (b)	18. (a)	19. (b)	20. (a)
21. (b)	22. (d)	23. (a)	24. (c)	25. (b)
26. (a)	27. (a)	28. (b)	29. (b)	30. (a)
31. (c)	32. (a)	33. (b)	34. (a)	35. (a)
36. (a)	37. (a)	38. (a)	39. (a)	40. (a)
41. (d)	42. (a)	43. (a)	44. (b)	45. (a)
46. (b)	47. (a)	48. (a)	49. (d)	50. (a)
51. (a)	52. (d)	53. (a)	54. (d)	55. (b)
56. (c)	57. (a)	58. (c)	59. (a)	60. (b)
61. (b)	62. (d)	63. (a)	64. (b)	65. (b)
66. (c)	67. (a)	68. (a)	69. (a)	70. (c)
71. (c)	72. (c)	73. (a)	74. (a)	75. (d)
76. (a)	77. (b)	78. (b)	79. (a)	80. (c)
81. (b)	82. (a)	83. (b)	84. (b)	85. (a)
86. (c)	87. (c)	88. (a)	89. (b)	90. (a)
91. (b)	92. (d)	93. (a)	94. (d)	95. (c)
96. (c)	97. (a)	98. (a)	99. (c)	100. (b)
101. (a)	102. (b)	103. (c)	104. (a)	105. (a)
106. (b)	107. (a)	108. (d)	109. (c)	110. (a)
111. (b)	112. (c)	113. (b)	114. (b)	115. (a)
116. (b)	117. (c)	118. (c)	119. (b)	120. (a)
121. (b)	122. (b)	123. (a)	124. (b)	125. (b)
126. (c)	127. (d)	128. (d)	129. (a)	130. (c)
131. (b)	132. (a)	133. (a)	134. (a)	135. (b)
136. (c)	137. (a)	138. (d)	139. (a)	140. (b)
141. (a)	142. (d)	143. (c)	144. (c)	145. (a)
146. (b)	147. (a)	148. (b)	149. (a)	150. (d)
151. (c)	152. (b)	153. (b)	154. (c)	155. (a)

156. (a)	**157.** (c)	**158.** (b)	**159.** (d)	**160.** (a)
161. (c)	**162.** (a)	**163.** (b)	**164.** (a)	**165.** (c)
166. (a)	**167.** (b)	**168.** (c)	**169.** (a)	**170.** (b)
171. (b)	**172.** (c)	**173.** (c)	**174.** (c)	**175.** (b)
176. (c)	**177.** (b)	**178.** (c)	**179.** (c)	**180.** (a)
181. (a)	**182.** (b)	**183.** (b)	**184.** (c)	**185.** (a)
186. (a)	**187.** (b)	**188.** (a)	**189.** (c)	**190.** (c)
191. (b)	**192.** (a)	**193.** (b)	**194.** (a)	**195.** (b)
196. (a)	**197.** (a)	**198.** (a)	**199.** (b)	**200.** (b)
201. (b)	**202.** (a)	**203.** (c)	**204.** (d)	**205.** (b)
206. (a)	**207.** (b)	**208.** (b)	**209.** (a)	**210.** (a)
211. (b)	**212.** (a)	**213.** (b)	**214.** (a)	**215.** (b)
216. (a)	**217.** (b)	**218.** (c)	**219.** (a)	**220.** (a)
221. (a)	**222.** (a)	**223.** (b)	**224.** (c)	**225.** (a)
226. (d)	**227.** (a)	**228.** (b)	**229.** (a)	**230.** (d)
231. (c)	**232.** (a)	**233.** (b)	**234.** (a)	**235.** (d)
236. (b)	**237.** (b)	**238.** (c)	**239.** (a)	**240.** (a)
241. (b)	**242.** (b)	**243.** (a)	**244.** (d)	**245.** (a)
246. (c)	**247.** (b)	**248.** (b)	**249.** (a)	**250.** (c)
251. (a)	**252.** (d)	**253.** (b)	**254.** (a)	**255.** (d)
256. (c)	**257.** (a)	**258.** (a)	**259.** (c)	**260.** (c)
261. (c)	**262.** (a)	**263.** (c)	**264.** (b)	**265.** (a)
266. (b)	**267.** (a)	**268.** (c)	**269.** (b)	**270.** (b)
271. (a)	**272.** (b)	**273.** (c)	**274.** (c)	**275.** (a)
276. (a)	**277.** (b)	**278.** (c)	**279.** (c)	**280.** (a)
281. (d)	**282.** (d)	**283.** (d)	**284.** (c)	**285.** (c)
286. (c)	**287.** (c)	**288.** (a)	**289.** (b)	

NOTES

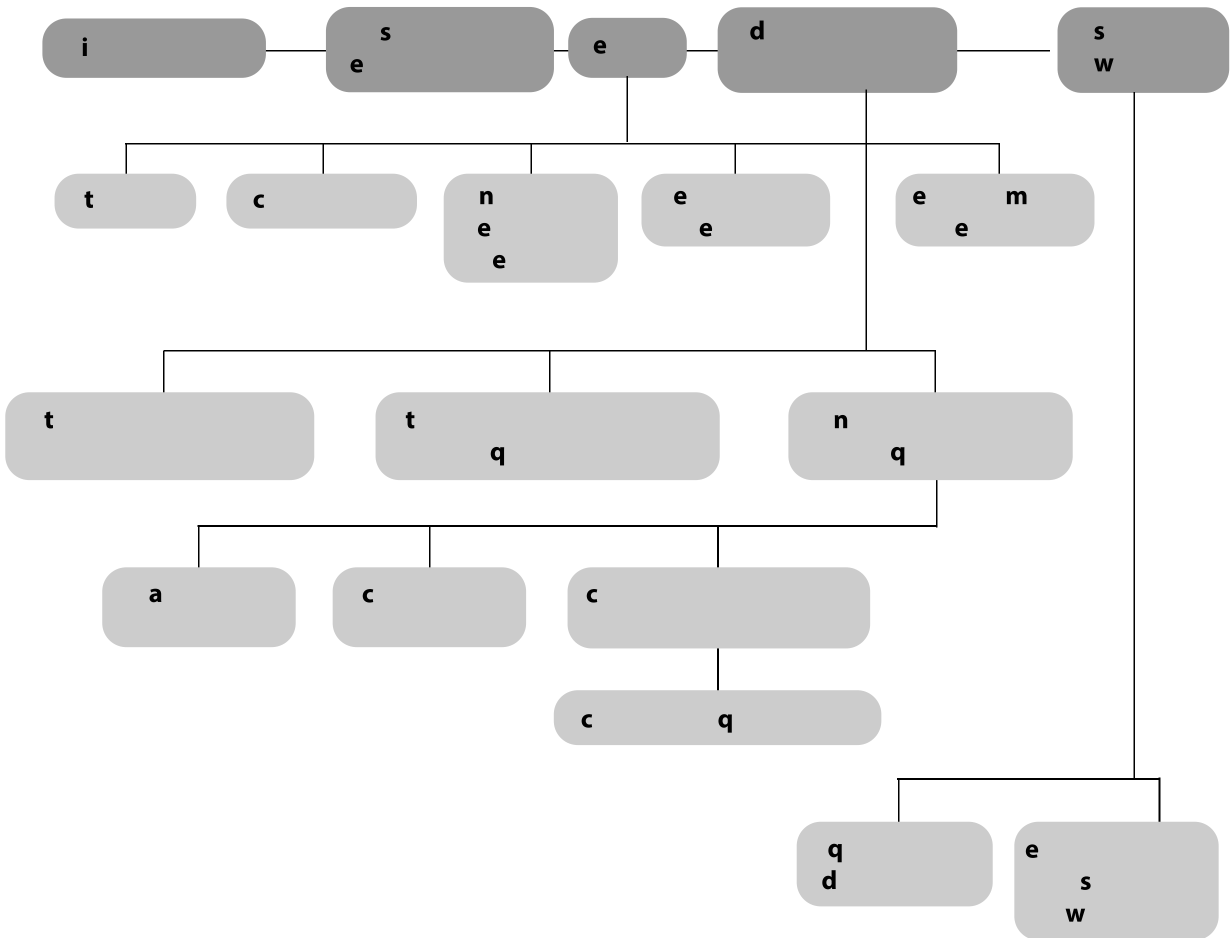
CHAPTER **BF**

PROBABILITY

LEARNING OBJECTIVES

Concept of probability is used in accounting and finance to understand the likelihood of occurrence or non-occurrence of a variable. It helps in developing financial forecasting in which you need to develop expertise at an advanced stage of chartered accountancy course.

CHAPTER OVERVIEW



15.1 INTRODUCTION

The terms 'Probably' 'in all likelihood', 'chance', 'odds in favour', 'odds against' are too familiar nowadays and they have their origin in a branch of Mathematics, known as Probability. In recent time, probability has developed itself into a full-fledged subject and become an integral part of statistics. The theories of Testing Hypothesis and Estimation are based on probability.

It is rather surprising to know that the first application of probability was made by a group of mathematicians in Europe about three hundreds years back to enhance their chances of winning in different games of gambling. Later on, the theory of probability was developed by Abraham De Moicere and Piere-Simon De Laplace of France, Reverend Thomas Bayes and R. A. Fisher of England, Chebyshev, Morkov, Khinchin, Kolmogorov of Russia and many other noted mathematicians as well as statisticians.

Two broad divisions of probability are Subjective Probability and Objective Probability. Subjective Probability is basically dependent on personal judgement and experience and, as such, it may be influenced by the personal belief, attitude and bias of the person applying it. However in the field of uncertainty, this would be quite helpful and it is being applied in the area of decision making management. This Subjective Probability is beyond the scope of our present discussion. We are going to discuss Objective Probability in the remaining sections.

15.2 RANDOM EXPERIMENT

In order to develop a sound knowledge about probability, it is necessary to get ourselves familiar with a few terms.

Experiment: An experiment may be described as a performance that produces certain results.

Random Experiment: An experiment is defined to be random if the results of the experiment depend on chance only. For example if a coin is tossed, then we get two outcomes—Head (H) and Tail (T). It is impossible to say in advance whether a Head or a Tail would turn up when we toss the coin once. Thus, tossing a coin is an example of a random experiment. Similarly, rolling a dice (or any number of dice), drawing items from a box containing both defective and non—defective items, drawing cards from a pack of well shuffled fifty two cards etc. are all random experiments.

Events: The results or outcomes of a random experiment are known as events. Sometimes events may be combination of outcomes. The events are of two types:

(i) Simple or Elementary,

(ii) Composite or Compound.

An event is known to be simple if it cannot be decomposed into further events. Tossing a coin once provides us two simple events namely Head and Tail. On the other hand, a composite event is one that can be decomposed into two or more events. Getting a head when a coin is tossed twice is an example of composite event as it can be split into the events HT and TH which are both elementary events.

Mutually Exclusive Events or Incompatible Events: A set of events A1, A2, A3, …… is known to be mutually exclusive if not more than one of them can occur simultaneously. Thus occurrence of one such event implies the non-occurrence of the other events of the set. Once a coin is tossed, we get two mutually exclusive events Head and Tail.

Exhaustive Events: The events A1, A2, A3, ………… are known to form an exhaustive set if one of these events must necessarily occur. As an example, the two events Head and Tail, when a coin is tossed once, are exhaustive as no other event except these two can occur.

Equally Likely Events or Mutually Symmetric Events or Equi-Probable Events: The events of a random experiment are known to be equally likely when all necessary evidence are taken into account, no event is expected to occur more frequently as compared to the other events of the set of events. The two events Head and Tail when a coin is tossed is an example of a pair of equally likely events because there is no reason to assume that Head (or Tail) would occur more frequently as compared to Tail (or Head).

15.3 CLASSICAL DEFINITION OF PROBABILITY OR A PRIORI DEFINITION

Let us consider a random experiment that result in n finite elementary events, which are assumed to be equally likely. We next assume that out of these n events, n_A $(\leq n)$ events are favourable to an event A. Then the probability of occurrence of the event A is defined as the ratio of the number of events favourable to A to the total number of events. Denoting this by P(A), we have

$$P(A) = \frac{n_A}{n} = \frac{\text{No. of equally likely events favourable to A}}{\text{Total no. of equally likely events}} \quad \ldots\ldots\ldots\ldots\ldots (15.1)$$

However if instead of considering all elementary events, we focus our attention to only those composite events, which are mutually exclusive, exhaustive and equally likely and if $m(\leq n)$ denotes such events and is furthermore $m_A(\leq n_A)$ denotes the no. of mutually exclusive, exhaustive and equally likely events favourable to A, then we have

$$P(A) = \frac{m_A}{m} = \frac{\text{No. of mutually exclusive, exhaustive and equally likely events favourable to A}}{\text{Total no. of mutually exclusive, exhaustive and equally likely events}}$$

……………… (15.2)

For this definition of probability, we are indebted to Bernoulli and Laplace. This definition is also termed as a priori definition because probability of the event A is defined on the basis of prior knowledge.

This classical definition of probability has the following demerits or limitations:

(i) It is applicable only when the total no. of events is finite.

(ii) It can be used only when the events are equally likely or equi-probable. This assumption is made well before the experiment is performed.

(iii) This definition has only a limited field of application like coin tossing, dice throwing, drawing cards etc. where the possible events are known well in advance. In the field of uncertainty or where no prior knowledge is provided, this definition is inapplicable.

In connection with classical definition of probability, we may note the following points:

(a) The probability of an event lies between 0 and 1, both inclusive.

i.e. $0 \leq P(A) \leq 1$ .... (15.3)

When P(A) = 0, A is known to be an impossible event and when P(A) = 1, A is known to be a sure event.

(b) Non-occurrence of event A is denoted by A′ or A^C or $\bar{A}$ and it is known as complimentary event of A. The event A along with its complimentary A′ forms a set of mutually exclusive and exhaustive events.

i.e. P(A) + P (A′) = 1

$$\Rightarrow \quad P(A') = 1 - P(A)$$

$$1 - \frac{m_A}{m}$$

$$= \frac{m - m_A}{m} \qquad \text{............... (15.4)}$$

(c) The ratio of no. of favourable events to the no. of unfavourable events is known as odds in favour of the event A and its inverse ratio is known as odds against the event A.

i.e. odds in favour of A $= m_A : (m - m_A)$ .... (15.5)

and odds against A $= (m - m_A) : m_A$ .... (15.6)

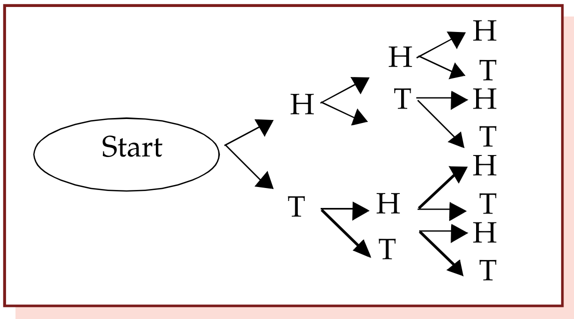
ILLUSTRATIONS:

Example 15.1: A coin is tossed three times. What is the probability of getting:

(i) 2 heads

(ii) at least 2 heads.

Solution: When a coin is tossed three times, first we need enumerate all the elementary events. This can be done using 'Tree diagram' as shown below:



Hence the elementary events are

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

Thus the number of elementary events (n) is 8.

(i) Out of these 8 outcomes, 2 heads occur in three cases namely HHT, HTH and THH. If we denote the occurrence of 2 heads by the event A and if assume that the coin as well as performer of the experiment is unbiased then this assumption ensures that all the eight elementary events are equally likely. Then by the classical definition of probability, we have

$$P(A) = \frac{n_A}{n}$$

$$= \frac{3}{8}$$

$$= 0.375$$

(ii) Let B denote occurrence of at least 2 heads i.e. 2 heads or 3 heads. Since 2 heads occur in 3 cases and 3 heads occur in only 1 case, B occurs in 3 + 1 or 4 cases. By the classical definition of probability,

$$P(B) = \frac{4}{8}$$

$$= 0.50$$

Example 15.2: A dice is rolled twice. What is the probability of getting a difference of 2 points?

Solution: If an experiment results in p outcomes and if the experiment is repeated q times, then the total number of outcomes is p^q. In the present case, since a dice results in 6 outcomes and the dice is rolled twice, total no. of outcomes or elementary events is 6^2 or 36. We assume that the dice is unbiased which ensures that all these 36 elementary events are equally likely. Now a difference of 2 points in the uppermost faces of the dice thrown twice can occur in the following cases:

1st Throw	2nd Throw	Difference
6	4	2
5	3	2
4	2	2
3	1	2
1	3	2
2	4	2
3	5	2
4	6	2

Thus denoting the event of getting a difference of 2 points by A, we find that the no. of outcomes favourable to A, from the above table, is 8. By classical definition of probability, we get

$$P(A) = \frac{8}{36}$$

$$= \frac{2}{9}$$

Example 15.3: Two dice are thrown simultaneously. Find the probability that the sum of points on the two dice would be 7 or more.

Solution: If two dice are thrown then, as explained in the last problem, total no. of elementary events is 6^2 or 36. Now a total of 7 or more i.e. 7 or 8 or 9 or 10 or 11 or 12 can occur only in the following combinations:

SUM = 7: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)

SUM = 8: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)

SUM = 9: (3, 6), (4, 5), (5, 4), (6, 3)

SUM = 10: (4, 6), (5, 5), (6, 4)

SUM = 11: (5, 6), (6, 5)

SUM = 12: (6, 6)

Thus the no. of favourable outcomes is 21. Letting A stand for getting a total of 7 points or more, we have

$$P(A) = \frac{21}{36}$$

$$= \frac{7}{12}$$

Example 15.4: What is the chance of picking a spade or an ace not of spade from a pack of 52 cards?

Solution: A pack of 52 cards contain 13 Spades, 13 Hearts, 13 Clubs and 13 Diamonds. Each of these groups of 13 cards has an ace. Hence the total number of elementary events is 52 out of which 13 + 3 or 16 are favourable to the event A representing picking a Spade or an ace not of Spade. Thus we have

$$P(A) = \frac{16}{52} = \frac{4}{13}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} = \frac{3}{52} - 0 = \frac{16}{52} = \frac{4}{13}$$

Example 15.5: Find the probability that a four digit number comprising the digits 2, 5, 6 and 7 would be divisible by 4.

Solution: Since there are four digits, all distinct, the total number of four digit numbers that can be formed without any restriction is 4! or 4 × 3 × 2 × 1 or 24. Now a four digit number would be divisible by 4 if the number formed by the last two digits is divisible by 4. This could happen when the four digit number ends with 52 or 56 or 72 or 76. If we fix the last two digits by 52, and then the 1st two places of the four digit number can be filled up using the remaining 2 digits in 2! or 2 ways. Thus there are 2 four digit numbers that end with 52. Proceeding in this manner, we find that the number of four digit numbers that are divisible by 4 is 4 × 2 or 8. If (A) denotes the event that any four digit number using the given digits would be divisible by 4, then we have

$$P(A) = \frac{8}{24}$$

$$= \frac{1}{3}$$

Example 15.6: A committee of 7 members is to be formed from a group comprising 8 gentlemen and 5 ladies. What is the probability that the committee would comprise:

(a) 2 ladies,

(b) at least 2 ladies.

Solution: Since there are altogether 8 + 5 or 13 persons, a committee comprising 7 members can be formed in

$$ {}^{13}C_7 \quad \text{or} \quad \frac{13!}{7!6!} \quad \text{or} \quad \frac{13\times12\times11\times10\times9\times8\times7!}{7!\times6\times5\times4\times3\times2\times1} $$

or $11 \times 12 \times 13$ ways.

(a) When the committee is formed taking 2 ladies out of 5 ladies, the remaining (7–2) or 5 committee members are to be selected from 8 gentlemen. Now 2 out of 5 ladies can be selected in ${}^{5}C_2$ ways and 5 out of 8 gentlemen can be selected in ${}^{8}C_5$ ways. Thus if A denotes the event of having the committee with 2 ladies, then A can occur in ${}^{5}C_2 \times {}^{8}C_5$ or

$$ \frac{5\times4}{2\times1}\times\frac{8\times7\times6}{3\times2} \text{ or } 10 \times 56 \text{ ways.} $$

Thus $P(A) = \frac{10\times56}{11\times12\times13}$

$$ = \frac{140}{429} $$

(b) Since the minimum number of ladies is 2, we can have the following combinations:

Population:	5L		8G
Sample:	2L	+	5G
or	3L	+	4G
or	4L	+	3G
or	5L	+	2G

Thus if B denotes the event of having at least two ladies in the committee, then B can occur in

$$ {}^{5}C_2 \times {}^{8}C_5 + {}^{5}C_3 \times {}^{8}C_4 + {}^{5}C_4 \times {}^{8}C_3 + {}^{5}C_5 \times {}^{8}C_2 $$

i.e. 1568 ways.

Hence $P(B) = \frac{1568}{11\times12\times13}$

$$ = \frac{392}{429} $$

15.4 RELATIVE FREQUENCY DEFINITION OF PROBABILITY

Owing to the limitations of the classical definition of probability, there are cases when we consider the statistical definition of probability based on the concept of relative frequency. This definition of probability was first developed by the British mathematicians in connection with the survival probability of a group of people.

Let us consider a random experiment repeated a very good number of times, say n, under an identical set of conditions. We next assume that an event A occurs f_A times. Then the limiting value of the ratio of f_A to n as n tends to infinity is defined as the probability of A.

i.e. $P(A) = \lim_{n \to \infty} \frac{F_A}{n}$ ………………. (15.7)

This statistical definition is applicable if the above limit exists and tends to a finite value.

Example 15.7: The following data relate to the distribution of wages of a group of workers:

Wages in ₹:	50-60	60-70	70-80	80-90	90-100	100-110	110-120
No. of workers:	15	23	36	42	17	12	5

If a worker is selected at random from the entire group of workers, what is the probability that

(a) his wage would be less than ₹ 50?

(b) his wage would be less than ₹ 80?

(c) his wage would be more than ₹ 100?

(d) his wages would be between ₹ 70 and ₹ 100?

Solution: As there are altogether 150 workers, n = 150.

(a) Since there is no worker with wage less than ₹ 50, the probability that the wage of a randomly selected worker would be less than ₹ 50 is $P(A) = \frac{0}{150} = 0$

(b) Since there are (15+23+36) or 74 worker having wages less than ₹ 80 out of a group of 150 workers, the probability that the wage of a worker, selected at random from the group, would be less than ₹ 80 is

$$P(B) = \frac{74}{150} = \frac{37}{75}$$

(c) There are (12+5) or 17 workers with wages more than ₹ 100. Thus the probability of finding a worker, selected at random, with wage more than ₹ 100 is

$$P(C) = \frac{17}{150}$$

(d) There are (36+42+17) or 95 workers with wages in between ₹ 70 and ₹ 100. Thus

$$P(D) = \frac{95}{150} = \frac{19}{30}$$

15.5 OPERATIONS ON EVENTS-SET THEORETIC APPROACH TO PROBABILITY

Applying the concept of set theory, we can give a new dimension to the classical definition of probability. A sample space may be defined as a non-empty set containing all the elementary events of a random experiment as sample points. A sample space is denoted by S or Ω. An event A may be defined as a non-empty subset of S. This is shown in Figure 15.1

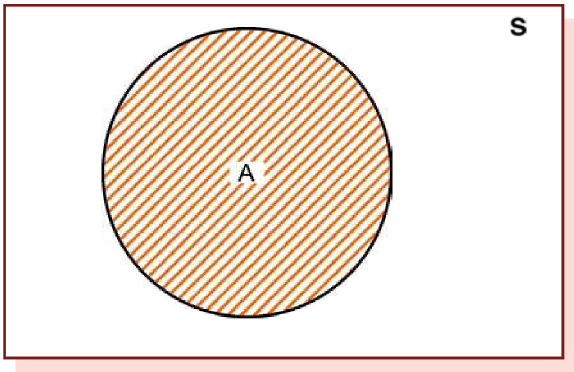


Figure 15.1

Showing an event A and the sample space S

As for example, if a dice is rolled once than the sample space is given by

S = {1, 2, 3, 4, 5, 6}.

Next, if we define the events A, B and C such that

A = {x: x is an even no. of points in S}

B = {x: x is an odd no. of points in S}

C = {x: x is a multiple of 3 points in S}

Then, it is quite obvious that

A = {2, 4, 6}, B = {1, 3, 5} and C = {3, 6}.

The classical definition of probability may be defined in the following way.

Let us consider a finite sample space S i.e. a sample space with a finite no. of sample points, n (S). We assume that all these sample points are equally likely. If an event A which is a subset of S, contains n (A) sample points, then the probability of A is defined as the ratio of the number of sample points in A to the total number of sample points in S. i.e.

$$P(A) = \frac{n(A)}{n(S)} \qquad \text{.....................} \ (15.8)$$

Union and Intersection of Two Events

Union of two events A and B is defined as a set of events containing all the sample points of event A or event B or both the events. This is shown in Figure 15.2 and

we have $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

where x denotes the sample points.

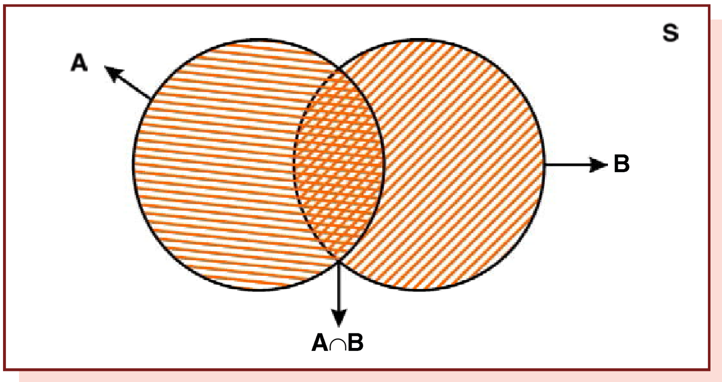


Figure 15.2

Showing the union of two events A and B and also their intersection

In the above example, we have $A \cup C = \{2, 3, 4, 6\}$

and $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

The intersection of two events A and B may be defined as the set containing all the sample points that are common to both the events A and B. This is shown in Figure 15.2. we have

$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

In the above example, $A \cap B = \phi$

$A \cap C = \{6\}$

Since the intersection of the events A and B is a null set ϕ, it is obvious that A and B are mutually exclusive events as they cannot occur simultaneously.

The difference of two events A and B, to be denoted by A – B, may be defined as the set of sample points present in set A but not in B. i.e.

$A - B = \{x : x \in A \text{ and } x \notin B\}$.

Similarly, B – A = {x:$x \in$ B and $x \notin$ A}.

In the above examples,

A – B = ϕ

And A – C = {2, 4}.

This is shown in Figure 15.3.

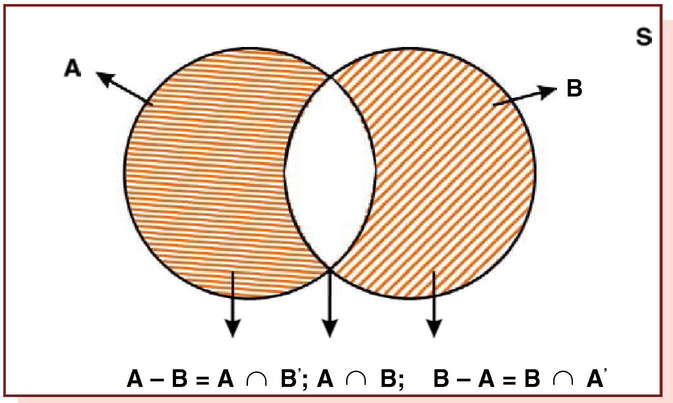


Figure 15.3

Showing (A – B) and (B – A)

The complement of an event A may be defined as the difference between the sample space S and the event A. i.e.

A′= {x: x $\in$ S and x $\notin$ A}.

In the above example A′= S – A

= {1, 3, 5}

Figure 15.4 depicts A′

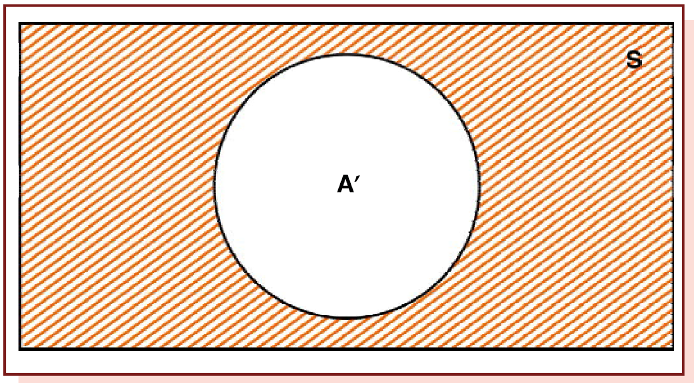


Figure 15.4

Showing A′

Now we are in a position to redefine some of the terms we have already discussed in this section.

Two events A and B are mutually exclusive if $P(A \cap B) = 0$ or more precisely, ...(15.9)

$P(A \cup B) = P(A) + P(B)$ ....(15.10)

Similarly three events A, B and C are mutually exclusive if

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ ....(15.11)

Two events A and B are exhaustive if

$P(A \cup B) = 1$ ....(15.12)

Similarly three events A, B and C are exhaustive if

$P(A \cup B \cup C) = 1$ .... (15.13)

Three events A, B and C are equally likely if

$P(A) = P(B) = P(C)$ .... (15.14)

Example 15.8: Three events A, B and C are mutually exclusive, exhaustive and equally likely. What is the probability of the complementary event of A?

Solution: Since A, B and C are mutually exclusive, we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ .... (1)

Since they are exhaustive, $P(A \cup B \cup C) = 1$ .... (2)

Since they are also equally likely, $P(A) = P(B) = P(C) = K$, Say ....(3)

Combining equations (1), (2) and (3), we have

$1 = K + K + K$

$\Rightarrow K = 1/3$

Thus $P(A) = P(B) = P(C) = 1/3$

Hence $P(A') = 1 - 1/3 = 2/3$

15.6 AXIOMATIC OR MODERN DEFINITION OF PROBABILITY

Let us consider a sample space S in connection with a random experiment and let A be an event defined on the sample space S i.e. $A \subseteq S$. Then a real valued function P defined on S is known as a probability measure and P(A) is defined as the probability of A if P satisfies the following axioms:

(i) $P(A) \geq 0$ for every $A \subseteq S$ (subset) .... (15.15)

(ii) $P(S) = 1$ .... (15.16)

(iii) For any sequence of mutually exclusive events $A_1, A_2, A_3,..$

$P(A_1 \cup A_2 \cup A_3 \cup ....) = P(A_1) + P(A_2) + P(A_3) +$ .... (15.17)

15.7 ADDITION THEOREMS OR THEOREMS ON TOTAL PROBABILITY

Theorem 1 For any two mutually exclusive events A and B, the probability that either A or B occurs is given by the sum of individual probabilities of A and B.

i.e. $P(A \cup B)$

or $P(A + B) = P(A) + P(B)$ .... (15.18)

or P(A or B) whenever A and B are mutually exclusive

This is illustrated in the following example.

Example 15.9: A number is selected from the first 25 natural numbers. What is the probability that it would be divisible by 4 or 7?

Solution: Let A be the event that the number selected would be divisible by 4 and B, the event that the selected number would be divisible by 7. Then AUB denotes the event that the number would be divisible by 4 or 7. Next we note that A = {4, 8, 12, 16, 20, 24} and B = {7, 14, 21} whereas S = {1, 2, 3, ……... 25}. Since $A \cap B = \phi$, the two events A and B are mutually exclusive and as such we have

$$P(A \cup B) = P(A) + P(A) \quad \text{.........} \quad (1)$$

Since $P(A) = \frac{n(A)}{n(S)} = \frac{6}{25}$

and $P(B) = \frac{n(B)}{n(S)} = \frac{3}{25}$

Thus from (1), we have

$$P(A \cup B) = \frac{6}{25} + \frac{3}{25}$$

$$= \frac{9}{25}$$

Hence the probability that the selected number would be divisible by 4 or 7 is 9/25 or 0.36

Example 15.10: A coin is tossed thrice. What is the probability of getting 2 or more heads?

Solution: If a coin is tossed three times, then we have the following sample space.

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} 2 or more heads imply 2 or 3 heads.

If A and B denote the events of occurrence of 2 and 3 heads respectively, then we find that

A = {HHT, HTH, THH} and B = {HHH}

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$

and $P(B) = \frac{n(B)}{n(S)} = \frac{1}{8}$

As A and B are mutually exclusive, the probability of getting 2 or more heads is

$P(A \cup B) = P(A) + P(B)$

$= \frac{3}{8} + \frac{1}{8}$

$= 0.50$

Theorem 2 For any $K(\geq 2)$ mutually exclusive events $A_1, A_2, A_3 \ldots, A_K$ the probability that at least one of them occurs is given by the sum of the individual probabilities of the K events.

i.e. $P(A_1 \cup A_2 \cup \ldots \cup A_K) = P(A_1) + P(A_2) + \ldots. P(A_K)$ .... (15.19)

Obviously, this is an extension of Theorem 1.

Theorem 3 For any two events A and B, the probability that either A or B occurs is given by the sum of individual probabilities of A and B less the probability of simultaneous occurrence of the events A and B.

i. e. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .... (15.20)

This theorem is stronger than Theorem 1 as we can derive Theorem 1 from Theorem 3 and not Theorem 3 from Theorem 1. For want of sufficient evidence, it is wiser to apply Theorem 3 for evaluating total probability of two events.

Example 15.11: A number is selected at random from the first 1000 natural numbers. What is the probability that it would be a multiple of 5 or 9?

Solution: Let A, B, $A \cup B$ and $A \cap B$ denote the events that the selected number would be a multiple of 5, 9, 5 or 9 and both 5 and 9 i.e. LCM of 5 and 9 i.e. 45 respectively.

multiple of 5 since $1000 = 5 \times 200$

multiple of $9 = 9 \times 111 + 1$

Both 5 and $9 = 45 \times 22 + 10$,

it is obvious that

$P(A) = \frac{200}{1000}, P(B) = \frac{111}{1000}, P(A \cap B) = \frac{22}{1000}$

Hence the probability that the selected number would be a multiple of 4 or 9 is given by

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{200}{1000} + \frac{111}{1000} - \frac{22}{1000}$

$= 0.29$

Example 15.12: The probability that an Accountant's job applicant has a B. Com. Degree is 0.85, that he is a CA is 0.30 and that he is both B. Com. and CA is 0.25 out of 500 applicants, how many would be B. Com. or CA?

Solution: Let the event that the applicant is a B. Com. be denoted by B and that he is a CA be denoted by C Then as given,

$P(B) = 0.85$, $P(C) = 0.30$ and $P(B \cap C) = 0.25$

The probability that an applicant is B. Com. or CA is given by

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$= 0.85 + 0.30 - 0.25$

$= 0.90$

Expected frequency = $N \times P(B \cup C)$

Expected frequency = $500 \times 0.90 = 450$

Example 15.13: If $P(A-B) = 1/5$, $P(A) = 1/3$ and $P(B) = 1/2$, what is the probability that out of the two events A and B, only B would occur?

Solution: A glance at Figure 15.3 suggests that

$P(A-B) = P(A \cap B') = P(A) - P(A \cap B)$ …………..(15.21)

And $P(B-A) = P(B \cap A') = P(B) - P(A \cap B)$ …………..(15.22)

Also (15.21) and (15.22) describe the probabilities of occurrence of the event only A and only B respectively.

As given $P(A-B) = \frac{1}{5}$

$\Rightarrow \quad P(A) - P(A \cap B) = \frac{1}{5}$

$\Rightarrow \quad \frac{1}{3} - P(A \cap B) = \frac{1}{5}$ [Since P(A) = 1/3]

$\Rightarrow \quad P(A \cap B) = \frac{2}{15}$

The probability that the event B only would occur

$= P(B-A)$

$= P(B) - P(A \cap B)$

$= \frac{1}{2} - \frac{2}{15}$ [Since $P(B) = \frac{1}{2}$]

$= \frac{11}{30}$

Theorem 4 For any three events A, B and C, the probability that at least one of the events occurs is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad \text{.......... (15.23)}$$

Following is an application of this theorem.

Example 15.14: There are three persons A, B and C having different ages. The probability that A survives another 5 years is 0.80, B survives another 5 years is 0.60 and C survives another 5 years is 0.50. The probabilities that A and B survive another 5 years is 0.46, B and C survive another 5 years is 0.32 and A and C survive another 5 years 0.48. The probability that all these three persons survive another 5 years is 0.26. Find the probability that at least one of them survives another 5 years.

Solution: As given P(A) = 0.80, P(B) = 0.60, P(C) = 0.50,

$P(A \cap B) = 0.46$, $P(B \cap C) = 0.32$, $P(A \cap C) = 0.48$ and

$P(A \cap B \cap C) = 0.26$

The probability that at least one of them survives another 5 years in given by

$P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$ .... (15.23)

$= 0.80 + 0.60 + 0.50 - 0.46 - 0.32 - 0.48 + 0.26$

$= 0.90$

Expected Frequency = N X $P(A \cup B \cup C)$
=500 X 0.90= 450

15.8 CONDITIONAL PROBABILITY AND COMPOUND THEOREM OF PROBABILITY

Compound Probability or Joint Probability

The probability of an event, discussed so far, is technically known as unconditional or marginal probability. However, there are situations that demand the probability of occurrence of more than one event. The probability of occurrence of two events A and B simultaneously is known as the Compound Probability or Joint Probability of the events A and B and is denoted by $P(A \cap B)$. In a similar manner, the probability of simultaneous occurrence of K events $A_1, A_2, \ldots. A_k$, is denoted by $P(A_1 \cap A_2 \cap \ldots. \cap A_k)$.

In case of compound probability of 2 events A and B, we may face two different situations. In the first case, if the occurrence of one event, say B, is influenced by the occurrence of another event A, then the two events A and B are known as dependent events. We use the notation P(B/A), to be read as 'probability of the event B given that the event A has already occurred (or 'the conditional probability of B given A) to suggest that another event B will happen if and only if the first event A has already happened. This is given by

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} \qquad \text{........ (15.24)}$$

Provided P(A) > 0 i.e. A is not an impossible event.

Similarly, $P(A/B) = \frac{P(A \cap B)}{P(B)}$ .... (15.25)

if P(B) > 0.

As an example if a box contains 5 red and 8 white balls and two successive draws of 2 balls are made from it without replacement then the probability of the event 'the second draw would result in 2 white balls given that the first draw has resulted in 2 Red balls' is an example of conditional probability since the drawings are made without replacement, the composition of the balls in the box changes and the occurrence of 2 white balls in the second draw (B_2) is dependent on the outcome of the first draw (R_2). This event may b denoted by

$P(B_2/R_2)$.

In the second scenario, if the occurrence of the second event B is not influenced by the occurrence of the first event A, then B is known to be independent of A. It also follows that in this case, A is also independent of B and A and B are known as mutually independent or just independent. In this case, we have

$P(B/A) = P(B)$ .... (15.26)

and also $P(A/B) = P(A)$ .... (15.27)

There by implying, $P(A \cap B) = P(A) \times P(B)$ .... (15.28)

[From (15.24) or (15.25)]

In the above example, if the balls are drawn with replacement, then the two events B_2 and R_2 are independent and we have

$P(B_2 / R_2) = P(B_2)$

(15.28) is the necessary and sufficient condition for the independence of two events. In a similar manner, three events A, B and C are known as independent if the following conditions hold :

$P(A \cap B) = P(A) \times P(B)$

$P(A \cap C) = P(A) \times P(C)$

$P(B \cap C) = P(B) \times P(C)$

$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$ .... (15.29)

It may be further noted that if two events A and B are independent, then the following pairs of events are also independent:

(i) A and B′

(ii) A′ and B

(iii) A′ and B′ .... (15.30)

Theorems of Compound Probability

Theorem 5 For any two events A and B, the probability that A and B occur simultaneously is given by the product of the unconditional probability of A and the conditional probability of B given that A has already occurred

i.e. $P(A \cap B) = P(A) \times P(B/A)$ Provided $P(A) > 0$ ……………(15.31)

Theorem 6 For any three events A, B and C, the probability that they occur jointly is given by

$P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/(A \cap B))$ Provided $P(A \cap B) > 0$ … (15.32)

In the event of independence of the events

(15.31) and (15.32) are reduced to

$P(A \cap B) = P(A) \times P(B)$

and $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$

which we have already discussed.

Example 15.15: Rupesh is known to hit a target in 5 out of 9 shots whereas David is known to hit the same target in 6 out of 11 shots. What is the probability that the target would be hit once they both try?

Solution: Let A denote the event that Rupesh hits the target and B, the event that David hits the target. Then as given,

$P(A) = \frac{5}{9}$, $P(B) = \frac{6}{11}$

and $P(A \cap B) = P(A) \times P(B)$

$= \frac{5}{9} \times \frac{6}{11}$

$= \frac{10}{33}$ (as A and B are independent)

The probability that the target would be hit is given by

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{5}{9} + \frac{6}{11} - \frac{10}{33}$

$= \frac{79}{99}$

Alternately $P(A \cup B) = 1 - P(A \cup B)'$

$= 1 - P(A' \cap B')$(by De-Morgan's Law)

$= 1 - P(A') \times P(B')$

$= 1 - [1 - P(A)] \times [1 - P(B)]$ (by 15.30)

$$= 1 - \left(1 - \frac{5}{9}\right) \times \left(1 - \frac{6}{11}\right)$$

$$= 1 - \left(\frac{4}{9} \times \frac{5}{11}\right)$$

$$= \frac{79}{99}$$

Example 15.16: A pair of dice is thrown together and the sum of points of the two dice is noted to be 10. What is the probability that one of the two dice has shown the point 4?

Solution: Let A denote the event of getting 4 points on one of the two dice and B denote the event of getting a total of 10 points on the two dice. Then we have

$$P(A) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

and $P(A \cap B) = \frac{2}{36}$

[Since a total of 10 points may result in (4, 6) or (5, 5) or (6, 4) and two of these combinations contain 4]

Thus $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$$= \frac{2/36}{1/12}$$

$$= \frac{2}{3}$$

Alternately The sample space for getting a total of 10 points when two dice are thrown simultaneously is given by

S = {(4, 6), (5, 5), (6, 4)}

Out of these 3 cases, we get 4 in 2 cases. Thus by the definition of probability, we have

$$P(B/A) = \frac{2}{3}$$

Example 15.17: In a group of 20 males and 15 females, 12 males and 8 females are service holders. What is the probability that a person selected at random from the group is a service holder given that the selected person is a male?

Solution: Let S and M stand for service holder and male respectively. We are to evaluate P (S / M).

We note that $(S \cap M)$ represents the event of both service holder and male.

Thus $P(S/M) = \dfrac{P(S \cap M)}{P(M)}$

$= \dfrac{12/35}{20/35}$

$= 0.60$

Example 15.18: In connection with a random experiment, it is found that

$P(A) = \frac{2}{3}$, $P(B)\ \frac{3}{5}$ = and $P(A \cup B) = \frac{5}{6}$

Evaluate the following probabilities:

(i) P(A/B) (ii) P(B/A) (iii) P(A′/ B) (iv) P(A/ B′) (v) P(A′/ B′)

Solution: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \frac{5}{6} = \frac{2}{3} + \frac{3}{5} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{2}{3} + \frac{3}{5} - \frac{5}{6}$$

$$= \frac{13}{30}$$

Hence (i) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{13/30}{3/5} = \dfrac{13}{18}$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{13/30}{2/3} = \dfrac{13}{20}$

(iii) $P(A'/B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{\frac{3}{5} - \frac{13}{30}}{\frac{3}{5}} = \dfrac{5}{18}$

(iv) $(A/B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{P(A) - P(A \cap B)}{1 - P(B)} = \dfrac{7}{12}$

(v) $P(A'/B') = \dfrac{P(A' \cap B')}{P(B')}$

$= \frac{P(A \cup B)'}{P(B')}$ [by De-Morgan's Law $A' \cap B' = (AUB)'$]

$= \frac{1 - P(A \cup B)}{1 - P(B)}$

$= \frac{1 - 5/6}{1 - 3/5}$

$= \frac{5}{12}$

Example 15.19: The odds in favour of an event is 2 : 3 and the odds against another event is 3 : 7. Find the probability that only one of the two events occurs.

Solution: We denote the two events by A and B respectively. Then by (15.5) and (15.6), we have

$P(A) = \frac{2}{2+3} = \frac{2}{5}$

and $P(B) = \frac{7}{7+3} = \frac{7}{10}$

As A and B are independent, $P(A \cap B) = P(A) \times P(B)$

$= \frac{2}{5} \times \frac{7}{10} = \frac{7}{25}$

Probability that either only A occurs or only B occurs

$= P(A - B) + P(B - A)$

$= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)]$

$= P(A) + P(B) - 2\,P(A \cap B)$

$= \frac{2}{5} + \frac{7}{10} - 2 \times \frac{7}{25}$

$= \frac{20 + 35 - 28}{50}$

$= \frac{27}{50}$

Example 15.20: There are three boxes with the following compositions :

Colour / Box	Blue	Red	White	Total
I	5	8	10	23
II	4	9	8	21
III	3	6	7	16

One ball is drawn from each box. What is the probability that they would be of the same colour?

Solution: Either the balls would be Blue or Red or White. Denoting Blue, Red and White balls by B, R and W respectively and the box by lower suffix, the required probability is

$$= P(B_1 \cap B_2 \cap B_3) + P(R_1 \cap R_2 \cap R_3) + P(W_1 \cap W_2 \cap W_3)$$

$$= P(B_1) \times P(B_2) \times P(B_3) + P(R_1) \times P(R_2) \times P(R_3) + P(W_1) \times P(W_2) \times P(W_3)$$

$$= \frac{5}{23} \times \frac{4}{21} \times \frac{3}{16} + \frac{8}{23} \times \frac{9}{21} \times \frac{6}{16} + \frac{10}{23} \times \frac{8}{21} \times \frac{7}{16}$$

$$= \frac{60 + 432 + 560}{7728}$$

$$= \frac{1052}{7728}$$

Example 15.21: Mr. Roy is selected for three separate posts. For the first post, there are three candidates, for the second, there are five candidates and for the third, there are 10 candidates. What is the probability that Mr. Roy would be selected?

Solution: Denoting the three posts by A, B and C respectively, we have

$$P(A) = \frac{1}{3}, P(B) = \frac{1}{5} \text{ and } P(C) = \frac{1}{10}$$

The probability that Mr. Roy would be selected (i.e. selected for at least one post).

$$= P(A \cup B \cup C)$$

$$= 1 - P[(A \cup B \cup C)']$$

$$= 1 - P(A' \cap B' \cap C')$$ (by De-Morgan's Law)

$$= 1 - P(A') \times P(B') \times P(C')$$ (As A , B and C are independent, so are their complements)

$$= 1 - \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{5}\right) \times \left(1 - \frac{1}{10}\right) = \frac{13}{25}$$

Example 15.22: The independent probabilities that the three sections of a costing department will encounter a computer error are 0.2, 0.3 and 0.1 per week respectively. What is the probability that there would be

(i) at least one computer error per week?

(ii) one and only one computer error per week?

Solution: Denoting the three sections by A, B and C respectively, the probabilities of encountering a computer error by these three sections are given by P(A) = 0.20, P(B) = 0.30 and P(C) = 0.10

(i) Probability that there would be at least one computer error per week.

= 1 – Probability of having no computer error in any at the three sections.

$= 1 - P(A' \cap B' \cap C')$

$= 1 - P(A') \times P(B') \times P(C')$ [Since A, B and C are independent]

$= 1 - (1 - 0.20) \times (1 - 0.30) \times (1 - 0.10)$

$= 0.50$

(ii) Probability of having one and only one computer error per week

$= P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C)$

$= P(A) \times P(B') \times P(C') + P(A') \times P(B) \times P(C') + P(A') \times P(B') \times P(C)$

$= 0.20 \times 0.70 \times 0.90 + 0.80 \times 0.30 \times 0.90 + 0.80 \times 0.70 \times 0.10$

$= 0.40$

Example 15.23: A lot of 10 electronic components is known to include 3 defective parts. If a sample of 4 components is selected at random from the lot, what is the probability that this sample does not contain more than one defectives?

Solution: Denoting defective component and non-defective components by D and D′ respectively, we have the following situation :

	D	D′	T
Lot	3	7	10
Sample (1)	0	4	4
(2)	1	3	4

Thus the required probability is given by

$$= ({}^{3}C_{0} \times {}^{7}C_{4} + {}^{3}C_{1} \times {}^{7}C_{3}) \,/\, {}^{10}C_{4}$$

$$= \frac{1 \times 35 + 3 \times 35}{210}$$

$$= \frac{2}{3}$$

Example 15.24: There are two urns containing 5 red and 6 white balls and 3 red and 7 white balls respectively. If two balls are drawn from the first urn without replacement and transferred to the second urn and then a draw of another two balls is made from it, what is the probability that both the balls drawn are red?

Solution: Since two balls are transferred from the first urn containing 5 red and 6 white balls to the second urn containing 3 red and 7 white balls, we are to consider the following cases :

Case A : Both the balls transferred are red. In this case, the second urn contains 5 red and 7 white balls.

Case B : The two balls transferred are of different colours. Then the second urn contains 4 red and 8 white balls.

Case C : Both the balls transferred are white. Now the second urn contains 3 red and 9 white balls.

The required probability is given by

$P(R \cap A) + P(R \cap B) + P(R \cap C)$

$= P(R/A) \times P(A) + P(R/B) \times P(B) + P(R/C) \times P(C)$

$$= \frac{{}^5C_2}{{}^{12}C_2} \times \frac{{}^5C_2}{{}^{11}C_2} + \frac{{}^4C_2}{{}^{12}C_2} \times \frac{{}^5C_1 \times {}^6C_1}{{}^{11}C_2} + \frac{{}^3C_2}{{}^{12}C_2} \times \frac{{}^6C_2}{{}^{11}C_2}$$

$$= \frac{10}{66} \times \frac{10}{55} + \frac{6}{66} \times \frac{30}{55} + \frac{3}{66} \times \frac{15}{55}$$

$$= \frac{325}{66 \times 55} = \frac{65}{726}$$

Example 15.25: If 8 balls are distributed at random among three boxes, what is the probability that the first box would contain 3 balls?

Solution: The first ball can be distributed to the 1st box or 2nd box or 3rd box i.e. it can be distributed in 3 ways. Similarly, the second ball also can be distributed in 3 ways. Thus the first two balls can be distributed in 3^2 ways. Proceeding in this way, we find that 8 balls can be distributed to 3 boxes in 3^8 ways which is the total number of elementary events.

Let A be the event that the first box contains 3 balls which implies that the remaining 5 balls must go to the remaining 2 boxes which, as we have already discussed, can be done in 2^5 ways. Since 3 balls out of 8 balls can be selected in ${}^8C_3$ ways, the event can occur in ${}^8C_3 \times 2^5$ ways, thus we have

$$P(A) = \frac{{}^8C_3 \times 2^5}{3^8}$$

$$= \frac{56 \times 32}{6561}$$

$$= \frac{1792}{6561}$$

Example 15.26: There are 3 boxes with the following composition :

Box I : 7 Red + 5 White + 4 Blue balls

Box II : 5 Red + 6 White + 3 Blue balls

Box III : 4 Red + 3 White + 2 Blue balls

One of the boxes is selected at random and a ball is drawn from it. What is the probability that the drawn ball is red?

Solution: Let A denote the event that the drawn ball is red. Since any of the 3 boxes may be drawn, we have $P(B_I) = P(B_{II}) = P(B_{III}) = \frac{1}{3}$

Also $P(R_1/B_{II})$ = probability of drawing a red ball from the first box

$$= \frac{7}{16}$$

$P(R_2 / B_{II}) = \frac{5}{14}$ and $P(R_3 / B_{III}) = \frac{4}{9}$

Thus we have

$$P(A) = P(R_1 \cap B_I) + P(R_2 \cap B_{II}) + P(R_3 \cap B_{III})$$

$$= P(R_1 / B_I) \times P(B_I) + P(R_2 / B_{II}) \times P(B_{II}) + P(R_3 / B_{III}) \times P(B_{III})$$

$$= \frac{7}{16} \times \frac{1}{3} + \frac{5}{14} \times \frac{1}{3} + \frac{4}{9} \times \frac{1}{3}$$

$$= \frac{7}{48} + \frac{5}{42} + \frac{4}{27}$$

$$= \frac{1249}{3024}$$

15.9 RANDOM VARIABLE - PROBABILITY DISTRIBUTION

A random variable or stochastic variable is a function defined on a sample space associated with a random experiment assuming any value from R and assigning a real number to each and every sample point of the random experiment. A random variable is denoted by a capital letter. For example, if a coin is tossed three times and if X denotes the number of heads, then X is a random variable. In this case, the sample space is given by

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

and we find that X = 0 if the sample point is TTT

X = 1 if the sample point is HTT, THT or TTH

X = 2 if the sample point is HHT, HTH or THH

and X = 3 if the sample point is HHH.

We can make a distinction between a discrete random variable and a continuous variable. A random variable defined on a discrete sample space is known as a discrete random variable and it can assume either only a finite number or a countably infinite number of values. The number of car accident, the number of heads etc. are examples of discrete random variables.

A continuous random variable, like height, weight etc. is a random variable defined on a continuous sample space and assuming an uncountably infinite number of values.

The probability distribution of a random variable may be defined as a statement expressing the different values taken by a random variable and the corresponding probabilities. Then if a random variable X assumes n finite values $X_1, X_2, X_3, \ldots\ldots, X_n$ with corresponding probabilities $P_1, P_2, P_3, \ldots\ldots, P_n$ such that

(i) $p_i \geq 0$ for every i …………… (15.33)

and (ii) $\sum p_i = 1$ (over all i) ………….. (15.34)

then the probability distribution of the random variable X is given by

Probability Distribution of X

X :	X_1	X_2	X_3	……X_n	Total
P :	P_1	P_2	P_3	……. P_n	1

For example, if an unbiased coin is tossed three times and if X denotes the number of heads then, as we have already discussed, X is a random variable and its probability distribution is given by

Probability Distribution of Head when a Coin is Tossed Thrice

X :	0	1	2	3	Total
P :	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

There are cases when it is possible to express the probability (P) as a function of X. In case X is a discrete variable and if such a function f(X) really exists, then f(X) is known as Probability Mass Function (PMF) of X, f(X), then, must satisfy the conditions:

(i) $f(X) \geq 0$ for every X …………… (15.35)

and (ii) $\sum_X f(X) = 1$ …………… (15.36)

Where f(X) is given by

$f(X) = P(X = x)$ …………… (15.37)

When x is a continuous random variable defined over an interval [α , β], where $\beta > \alpha$, then x can assume an infinite number of values from its interval and instead of assigning individual probability to every mass point x, we assign probabilities to interval of values. Such a function of x, provided it exists, is known as probability density function (pdf) of x. f(x) satisfies the following conditions:

(i) $f(x) \geq 0$ for $x \in [\alpha, \beta]$ .... (15.38)

(ii) $\int_{\alpha}^{\beta} f(x)dx = 1$ .... (15.39)

and the probability that x lies between two specified values a and b, where $\alpha \leq a < b \leq \beta$, is given by

$\int_{a}^{b} f(x)dx$ .... (15.40)

15.10 EXPECTED VALUE OF A RANDOM VARIABLE

Expected value or Mathematical Expectation or Expectation of a random variable may be defined as the sum of products of the different values taken by the random variable and the corresponding probabilities. Hence, if a random variable x assumes n values x_1, x_2, x_3 ...., x_n with corresponding probabilities p_1, p_2, p_3 ...., p_n, where p_i's satisy (15.33) and (15.34), then the expected value of x is given by

$\mu = E(x) = \sum p_i x_i$ .... (15.41)

Expected value of x^2 in given by

$E(x^2) = \sum p_i x_i^2$ .... (15.42)

In particular expected value of a monotonic function g (x) is given by

$E[g(x)] = \sum p_i g(x_i)$ .... (15.43)

Variance of x, to be denoted by , σ^2 is given by

$V(x) = \sigma^2 = E(x-\mu)^2$

$= E(x^2) - \mu^2$ .... (15.44)

The positive square root of variance is known as standard deviation and is denoted by σ.

If y = a + b x, for two random variables x and y and for a pair of constants a and b, then the mean i.e. expected value of y is given by

$\mu_y = a + b\ \mu_x$ .... (15.45)

and the standard deviation of y is

$\sigma_y = |b| \times \sigma_x$ .... (15.46)

When x is a discrete random variable with probability mass function f(x), then its expected value is given by

$$\mu = \sum_x x f(x) \quad \text{......... (15.47)}$$

and its variance is

$$\sigma^2 = E(x^2) - \mu^2$$

Where $E(x^2) = \sum_x x^2 f(x)$ ....(15.48)

For a continuous random variable x defined in [–∞, ∞], its expected value (i.e. mean) and variance are given by

$$E(x) = \int_{-\infty}^{\infty} x f(x)dx \quad \text{......... (15.49)}$$

and $\sigma^2 = E(x^2) - \mu^2$

where $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$ .... (15.50)

Properties of Expected Values

1. Expectation of a constant k is k

   i.e. E(k) = k for any constant k. ....(15.51)

2. Expectation of sum of two random variables is the sum of their expectations.

   i.e. E(x + y) = E(x) + E(y) for any two random variables x and y. .... (15.52)

3. Expectation of the product of a constant and a random variable is the product of the constant and the expectation of the random variable.

   i.e. E(k x) = k.E(x) for any constant k .... (15.53)

4. Expectation of the product of two random variables is the product of the expectation of the two random variables, provided the two variables are independent.

   i.e. $E(xy) = E(x) \times E(y)$ .... (15.54)

   Whenever x and y are independent.

Example 15.27: An unbiased coin is tossed three times. Find the expected value of the number of heads and also its standard deviation.

Solution: If x denotes the number of heads when an unbiased coin is tossed three times, then the probability distribution of x is given by

X :	0	1	2	3
P :	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The expected value of x is given by

$$\mu = E(x) = \sum p_i x_i$$

$$= \frac{1}{8}\times 0+\frac{3}{8}\times 1+\frac{3}{8}\times 2+\frac{1}{8}\times 3$$

$$= \frac{0+3+6+3}{8} = 1.50$$

Also $\quad E(x^2) = \sum p_i x_i^2$

$$= \frac{1}{8}\times 0^2+\frac{3}{8}\times 1^2+\frac{3}{8}\times 2^2+\frac{1}{8}\times 3^2$$

$$= \frac{0+3+12+9}{8} = 3$$

$$= \sigma^2 = E(x^2) - \mu^2$$

$$= 3 - (1.50)^2$$

$$= 0.75$$

$$\therefore SD = \sigma = 0.87$$

Example 15.28: A random variable has the following probability distribution:

X :	4	5	7	8	10
P :	0.15	0.20	0.40	0.15	0.10

Find $E[x - E(x)]^2$. Also obtain $v(3x - 4)$

Solution: The expected value of x is given by

$$E(x) = \sum p_i x_i$$

$$= 0.15 \times 4 + 0.20 \times 5 + 0.40 \times 7 + 0.15 \times 8 + 0.10 \times 10$$

$$= 6.60$$

Also, $\quad E[x - E(x)]^2 = \sum \mu_i^2 P_i$ where = $\mu_i = x_i - E(x)$

Let $y = 3x - 4 = (-4) + (3)x$. Then variance of $y = \text{var } y = b^2 \times \sigma_x^2 = 9 \times \mu_x^2$ (From 15.46)

Table 15.1

Computation of E [x – E(x)]²

x_i	p_i	$\mu_i = x_i - E(x)$	μ_i^2	$\mu_i^2 p_i$
4	0.15	–2.60	6.76	1.014
5	0.20	–1.60	2.56	0.512
7	0.40	0.40	0.16	0.064
8	0.15	1.40	1.96	0.294
10	0.10	3.40	11.56	1.156
Total	1.00	–	–	3.040

Thus $E[x - E(x)]^2 = 3.04$

As $\mu_x^2 = 3.04$, $v(y) = 9 \times 3.04 = 27.36$

Example 15.29: In a business venture, a man can make a profit of ₹ 50,000 or incur a loss of ₹ 20,000. The probabilities of making profit or incurring loss, from the past experience, are known to be 0.75 and 0.25 respectively. What is his expected profit?

Solution: If the profit is denoted by x, then we have the following probability distribution of x:

X :	₹ 50,000	₹ –20,000
P :	0.75	0.25

Thus his expected profit

$E(x) = p_1x_1 + p_2x_2$

$= 0.75 \times$ ₹ 50,000 + $0.25 \times$ (₹ – 20,000)

= ₹ 32,500

Example 15.30: A box contains 12 electric lamps of which 5 are defectives. A man selects three lamps at random. What is the expected number of defective lamps in his selection?

Solution: Let x denote the number of defective lamps x can assume the values 0, 1, 2 and 3.

P(x = 0) = Prob. of having 0 defective out of 5 defectives and 3 non defective out of 7 non defectives

$$= \frac{{}^5C_0 \text{ x } {}^7C_3}{{}^{12}C_3} = \frac{35}{220}$$

Similarly $$P(x = 1) = \frac{{}^5C_1 \text{ x } {}^7C_2}{{}^{12}C_3} = \frac{105}{220}$$

$$P(x = 2) = \frac{{}^5C_2 \times {}^7C_1}{{}^{12}C_3} = \frac{70}{220}$$

and $$P(x = 3) = \frac{{}^5C_3 \times {}^7C_0}{{}^{12}C_3} = \frac{10}{220}$$

Probability Distribution of No. of Defective Lamp

X :	0	1	2	3
P :	$\frac{35}{220}$	$\frac{105}{220}$	$\frac{70}{220}$	$\frac{10}{220}$

Thus the expected number of defectives is given by

$$\frac{35}{220} \times 0 + \frac{105}{220} \times 1 + \frac{70}{220} \times 2 + \frac{10}{220} \times 3$$

= 1.25

Example 15.31: Moidul draws 2 balls from a bag containing 3 white and 5 Red balls. He gets ₹ 500 if he draws a white ball and ₹ 200 if he draws a red ball. What is his expectation? If he is asked to pay ₹ 400 for participating in the game, would he consider it a fair game and participate?

Solution: We denote the amount by x. Then x assumes the value 2 x ₹ 500 i.e. ₹ 1000 if 2 white balls are drawn, the value ₹ 500 + ₹ 200 i.e. ₹ 700 if 1 white and 1 red balls are drawn and the value 2 x ₹ 200 i.e. ₹ 400 if 2 red balls are drawn. The respective probabilities are given by

$$P(WW) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$$

$$P(WR) = \frac{{}^3C_1 \times {}^5C_1}{{}^8C_2} = \frac{15}{28}$$

$$\text{and } P(RR) = \frac{{}^5C_2}{{}^8C_2} = \frac{10}{28}$$

Probability Distribution of x

X :	₹ 1000	₹ 700	₹ 400
P :	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

$$\text{Hence } E(x) = \frac{3}{28} \times ₹\,1000 + \frac{15}{28} \times ₹\,700 + \frac{10}{28} \times ₹\,400$$

$$= \frac{₹\,3000 + ₹\,10500 + ₹\,4000}{28}$$

= ₹ 625 > 400. Therefore the game is fair and he would participate.

Example 15.32: A number is selected at random from a set containing the first 100 natural numbers and another number is selected at random from another set containing the first 200 natural numbers. What is the expected value of the product?

Solution: We denote the number selected from the first set by x and the number selected from the second set by y. Since the selections are independent of each other, the expected value of the product is given by

$E(xy) = E(x) \times E(y)$ .... (1)

Now x can assume any value between 1 to 100 with the same probability 1/100 and as such the probability distribution of x is given by

X :	1	2	3	100
P :	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	.... $\frac{1}{100}$

Thus $E(x) = \frac{1}{100} \times 1 + \frac{1}{100} \times 2 + \frac{1}{100} \times 3 + \ldots\ldots\ldots\ldots \frac{1}{100} \times 100$

$$= \frac{1+2+3+..........+100}{100}$$

$$= \frac{100 \times 101}{2 \times 100} \qquad \left[\text{Since } 1+2+\ldots\ldots + n = \frac{n(n+1)}{2}\right]$$

$$= \frac{101}{2}$$

Similarly, $E(y) = \frac{201}{2}$

$$\therefore E(xy) = \frac{101}{2} \times \frac{201}{2} \qquad \text{[From (1)]}$$

$$= \frac{20301}{4}$$

$= 5075.25$

Example 15.33: A dice is thrown repeatedly till a 'six' appears. Write down the sample space. Also find the expected number of throws.

Solution: Let p denote the probability of getting a six and q = 1 – p, the probability of not getting a six. If the dice is unbiased then

$p = \frac{1}{6}$ and $q = \frac{5}{6}$

If a six obtained with the very first throw then the experiment ends and the probability of getting a six, as we have already seen, is p. However, if the first throw does not produce a six, the dice is thrown again and if a six appears with the second throw, the experiment ends. The probability of getting a six preceded by a non–six is qp. If the second thrown does not yield a six, we go for a third throw and if the third throw produces a six, the experiment ends and the probability of getting a Six in the third attempt is q^2p. The experiment is carried on and we get the following countably infinite sample space.

$S = \{ p, qp, q^2p, q^3p, \ldots\}$

If x denotes the number of throws necessary to produce a six, then x is a random variable with the following probability distribution :

X :	1	2	3	4 ....
P :	p	qp	q^2p	q^3p ....

Thus $E(x) = p \times 1 + qp \times 2 + q^2p \times 3 + q^3p \times 4 + \ldots\ldots$

$= p(1+ 2q + 3q^2 + 4q^3 + \ldots\ldots)$

$= p\,(1 - q)^{-2}$

$= \frac{p}{p^2}$ (as $1-q = p$)

$= \frac{1}{p}$

In case of an unbiased dice, $p = 1/6$ and $E(x) = 6$

Example 15.34: A random variable x has the following probability distribution :

X	:	0	1	2	3	4	5	6	7
P(X)	:	0	2k	3k	k	2k	k^2	$7k^2$	$2k^2+k$

Find (i) the value of k

(ii) $P(x < 3)$

(iii) $P(x \geq 4)$

(iv) $P(2 < x \leq 5)$

Solution: By virtue of (15.36), we have

$\sum P(x) = 1$

$\Rightarrow 0 + 2k + 3k + k + 2k + k^2 + 7k^2 + 2k^2 + k = 1$

$\Rightarrow$ $10k^2 + 9k - 1 = 0$

$\Rightarrow$ $(k + 1)(10k - 1) = 0$

$\Rightarrow$ $k = 1/10$ (as $k \neq -1$ by virtue of (15.36))

(i) Thus the value of k is 0.10

(ii) $P(x < 3) = P(x = 0) + P(x = 1) + P(x = 2)$

$= 0 + 2k + 3k$

$= 5k$

$= 0.50$ (as $k = 0.10$)

(iii) $P(x \geq 4) = P(x = 4) + P(x = 5) + P(x = 6) + P(x = 7)$

$= 2k + k^2 + 7k^2 + (2k^2 + k)$

$= 10k^2 + 3k$

$= 10 \text{ x } (0.10)^2 + 3 \times 0.10$

$= 0.40$

(iv) $P(2 < x \leq 5) = P(x = 3) + P(x = 4) + P(x = 5)$

$= k + 2k + k^2$

$= k^2 + 3k$

$= (0.10)^2 + 3 \times 0.10$

$= 0.31$

SUMMARY

- **Experiment:** An experiment may be described as a performance that produces certain results.
- **Random Experiment:** An experiment is defined to be random if the results of the experiment depend on chance only.
- **Events:** The results or outcomes of a random experiment are known as events. Sometimes events may be combination of outcomes. The events are of two types:

  (i) Simple or Elementary,

  (ii) Composite or Compound.
- **Mutually Exclusive Events or Incompatible Events:** A set of events A_1, A_2, A_3, …… is known to be mutually exclusive if not more than one of them can occur simultaneously
- **Exhaustive Events:** The events A_1, A_2, A_3, ………… are known to form an exhaustive set if one of these events must necessarily occur.

- **Equally Likely Events or Mutually Symmetric Events or Equi-Probable Events:** The events of a random experiment are known to be equally likely when all necessary evidence are taken into account, no event is expected to occur more frequently as compared to the other events of the set of events.
- The probability of occurrence of the event A is defined as the ratio of the number of events favourable to A to the total number of events. Denoting this by P(A), we have

$$P(A) = \frac{n_A}{n} = \frac{\text{No. of equally likely events favourable to A}}{\text{Total no. of equally likely events}}$$

(a) The probability of an event lies between 0 and 1, both inclusive.

i.e. $0 \le P(A) \le 1$

When P(A) = 0, A is known to be an impossible event and when P(A) = 1, A is known to be a sure event.

(b) Non-occurrence of event A is denoted by A′ or A^C or $\bar{A}$ and it is known as complimentary event of A. The event A along with its complimentary A′ forms a set of mutually exclusive and exhaustive events.

i.e. $P(A) + P(A') = 1$

$\Rightarrow \quad P(A') = 1 - P(A)$

$$1 - \frac{m_A}{m}$$

$$= \frac{m - m_A}{m}$$

(c) The ratio of no. of favourable events to the no. of unfavourable events is known as odds in favour of the event A and its inverse ratio is known as odds against the event A.

i.e. odds in favour of A $= m_A : (m - m_A)$

and odds against A $= (m - m_A) : m_A$

(d) For any two mutually exclusive events A and B, the probability that either A or B occurs is given by the sum of individual probabilities of A and B.

i.e. $P(A \cup B)$

or $P(A + B) = P(A) + P(B)$

(e) For any K(≥ 2) mutually exclusive events $A_1, A_2, A_3 \ldots, A_K$ the probability that at least one of them occurs is given by the sum of the individual probabilities of the K events.

i.e. $P(A_1 \cup A_2 \cup \ldots \cup A_K) = P(A_1) + P(A_2) + \ldots . P(A_K)$

(f) For any two events A and B, the probability that either A or B occurs is given by the sum of individual probabilities of A and B less the probability of simultaneous occurrence of the events A and B.

i.e. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(g) For any three events A, B and C, the probability that at least one of the events occurs is given by

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

(h) For any two events A and B, the probability that A and B occur simultaneously is given by the product of the unconditional probability of A and the conditional probability of B given that A has already occurred

i.e. $P(A \cap B) = P(A) \times P(B/A)$ Provided $P(A) > 0$

(i) Compound Probability or Joint Probability

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

(j) For any three events A, B and C, the probability that they occur jointly is given by $P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/(A \cap B))$ Provided $P(A \cap B) > 0$

(k) $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A'/B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$$

(l) $$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

(m) $$P(A'/B') = \frac{P(A' \cap B')}{P(B')}$$

$$= \frac{P(A \cup B)'}{P(B')}$$ [by De-Morgan's Law $A' \cap B' = (AUB)'$]

$$= \frac{1 - P(A \cup B)}{1 - P(B)}$$

- A random variable or stochastic variable is a function defined on a sample space associated with a random experiment assuming any value from R and assigning a real number to each and every sample point of the random experiment.
- Expected value or Mathematical Expectation or Expectation of a random variable may be defined as the sum of products of the different values taken by the random variable and the corresponding probabilities.

When x is a discrete random variable with probability mass function f(x), then its expected value is given by

$$\mu = \sum_x x f(x)$$

and its variance is

$$\sigma^2 = E(x^2) - \mu^2$$

Where $E(x^2) = \sum_x x^2 f(x)$

For a continuous random variable x defined in [–∞, ∞], its expected value (i.e. mean) and variance are given by

$$E(x) = \int_{-\infty}^{\infty} x f(x)dx$$

$$\text{and } \sigma^2 = E(x^2) - \mu^2$$

$$\text{where } E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

Properties of Expected Values

(i) Expectation of a constant k is k

i.e. E(k) = k for any constant k.

(ii) Expectation of sum of two random variables is the sum of their expectations.

i.e. E(x + y) = E(x) + E(y) for any two random variables x and y.

(iii) Expectation of the product of a constant and a random variable is the product of the constant and the expectation of the random variable.

i.e. E(k x) = k.E(x) for any constant k

(iv) Expectation of the product of two random variables is the product of the expectation of the two random variables, provided the two variables are independent.

i.e. $E(xy) = E(x) \times E(y)$

Whenever x and y are independent.

EXERCISE

Set A

Write down the correct answers. Each question carRies 1 mark.

1. Initially, probability was a branch of
   (a) Physics (b) Statistics
   (c) Mathematics (d) Economics.
2. Two broad divisions of probability are
   (a) Subjective probability and objective probability
   (b) Deductive probability and non–deductive probability
   (c) Statistical probability and Mathematical probability
   (d) None of these.
3. Subjective probability may be used in
   (a) Mathematics (b) Statistics
   (c) Management (d) Accountancy.
4. An experiment is known to be random if the results of the experiment
   (a) Can not be predicted (b) Can be predicted
   (c) Can be split into further experiments (d) Can be selected at random.
5. An event that can be split into further events is known as
   (a) Complex event (b) Mixed event
   (c) Simple event (d) Composite event.
6. Which of the following pairs of events are mutually exclusive?
   (a) A : The student reads in a school. B : He studies Philosophy.
   (b) A : Raju was born in India. B : He is a fine Engineer.
   (c) A : Ruma is 16 years old. B : She is a good singer.
   (d) A : Peter is under 15 years of age. B : Peter is a voter of Kolkata.
7. If P(A) = P(B), then
   (a) A and B are the same events (b) A and B must be same events
   (c) A and B may be different events (d) A and B are mutually exclusive events.
8. If $P(A \cap B) = 0$, then the two events A and B are
   (a) Mutually exclusive (b) Exhaustive
   (c) Equally likely (d) Independent.

9. If for two events A and B, P(AUB) = 1, then A and B are
   (a) Mutually exclusive events (b) Equally likely events
   (c) Exhaustive events (d) Dependent events.
10. If an unbiased coin is tossed once, then the two events Head and Tail are
   (a) Mutually exclusive (b) Exhaustive
   (c) Equally likely (d) All these (a), (b) and (c).
11. If P(A) = P(B), then the two events A and B are
   (a) Independent (b) Dependent
   (c) Equally likely (d) Both (a) and (c).
12. If for two events A and B, $P(A \cap B) \neq P(A) \times P(B)$, then the two events A and B are
   (a) Independent (b) Dependent
   (c) Not equally likely (d) Not exhaustive.
13. If P(A/B) = P(A), then
   (a) A is independent of B (b) B is independent of A
   (c) B is dependent of A (d) Both (a) and (b).
14. If two events A and B are independent, then
   (a) A and the complement of B are independent
   (b) B and the complement of A are independent
   (c) Complements of A and B are independent
   (d) All of these (a), (b) and (c).
15. If two events A and B are independent, then
   (a) They can be mutually exclusive (b) They can not be mutually exclusive
   (c) They can not be exhaustive (d) Both (b) and (c).
16. If two events A and B are mutually exclusive, then
   (a) They are always independent (b) They may be independent
   (c) They can not be independent (d) They can not be equally likely.
17. If a coin is tossed twice, then the events 'occurrence of one head', 'occurrence of 2 heads' and 'occurrence of no head' are
   (a) Independent (b) Equally likely
   (c) Not equally likely (d) Both (a) and (b).
18. The probability of an event can assume any value between
   (a) – 1 and 1 (b) 0 and 1, including 0 and 1
   (c) – 1 and 0 (d) none of these.

19. If P(A) = 0, then the event A
    (a) will never happen
    (b) will always happen
    (c) may happen
    (d) may not happen.
20. If P(A) = 1, then the event A is known as
    (a) symmetric event
    (b) dependent event
    (c) improbable event
    (d) sure event.
21. If p : q are the odds in favour of an event, then the probability of that event is
    (a) $\frac{p}{q}$
    (b) $\frac{p}{p+q}$
    (c) $\frac{q}{p+q}$
    (d) none of these.
22. If P(A) = 5/9, then the odds against the event A is
    (a) 5 : 9
    (b) 5 : 4
    (c) 4 : 5
    (d) 5 : 14
23. If A, B and C are mutually exclusive and exhaustive events, then P(A) + P(B) + P(C) equals to
    (a) $\frac{1}{3}$
    (b) 1
    (c) 0
    (d) any value between 0 and 1.
24. If A denotes that a student reads in a school and B denotes that he plays cricket, then
    (a) $P(A \cap B) = 1$
    (b) $P(A \cup B) = 1$
    (c) $P(A \cap B) = 0$
    (d) P(A) = P(B).
25. P(B/A) is defined only when
    (a) A is a sure event
    (b) B is a sure event
    (c) A is not an impossible event
    (d) B is an impossible event.
26. P(A/B') is defined only when
    (a) B is not a sure event
    (b) B is a sure event
    (c) B is an impossible event
    (d) B is not an impossible event.
27. For two events A and B, $P(A \cup B) = P(A) + P(B)$ only when
    (a) A and B are equally likely events
    (b) A and B are exhaustive events
    (c) A and B are mutually independent
    (d) A and B are mutually exclusive.

28. Addition Theorem of Probability states that for any two events A and B,

(a) $P(A \cup B) = P(A) + P(B)$
(b) $P(A \cup B) = P(A) + P(B) + P(A \cap B)$
(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(d) $P(A \cup B) = P(A) \times P(B)$

29. For any two events A and B,

(a) $P(A) + P(B) > P(A \cap B)$
(b) $P(A) + P(B) < P(A \cap B)$
(c) $P(A) + P(B) \geq P(A \cap B)$
(d) $P(A) \text{ x } P(B) \leq P(A \cap B)$

30. For any two events A and B,

(a) $P(A–B) = P(A) – P(B)$
(b) $P(A–B) = P(A) – P(A \cap B)$
(c) $P(A–B) = P(B) – P(A \cap B)$
(d) $P(B–A) = P(B) + P(A \cap B)$.

31. The limitations of the classical definition of probability

(a) it is applicable when the total number of elementary events is finite
(b) it is applicable if the elementary events are equally likely
(c) it is applicable if the elementary events are mutually independent
(d) (a) and (b).

32. According to the statistical definition of probability, the probability of an event A is the

(a) limiting value of the ratio of the no. of times the event A occurs to the number of times the experiment is repeated
(b) the ratio of the frequency of the occurrences of A to the total frequency
(c) the ratio of the frequency of the occurrences of A to the non-occurrence of A
(d) the ratio of the favourable elementary events to A to the total number of elementary events.

33. The Theorem of Compound Probability states that for any two events A and B.

(a) $P(A \cap B) = P(A) \times P(B/A)$
(b) $P(A \cup B) = P(A) \times P(B/A)$
(c) $P(A \cap B) = P(A) \times P(B)$
(d) $P(A \cup B) = P(B) + P(B) – P(A \cap B)$.

34. If A and B are mutually exclusive events, then

(a) $P(A) = P(A–B)$.
(b) $P(B) = P(A–B)$.
(c) $P(A) = P(A \cap B)$.
(d) $P(B) = P(A \cap B)$.

35. If $P(A–B) = P(B–A)$, then the two events A and B satisfy the condition

(a) $P(A) = P(B)$.
(b) $P(A) + P(B) = 1$
(c) $P(A \cap B) = 0$
(d) $P(A \cup B) = 1$

36. The number of conditions to be satisfied by three events A, B and C for complete independence is

(a) 2
(b) 3
(c) 4
(d) any number.

37. If two events A and B are independent, then $P(A \cap B)$

(a) equals to P(A) + P(B)
(b) equals to P(A) $\times$ P(B)
(c) equals to P(A) $\times$ P(B/A)
(d) equals to P(B) $\times$ P(A/B).

38. Values of a random variable are

(a) always positive numbers.
(b) always positive real numbers.
(c) real numbers.
(d) natural numbers.

39. Expected value of a random variable

(a) is always positive
(b) may be positive or negative
(c) may be positive or negative or zero
(d) can never be zero.

40. If all the values taken by a random variable are equal then

(a) its expected value is zero
(b) its standard deviation is zero
(c) its standard deviation is positive
(d) its standard deviation is a real number.

41. If x and y are independent, then

(a) E(xy) = E(x) $\times$ E(y)
(b) E(xy) = E(x) + E(y)
(c) E(x – y) = E(x) + E(y)
(d) E(x – y) = E(x) + x E(y)

42. If a random variable x assumes the values x_1, x_2, x_3, x_4 with corresponding probabilities p_1, p_2, p_3, p_4 then the expected value of x is

(a) $p_1 + p_2 + p_3 + p_4$
(b) $x_1 p_1 + x_2 p_3 + x_3 p_2 + x_4 p_4$
(c) $p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4$
(d) none of these.

43. f(x), the probability mass function of a random variable x satisfies

(a) f(x) > 0
(b) $\sum_x f(x)=1$
(c) both (a) and (b)
(d) f(x) ≥ 0 and $\sum_x f(x)=1$

44. Variance of a random variable x is given by

(a) $E(x-\mu)^2$
(b) $E[x-E(x)]^2$
(c) $E(x^2-\mu)$
(d) (a) and (b)

45. If two random variables x and y are related by y = 2 – 3x, then the SD of y is given by

(a) –3 $\times$ SD of x
(b) 3 $\times$ SD of x.
(c) 9 $\times$ SD of x
(d) 2 $\times$ SD of x.

46. Probability of getting a head when two unbiased coins are tossed simultaneously is

(a) 0.25
(b) 0.50
(c) 0.20
(d) 0.75

47. If an unbiased coin is tossed twice, the probability of obtaining at least one tail is

(a) 0.25
(b) 0.50
(c) 0.75
(d) 1.00

48. If an unbiased die is rolled once, the odds in favour of getting a point which is a multiple of 3 is

(a) 1:2 (b) 2:1

(c) 1:3 (d) 3:1

49. A bag contains 15 one rupee coins, 25 two rupee coins and 10 five rupee coins. If a coin is selected at random from the bag, then the probability of not selecting a one rupee coin is

(a) 0.30 (b) 0.70

(c) 0.25 (d) 0.20

50. A, B, C are three mutually independent with probabilities 0.3, 0.2 and 0.4 respectively. What is $P(A \cap B \cap C)$?

(a) 0.400 (b) 0.240

(c) 0.024 (d) 0.500

51. If two letters are taken at random from the word HOME, what is the Probability that none of the letters would be vowels?

(a) 1/6 (b) 1/2

(c) 1/3 (d) 1/4

52. If a card is drawn at random from a pack of 52 cards, what is the chance of getting a Spade or an ace?

(a) 4/13 (b) 5/13

(c) 0.25 (d) 0.20

53. If x and y are random variables having expected values as 4.5 and 2.5 respectively, then the expected value of (x–y) is

(a) 2 (b) 7

(c) 6 (d) 0

54. If variance of a random variable x is 23, then what is the variance of 2x+10?

(a) 56 (b) 33

(c) 46 (d) 92

55. What is the probability of having at least one 'six' from 3 throws of a perfect die?

(a) 5/6 (b) $(5/6)^3$

(c) $1-(1/6)^3$ (d) $1-(5/6)^3$

Set B

Write down the correct answers. Each question carries 2 marks.

1. Two balls are drawn from a bag containing 5 white and 7 black balls at random. What is the probability that they would be of different colours?

(a) 35/66 (b) 30/66

(c) 12/66 (d) None of these

2. What is the chance of throwing at least 7 in a single cast with 2 dice?

(a) 5/12 (b) 7/12

(c) 1/4 (d) 17/36

3. What is the chance of getting at least one defective item if 3 items are drawn randomly from a lot containing 6 items of which 2 are defective item?

(a) 0.30 (b) 0.20

(c) 0.80 (d) 0.50

4. If two unbiased dice are rolled together, what is the probability of getting no difference of points?

(a) 1/2 (b) 1/3

(c) 1/5 (d) 1/6

5. If A, B and C are mutually exclusive independent and exhaustive events then what is the probability that they occur simultaneously?

(a) 1 (b) 0.50

(c) 0 (d) any value between 0 and 1

6. There are 10 balls numbered from 1 to 10 in a box. If one of them is selected at random, what is the probability that the number printed on the ball would be an odd number greater that 4?

(a) 0.50 (b) 0.40

(c) 0.60 (d) 0.30

7. Following are the wages of 8 workers in rupees:

50, 62, 40, 70, 45, 56, 32, 45

If one of the workers is selected at random, what is the probability that his wage would be lower than the average wage?

(a) 0.625 (b) 0.500

(c) 0.375 (d) 0.450

8. A, B and C are three mutually exclusive and exhaustive events such that P (A) = 2 P (B) = 3P(C). What is P (B)?

(a) 6/11 (b) 3/11

(c) 1/6 (d) 1/3

9. For two events A and B, P (B) = 0.3, P (A but not B) = 0.4 and P (not A) = 0.6. The events A and B are

(a) exhaustive (b) independent

(c) equally likely (d) mutually exclusive

10. A bag contains 12 balls which are numbered from 1 to 12. If a ball is selected at random, what is the probability that the number of the ball will be a multiple of 5 or 6 ?

(a) 0.30 (b) 0.25

(c) 0.20 (d) 1/3

11. Given that for two events A and B, P (A) = 3/5, P (B) = 2/3 and P $(A \cup B)$ = 3/4, what is P (A/B)?

(a) 0.655 (b) 13/60

(c) 31/60 (d) 0.775

12. For two independent events A and B, what is P (A+B), given P(A) = 3/5 and P(B) = 2/3?

(a) 11/15 (b) 13/15

(c) 7/15 (d) 0.65

13. If P (A) = p and P (B) = q, then

(a) $P(A/B) \leq p/q$ (b) $P(A/B) < p/q$

(c) $P(A/B) \leq q/p$ (d) None of these

14. If P $(\bar{A} \cup \bar{B})$ = 5/6, P(A) = ½ and P $(\bar{B})$ = 2/3, , what is P $(A \cup B)$?

(a) 1/3 (b) 5/6

(c) 2/3 (d) 4/9

15. If for two independent events A and B, P $(A \cup B)$ = 2/3 and P (A) = 2/5, what is P (B)?

(a) 4/15 (b) 4/9

(c) 5/9 (d) 7/15

16. If P (A) = 2/3, P (B) =3/4, P (A/B) = 2/3, then what is P (B / A)?

(a) 1/3 (b) 2/3

(c) 3/4 (d) 1/2

17. If P (A) = a, P (B) = b and P $(A \cap B)$ = c then the expression of P $(A' \cap B')$ in terms of a, b and c is

(a) 1 – a – b – c (b) a + b – c

(c) 1 + a – b – c (d) 1 – a – b + c

18. For three events A, B and C, the probability that only A occur is

(a) P (A) (b) $P(A \cup B \cup C)$

(c) $P(A' \cap B \cap C)$ (d) $P(A \cap B' \cap C')$

19. It is given that a family of 2 children has a girl, what is the probability that the other child is also a girl ?

(a) 0.50 (b) 0.75

(c) 1/3 (d) 2/3

20. Two coins are tossed simultaneously. What is the probability that the second coin would show a tail given that the first coin has shown a head?

(a) 0.50 (b) 0.25

(c) 0.75 (d) 0.125

21. If a random variable x assumes the values 0, 1 and 2 with probabilities 0.30, 0.50 and 0.20, then its expected value is

(a) 1.50 (b) 3

(c) 0.90 (d) 1

22. If two random variables x and y are related as $y = -3x + 4$ and standard deviation of x is 2, then the standard deviation of y is

(a) – 6 (b) 6

(c) 18 (d) 3.50

23. If $2x + 3y + 4 = 0$ and $v(x) = 6$ then $v(y)$ is

(a) 8/3 (b) 9

(c) – 9 (d) 6

Set C

Write down the correct answers. Each question carries 5 marks.

1. What is the probability that a leap year selected at random would contain 53 Saturdays?

(a) 1/7 (b) 2/7

(c) 1/12 (d) 1/4

2. If an unbiased coin is tossed three times, what is the probability of getting more that one head?

(a) 1/8 (b) 3/8

(c) 1/2 (d) 1/3

3. If two unbiased dice are rolled, what is the probability of getting points neither 6 nor 9?

(a) 0.25 (b) 0.50

(c) 0.75 (d) 0.80

4. What is the probability that 4 children selected at random would have different

birthdays?

(a) $\frac{364 \times 363 \times 362}{(365)^3}$ (b) $\frac{6 \times 5 \times 4}{7^3}$

(c) 1/365 (d) $(1/7)^3$

5. A box contains 5 white and 7 black balls. Two successive drawn of 3 balls are made (i) with replacement (ii) without replacement. The probability that the first draw would produce white balls and the second draw would produce black balls are respectively

(a) 6/321 and 3/926 (b) 1/20 and 1/30

(c) 35/144 and 35/108 (d) 7/968 and 5/264

6. There are three boxes with the following composition:

Box I: 5 Red + 7 White + 6 Blue balls Box II: 4 Red + 8 White + 6 Blue balls

Box III: 3 Red + 4 White + 2 Blue balls

If one ball is drawn at random, then what is the probability that they would be of same colour?

(a) 89/729 (b) 97/729

(c) 82/729 (d) 23/32

7. A number is selected at random from the first 1000 natural numbers. What is the probability that the number so selected would be a multiple of 7 or 11?

(a) 0.25 (b) 0.32

(c) 0.22 (d) 0.33

8. A bag contains 8 red and 5 white balls. Two successive draws of 3 balls are made without replacement. The probability that the first draw will produce 3 white balls and the second 3 red balls is

(a) 5/223 (b) 6/257

(c) 7/429 (d) 3/548

9. There are two boxes containing 5 white and 6 blue balls and 3 white and 7 blue balls respectively. If one of the the boxes is selected at random and a ball is drawn from it, then the probability that the ball is blue is

(a) 115/227 (b) 83/250

(c) 137/220 (d) 127/250

10. A problem in probability was given to three CA students A, B and C whose chances of solving it are 1/3, 1/5 and 1/2 respectively. What is the probability that the problem would be solved?

(a) 4/15 (b) 7/8

(c) 8/15 (d) 11/15

11. There are three persons aged 60, 65 and 70 years old. The survival probabilities for these three persons for another 5 years are 0.7, 0.4 and 0.2 respectively. What is the probability that at least two of them would survive another five years?

(a) 0.425 (b) 0.456

(c) 0.392 (d) 0.388

12. Tom speaks truth in 30 percent cases and Dick speaks truth in 25 percent cases. What is the probability that they would contradict each other?

(a) 0.325 (b) 0.400

(c) 0.925 (d) 0.075

13. There are two urns. The first urn contains 3 red and 5 white balls whereas the second urn contains 4 red and 6 white balls. A ball is taken at random from the first urn and is transferred to the second urn. Now another ball is selected at random from the second urn. The probability that the second ball would be red is

(a) 7/20 (b) 35/88

(c) 17/52 (d) 3/20

14. For a group of students, 30 %, 40% and 50% failed in Physics , Chemistry and at least one of the two subjects respectively. If an examinee is selected at random, what is the probability that he passed in Physics if it is known that he failed in Chemistry?

(a) 1/2 (b) 1/3

(c) 1/4 (d) 1/6

15. A packet of 10 electronic components is known to include 2 defectives. If a sample of 4 components is selected at random from the packet, what is the probability that the sample does not contain more than 1 defective?

(a) 1/3 (b) 2/3

(c) 13/15 (d) 3/15

16. 8 identical balls are placed at random in three bags. What is the probability that the first bag will contain 3 balls?

(a) 0.2731 (b) 0.3256

(c) 0.1924 (d) 0.3443

17. X and Y stand in a line with 6 other people. What is the probability that there are 3 persons between them?

(a) 1/5 (b) 1/6

(c) 1/7 (d) 1/3

18. Given that P (A) = 1/2, P (B) = 1/3, P ($A \cap B$) = 1/4, what is P (A'/B')

(a) 1/2 (b) 7/8

(c) 5/8 (d) 2/3

19. Four digits 1, 2, 4 and 6 are selected at random to form a four digit number. What is the probability that the number so formed, would be divisible by 4?

(a) 1/2 (b) 1/5

(c) 1/4 (d) 1/3

20. The probability distribution of a random variable x is given below:

x :	1	2	4	5	6
P :	0.15	0.25	0.20	0.30	0.10

What is the standard deviation of x?

(a) 1.49 (b) 1.56

(c) 1.69 (d) 1.72

21. A packet of 10 electronic components is known to include 3 defectives. If 4 components are selected from the packet at random, what is the expected value of the number of defective?

(a) 1.20 (b) 1.21

(c) 1.69 (d) 1.72

22. The probability that there is at least one error in an account statement prepared by 3 persons A, B and C are 0.2, 0.3 and 0.1 respectively. If A, B and C prepare 60, 70 and 90 such statements, then the expected number of correct statements

(a) 170 (b) 176

(c) 178 (d) 180

23. A bag contains 6 white and 4 red balls. If a person draws 2 balls and receives ₹ 10 and ₹ 20 for a white and red balls respectively, then his expected amount is

(a) ₹ 25 (b) ₹ 26

(c) ₹ 29 (d) ₹ 28

24. The probability distribution of a random variable is as follows:

x :	1	2	4	6	8
P :	k	2k	3k	3k	k

The variance of x is

(a) 2.1 (b) 4.41

(c) 2.32 (d) 2.47

ANSWERS

Set A

1.	(c)	**2.**	(a)	**3.**	(c)	**4.**	(d)	**5.**	(d)	**6.**	(d)
7.	(c)	**8.**	(a)	**9.**	(c)	**10.**	(d)	**11.**	(c)	**12.**	(b)
13.	(d)	**14.**	(d)	**15.**	(b)	**16.**	(c)	**17.**	(c)	**18.**	(b)
19.	(a)	**20.**	(d)	**21.**	(b)	**22.**	(c)	**23.**	(b)	**24.**	(c)
25.	(c)	**26.**	(a)	**27.**	(d)	**28.**	(c)	**29.**	(c)	**30.**	(b)
31.	(d)	**32.**	(a)	**33.**	(a)	**34.**	(a)	**35.**	(a)	**36.**	(c)
37.	(b)	**38.**	(c)	**39.**	(c)	**40**	(b)	**41.**	(a)	**42.**	(c)
43.	(d)	**44.**	(d)	**45.**	(b)	**46.**	(b)	**47.**	(c)	**48.**	(a)
49.	(b)	**50.**	(c)	**51.**	(a)	**52.**	(a)	**53.**	(a)	**54.**	(d)
55.	(d)										

Set B

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(d)	**5.**	(c)	**6.**	(d)
7.	(b)	**8.**	(b)	**9.**	(d)	**10.**	(d)	**11.**	(d)	**12.**	(b)
13.	(a)	**14.**	(c)	**15.**	(b)	**16.**	(c)	**17.**	(d)	**18.**	(d)
19.	(c)	**20.**	(a)	**21.**	(c)	**22.**	(b)	**23.**	(a)		

Set C

1.	(b)	**2.**	(c)	**3.**	(c)	**4.**	(a)	**5.**	(d)	**6.**	(a)
7.	(c)	**8.**	(c)	**9.**	(c)	**10.**	(d)	**11.**	(d)	**12.**	(b)
13.	(b)	**14.**	(a)	**15.**	(c)	**16.**	(a)	**17.**	(c)	**18.**	(c)
19.	(d)	**20.**	(c)	**21.**	(a)	**22.**	(c)	**23.**	(d)	**24.**	(b)

ADDITIONAL QUESTION BANK

1. All possible outcomes of a random experiment forms the
(a) events (b) sample space (c) both (d) none
2. If one of outcomes cannot be expected to occur in preference to the other in an experiment the events are
(a) simple events (b) compound events
(c) favourable events (d) equally likely events
3. If two events cannot occur simultaneously in the same trial then they are
(a) mutually exclusive events (b) simple events
(c) favourable events (d) none
4. When the number of cases favourable to the event A is none then P(A) is equal to
(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) none
5. A card is drawn from a well-shuffled pack of playing cards. The probability that it is a spade is
(a) $\frac{1}{13}$ (b) $\frac{1}{4}$ (c) $\frac{3}{13}$ (d) none
6. A card is drawn from a well-shuffled pack of playing cards. The probability that it is a king is
(a) $\frac{1}{13}$ (b) $\frac{1}{4}$ (c) $\frac{4}{13}$ (d) none
7. A card is drawn from a well-shuffled pack of playing cards. The probability that it is the ace of clubs is
(a) $\frac{1}{13}$ (b) $\frac{1}{4}$ (c) $\frac{1}{52}$ (d) none
8. In a single throw with two dice the probability of getting a sum of five on the two dice is
(a) $\frac{1}{9}$ (b) $\frac{5}{36}$ (c) $\frac{5}{9}$ (d) none
9. In a single throw with two dice, the probability of getting a sum of six on the two dice is
(a) $\frac{1}{9}$ (b) $\frac{5}{36}$ (c) $\frac{5}{9}$ (d) none
10. The probability that exactly one head appears in a single throw of two fair coins is
(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) none
11. The probability that at least one head appears in a single throw of three fair coins is
(a) $\frac{1}{8}$ (b) $\frac{7}{8}$ (c) $\frac{1}{3}$ (d) none
12. The definition of probability fails when the no of possible outcomes of the experiment is infinite
(a) True (b) false (c) both (d) none

13. The following table gives distribution of wages of 100 workers –

Wages (in ₹)	120–140	140–160	160–180	180–200	200–220	220–240	240–260
No. of workers	9	20	0	10	8	35	18

The probability that his wages are under ₹ 140 is
(a) 20/100 (b) 9/100 (c) 29/100 (d) none

14. An individual is selected at random from the above group. The probability that his wages are under ₹160 is
(a) 9/100 (b) 20/100 (c) 29/100 (d) none

15. For the above table the probability that his wages are above ₹ 200 is
(a) 43/100 (b) 35/100 (c) 53/100 (d) 61/100

16. For the above table the probability that his wages between ₹ 160 and 220 is
(a) 30/100 (b) 10/100 (c) 38/100 (d) 18/100

17. The table below shows the history of 1000 men :

Life (in years) :	60	70	80	90
No. survived :	1000	500	100	60

The probability that a man will survived to age 90 is
(a) 60/1000 (b) 160/1000 (c) 660/1000 (d) none

18. The terms "chance" and probability are synonymous
(a) True (b) false (c) both (d) none

19. If probability of drawing a spade from a well-shuffled pack of playing cards is $\frac{1}{4}$ then the probability that of the card drawn from a well-shuffled pack of playing cards is 'not a spade' is
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

20. Sum of probability of events in sample space is
(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) none

21. Sum of all probabilities of mutually exclusive and exhaustive events is equal to
(a) 0 (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) 1

22. Let a sample space be $S = \{X_1, X_2, X_3\}$ which of the following defines probability space on S ?
(a) $P(X_1)= \frac{1}{4}$, $P(X_2)= \frac{1}{3}$, $P(X_3)= \frac{1}{3}$ (b) $P(X_1)= 0$, $P(X_2)= \frac{1}{3}$, $P(X_3)= \frac{2}{3}$
(c) $P(X_1)= \frac{2}{3}$, $P(X_2)= \frac{1}{3}$, $P(X_3)= \frac{2}{3}$ (d) none

23. Let P be a probability function on S = {X_1 , X_2 , X_3} if $P(X_1)= \frac{1}{4}$ and $P(X_3) = \frac{1}{3}$ then P (X_2) is equal to
(a) 5/12 (b) 7/12 (c) 3/4 (d) none
24. The chance of getting a sum of 10 in a single throw with two dice is
(a) 10/36 (b) 1/12 (c) 5/36 (d) none
25. The chance of getting a sum of 6 in a single throw with two dice is
(a) 3/36 (b) 4/36 (c) 6/36 (d) 5/36
26. P (B/A) defines the probability that event B occurs on the assumption that A has happened
(a) Yes (b) no (c) both (d) none
27. The complete group of all possible outcomes of a random experiment given an ________ set of events.
(a) mutually exclusive (b) exhaustive (c) both (d) none
28. When the event is 'certain' the probability of it is
(a) 0 (b) 1/2 (c) 1 (d) none
29. The classical definition of probability is based on the feasibility at subdividing the possible outcomes of the experiments into
(a) mutually exclusive and exhaustive
(b) mutually exclusive and equally likely
(c) exhaustive and equally likely
(d) mutually exclusive,exhaustive and equally likely cases.
30. Two unbiased coins are tossed. The probability of obtaining 'both heads' is
(a) $\frac{1}{4}$ (b) $\frac{2}{4}$ (c) $\frac{3}{4}$ (d) none
31. Two unbiased coins are tossed. The probability of obtaining one head and one tail is
(a) $\frac{1}{4}$ (b) $\frac{2}{4}$ (c) $\frac{3}{4}$ (d) none
32. Two unbiased coins are tossed. The probability of obtaining both tail is
(a) $\frac{2}{4}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) none
33. Two unbiased coins are tossed. The probability of obtaining at least one head is
(a) $\frac{1}{4}$ (b) $\frac{2}{4}$ (c) $\frac{3}{4}$ (d) none
34. When two unbiased coins are tossed, the probability of obtaining 3 heads is
(a) $\frac{2}{4}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) 0
35. When two unbiased coins are tossed, the probability of obtaining not more than 3 heads is
(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

36. When two unbiased coins are tossed, the probability of getting both heads or both tails is

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) none

37. Two dice with face marked 1, 2, 3, 4, 5, 6 are thrown simultaneously and the points on the dice are multiplied together. The probability that product is 12 is

(a) 4/36 (b) 5/36 (c) 12/36 (d) none

38. A bag contain 6 white and 5 black balls. One ball is drawn. The probability that it is white is

(a) 5/11 (b) 1 (c) 6/11 (d) 1/11

39. Probability of occurrence of at least one of the events A and B is denoted by

(a) P(AB) (b) P(A+B) (c) P(A/B) (d) none

40. Probability of occurrence of A as well as B is denoted by

(a) P(AB) (b) P(A+B) (c) P(A/B) (d) none

41. Which of the following relation is true ?

(a) $P(A) - P(A^C) = 1$ (b) $P(A) + P(A^C) = 1$ (c) $P(A)\,P(A^C) = 1$ (d) none

42. If events A and B are mutually exclusive, the probability that either A or B occurs is given by

a) P (A+B)= P(A)– P(B)
(b) P (A+B)=P(A)+ P(B)– P(AB)
c) P (A+B)= P(A)– P(B)+ P(AB)
(d) P (A+B)= P(A)+ P(B)

43. The probability of occurrence of at least one of the 2 events A and B (which may not be mutually exclusive) is given by

a) P(A+B)= P(A)– P(B)
(b) P(A+B)= P(A)+ P(B)– P(AB)
c) P(A+B)= P(A)– P(B)+ P(AB)
(d) P(A+B)= P(A)+P (B)

44. If events A and B are independent, the probability of occurrence of A as well as B is given by

(a) P(AB)= P(A/B)
(b) P(AB)= P(A)/P(B)
(c) P(AB)= P(A)P(B)
(d) None

45. For the condition P(AB)= P(A)P(B)two events A and B are said to be

(a) dependent (b) independent (c) equally like (d) none

46. The conditional probability of an event B on the assumption that another event A has actually occurred is given by

(a) P(B/A)= P(AB)/P(A)
(b) P(A/B)= P(AB)/ P(B)
(c) P(B/A)= P(AB)
(d) P(A/B)= P(AB)/ P(A)P(B)

47. Given $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(AB) = \frac{1}{4}$, the value of P(A+B) is

a) $\frac{3}{4}$ b) $\frac{7}{12}$ c) $\frac{5}{6}$ d) $\frac{1}{6}$

48. Given P(A)= $\frac{1}{2}$, P(B)= $\frac{1}{3}$, P (AB)= $\frac{1}{4}$, the value of P (A/B) is

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

49. If P (A)= $\frac{1}{3}$, P(B)= $\frac{1}{4}$, the events A & B are

a) not equally likely b) mutually exclusive
c) equally likely d) none

50. If events A and B are independent then

a) A^C and B^C are dependent b) A^C and B are dependent
c) A and B^C are dependent d) A^C and B^C are also independent

51. A card is drawn from each of two well-shuffled packs of cards.The probability that at least one of them is an ace is

a) $\frac{1}{69}$ b) $\frac{25}{169}$ c) $\frac{2}{13}$ d) none

52. When a die is tossed, the sample space is

a) S ={1,2,3,4,5} b) S ={1,2,3,4} c) S ={1,2,3,4,5,6} d) none

53. If P (A)= $\frac{1}{4}$, P(B)= $\frac{2}{5}$, P (A+B)= $\frac{1}{2}$ then P(AB)is equal to

a) $\frac{3}{4}$ b) $\frac{2}{20}$ c) $\frac{13}{20}$ d) $\frac{3}{20}$

54. If events A and B are independent and P(A)= 2/3 , P(B)= 3/5 then P(A+B)is equal to

a) $\frac{13}{15}$ b) $\frac{6}{15}$ c) $\frac{1}{15}$ d) none

55. The expected number of head in 100 tosses of an unbiased coin is

a) 100 b) 50 c) 25 d) none

56. A and B are two events such that P(A)= $\frac{1}{3}$, P(B) = $\frac{1}{4}$, P(A+B)= $\frac{1}{2}$, than P(B/A) is equal to

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) none

57. Probability mass function is always

a) 0 b) greater than 0
c) greater than equal to 0 d) less than 0

58. The sum of probability mass function is equal to

a) –1 b) 0 c) 1 d) none

59. When X is a continuous function f(x)is called

a) probability mass function b) probability density function
c) both d) none

60. Which of the following set of function define a probability space on S = $\{a_1, a_2, a_3\}$

a) $P(a_1)= \frac{1}{3}, P(a_2) = \frac{1}{2}, P(a_3)= \frac{1}{4}$
b) $P(a_1)= \frac{1}{3}, P(a_2)= \frac{1}{6}, P(a_3)= \frac{1}{2}$
c) $P(a_1)= P(a_2)= \frac{2}{3}, P(a_3)= \frac{1}{4}$
d) None

61. If $P(a_1)= 0, P(a_2)= \frac{1}{3}, P(a_3) = \frac{2}{3}$ then S = $\{a_1, a_2, a_3\}$ is a probability space

a) true
b) false
c) both
d) none

62. If two events are independent then

a) P(B/A)= P(AB) P(A)
b) P(B/A)= P(AB) P(B)
c) P(B/A)= P(B)
d) P(B/A)= P(A)

63. When expected value is negative the result is

a) favourable
b) unfavourable
c) both
d) none to the above

64. The expected value of X, the sum of the scores, when two dice are rolled is

a) 9
b) 8
c) 6
d) 7

65. Let A and B be the events with P(A)= 1/3, P(B) = 1/4 and P(AB)= 1/12 then P(A/B) is equal to

a) $\frac{1}{3}$
b) $\frac{1}{4}$
c) $\frac{3}{4}$
d) $\frac{2}{3}$

66. Let A and B be the events with P(A)= 2/3, P(B)= 1/4 and P(AB)= 1/12 then P(B/A) is equal to

a) $\frac{7}{8}$
b) $\frac{1}{3}$
c) $\frac{1}{8}$
d) none

67. The odds in favour of one student passing a test are 3:7.The odds against another student passing at are 3:5.The probability that both pass is

a) $\frac{7}{16}$
b) $\frac{21}{80}$
c) $\frac{9}{80}$
d) $\frac{3}{16}$

68. The odds in favour of one student passing a test are 3:7.The odds against another student passing at are 3:5. The probability that both fail is

a) $\frac{7}{16}$
b) $\frac{21}{80}$
c) $\frac{9}{80}$
d) $\frac{3}{16}$

69. In formula P(B/A), P(A) is

a) greater than zero
b) less than zero
c) equal to zero
d) greater than equal to zero

70. Two events A and B are mutually exclusive means they are

a) not disjoint
b) disjoint
c) equally likely
d) none

71. A bag contains 10 white and 10 black balls A ball is drawn from it. The probability that it will be white is

(a) $\frac{1}{10}$ (b) 1 (c) $\frac{1}{2}$ (d) none

72. Two dice are thrown at a time. The probability that the numbers shown are equal is

(a) $\frac{2}{6}$ (b) $\frac{5}{6}$ (c) $\frac{1}{6}$ (d) none

73. Two dice are thrown at a time. The probability that 'the difference of numbers shown is 1' is

(a) $\frac{11}{18}$ (b) $\frac{5}{18}$ (c) $\frac{7}{18}$ (d) none

74. Two dice are thrown together. The probability that 'the event the difference of numbers shown is 2' is

(a) 2/9 (b) 5/9 (c) 4/9 (d) 7/9

75. The probability space in tossing two coins is

(a) {(H,H),(H,T),(T,H)} (b) {(H,T),(T,H),(T,T)}

(c) {(H,H),(H,T),(T,H), (T,T)} (d) none

76. The probability of drawing a white ball from a bag containing 3 white and 8 blue balls is

(a) 3/5 (b) 3/11 (c) 8/11 (d) none

77. Two dice are thrown together. The probability of the event that the sum of numbers shown is greater than 5 is

(a) 13/18 (b) 15/18 (c) 1 (d) none

78. A traffic census show that out of 1000 vehicles passing a junction point on a highway 600 turned to the right. The probability of an automobile turning the right is

(a) 2/5 (b) 3/5 (c) 4/5 (d) none

79. Three coins are tossed together. The probability of getting three tails is

(a) 5/8 (b) 3/8 (c) 1/8 (d) none

80. Three coins are tossed together.The probability of getting exactly two heads is

(a) 5/8 (b) 3/8 (c) 1/8 (d) none

81. Three coins are tossed together. The probability of getting at least two heads is

(a) 1/2 (b) 3/8 (c) 1/8 (d) none

82. 4 coins are tossed. The probability that there are 2 heads is

(a) 1/2 (b) 3/8 (c) 1/8 (d) none

83. If 4 coins are tossed. The chance that there should be two tails is

(a) 1/2 (b) 3/8 (c) 1/8 (d) none

84. If A is an event and A^C its complementary event then

(a) $P(A)=P(A^C)-1$ (b) $P(A^C)=1-P(A)$ (c) $P(A)=1+P(A^C)$ (d) none

85. If P(A)= 3/8, P(B)= 1/3 and P(AB)= $\frac{1}{4}$ then $P(A^C)$ is equal to

(a) 5/8 (b) 3/8 (c) 1/8 (d) none

86. If P(A)= 3/8, P(B)= 1/3 then $P(\bar{B})$ is equal to

(a) 1 (b) 1/3 (c) 2/3 (d) none

87. If P(A)= 3/8, P(B)= 1/3 and P(AB)= $\frac{1}{4}$ then P(A + B)is

(a) 13/24 (b) 11/24 (c) 17/24 (d) none

88. If P(A)= 1/5, P(B)= 1/2 and A and B are mutually exclusive then P(AB) is

(a) 7/10 (b) 3/10 (c) 1/5 (d) none

89. The probability of throwing more than 4 in a single throw from an ordinary die is

(a) 2/3 (b) 1/3 (c) 1 (d) none

90. The probability that a card drawn at random from the pack of playing cards may be either a queen or an ace is

(a) 2/13 (b) 11/13 (c) 9/13 (d) none

91. The chance of getting 7 or 11 in a throw of 2 dice is

(a) 7/9 (b) 5/9 (c) 2/9 (d) none

92. If the probability of a horse A winning a race is 1/6 and the probability of a horse B winning the same race is 1/4 , what is the probability that one of the horses will win

(a) 5/12 (b) 7/12 (c) 1/12 (d) none

93. If the probability of a horse A winning a race is 1/6 and the probability of a horse B winning the same race is 1/4 , What is the probability that none of them will win

(a) 5/12 (b) 7/12 (c) 1/12 (d) none

94. If P (A)= 7/8 then(P(A^C) is equal to

(a) 1 (b) 0 (c) 7/8 (d) 1/8

95. The value of P(S) were S is the sample space is

(a) –1 (b) 0 (c) 1 (d) none

96. A man can kill a bird once in three shots.The probabilities that a bird is not killed is

(a) 1/3 (b) 2/3 (c) 1 (d) 0

97. If on an average 9 shops out of 10 return safely to a port, the probability of one ship returns safely is

(a) 1/10 (b) 8/10 (c) 9/10 (d) none

98. If on an average 9 shops out of 10 return safely to a port, the probability of one ship does not reach safely is

(a) 1/10 (b) 8/10 (c) 9/10 (d) none

99. The probability of winning of a person is 6/11 and at a result he gets ₹ 77/-. The expectation of this person is

(a) ₹ 35/- (b) ₹ 42/- (c) ₹ 58/- (d) none

100. A family has 2 children. The probability that both of them are boys if it is known that one of them is a boy

(a) 1 (b) 1/2 (c) 3/4 (d) none

101. The Probability of the occurrence of a number greater then 2 in a throw of a die if it is known that only even numbers can occur is

(a) 1/3 (b) 1/2 (c) 2/3 (d) none

102. A player has 7 cards in hand of which 5 are red and of these five 2 are kings. A card is drawn at random. The probability that it is a king, it being known that it is red is

(a) 2/5 (b) 3/5 (c) 4/5 (d) none

103. In a class 40 % students read Mathematics, 25 % Biology and 15 % both Mathematics and Biology. One student is select at random. The probability that he reads Mathematics if it is known that he reads Biology is

(a) 2/5 (b) 3/5 (c) 4/5 (d) none

104. In a class 40 % students read Mathematics, 25 % Biology and 15 % both Mathematics and Biology. One student is select at random.The probability that he reads Biology if he reads Mathematics

(a) 7/8 (b) 1/8 (c) 3/8 (d) none

105. Probability of throwing an odd no with an ordinary six faced die is

(a) 1/2 (b) 1 (c) –1/2 (d) 0

106. For a event A which is certain, P (A) is equal to

(a) 1 (b) 0 (c) –1 (d) none

107. When none of the outcomes is favourable to the event then the event is said to be

(a) certain (b) sample (c) impossible (d) none

ANSWERS

1.	(b)	**2.**	(d)	**3.**	(a)	**4.**	(b)	**5.**	(b)
6.	(a)	**7.**	(c)	**8.**	(a)	**9.**	(b)	**10.**	(b)
11.	(b)	**12.**	(a)	**13.**	(b)	**14.**	(c)	**15.**	(d)
16.	(d)	**17.**	(a)	**18.**	(a)	**19.**	(d)	**20.**	(c)
21.	(d)	**22.**	(b)	**23.**	(a)	**24.**	(b)	**25.**	(d)
26.	(a)	**27.**	(b)	**28.**	(c)	**29.**	(d)	**30.**	(a)
31.	(b)	**32.**	(c)	**33.**	(c)	**34.**	(d)	**35.**	(c)
36.	(a)	**37.**	(a)	**38.**	(c)	**39.**	(b)	**40.**	(a)
41.	(b)	**42.**	(d)	**43.**	(b)	**44.**	(c)	**45.**	(b)
46.	(a)	**47.**	(b)	**48.**	(d)	**49.**	(a)	**50.**	(d)
51.	(b)	**52.**	(c)	**53.**	(d)	**54.**	(a)	**55.**	(b)
56.	(a)	**57.**	(c)	**58.**	(c)	**59.**	(b)	**60.**	(b)
61.	(a)	**62.**	(c)	**63.**	(b)	**64.**	(d)	**65.**	(a)
66.	(c)	**67.**	(d)	**68.**	(b)	**69.**	(a)	**70.**	(b)
71.	(c)	**72.**	(c)	**73.**	(b)	**74.**	(a)	**75.**	(c)
76.	(b)	**77.**	(a)	**78.**	(b)	**79.**	(c)	**80.**	(b)
81.	(a)	**82.**	(b)	**83.**	(b)	**84.**	(b)	**85.**	(a)
86.	(c)	**87.**	(b)	**88.**	(d)	**89.**	(b)	**90.**	(a)
91.	(c)	**92.**	(a)	**93.**	(b)	**94.**	(d)	**95.**	(c)
96.	(b)	**97.**	(c)	**98.**	(a)	**99.**	(b)	**100.**	(d)
101.	(c)	**102.**	(a)	**103.**	(b)	**104.**	(c)	**105.**	(a)
106.	(a)	**107.**	(c)						

NOTES

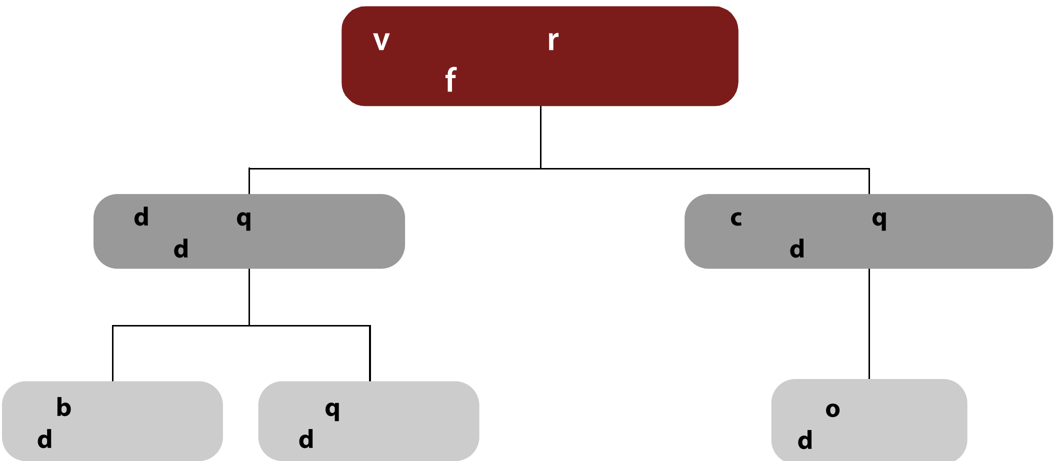
CHAPTER BI

THEORETICAL DISTRIBUTIONS

LEARNING OBJECTIVES

The Students will be introduced in this chapter to the techniques of developing discrete and continuous probability distributions and its applications.

CHAPTER OVERVIEW



16.1 INTRODUCTION

In chapter thirteen, it may be recalled, we discussed frequency distribution. In a similar manner, we may think of a probability distribution where just like distributing the total frequency to different class intervals, the total probability (i.e. one) is distributed to different mass points in case of a discrete random variable or to different class intervals in case of a continuous random variable. Such a probability distribution is known as Theoretical Probability Distribution, since such a distribution exists in theory. We need to study theoretical probability distribution for the following important factors:

(a) An observed frequency distribution, in many a case, may be regarded as a sample i.e. a representative part of a large, unknown, boundless universe or population and we may be interested to know the form of such a distribution. By fitting a theoretical probability distribution to an observed frequency distribution of, say, the lamps produced by a manufacturer, it may be possible for the manufacturer to specify the length of life of the lamps produced by him up to a reasonable degree of accuracy. By studying the effect of a particular type of missiles, it may be possible for our scientist to suggest the number of such missiles necessary to destroy an army position. By knowing the distribution of smokers, a social activist may warn the people of a locality about the nuisance of active and passive smoking and so on.

(b) Theoretical probability distribution may be profitably employed to make short term projections for the future.

(c) Statistical analysis is possible only on the basis of theoretical probability distribution. Setting confidence limits or testing statistical hypothesis about population parameter(s) is based on the probability distribution of the population under consideration.

A probability distribution also possesses all the characteristics of an observed distribution. We define mean (μ), median $(\tilde{\mu})$, mode (μ_0), standard deviation (σ) etc. exactly same way we have done earlier. Again a probability distribution may be either a discrete probability distribution or a Continuous probability distribution depending on the random variable under study. **Two important discrete probability distributions are (a) Binomial Distribution and (b) Poisson distribution.**

Some important continuous probability distributions are

Normal Distribution

16.2 BINOMIAL DISTRIBUTION

One of the most important and frequently used discrete probability distribution is Binomial Distribution. It is derived from a particular type of random experiment known as Bernoulli process named after the famous mathematician Bernoulli. Noting that a 'trial' is an attempt to produce a particular outcome which is neither certain nor impossible, the characteristics of Bernoulli trials are stated below:

(i) Each trial is associated with two mutually exclusive and exhaustive outcomes, the occurrence of one of which is known as a 'success' and as such its non occurrence as a 'failure'. As an example, when a coin is tossed, usually occurrence of a head is known as a success and its non–occurrence i.e. occurrence of a tail is known as a failure.

(ii) The trials are independent.

(iii) The probability of a success, usually denoted by p, and hence that of a failure, usually denoted by q = 1–p, remain unchanged throughout the process.

(iv) The number of trials is a finite positive integer.

A discrete random variable x is defined to follow binomial distribution with parameters n and p, to be denoted by $x \sim B(n, p)$, if the probability mass function of x is given by

$$f(x) = p(X = x) = {}^{n}c_{x}\, p^{x} q^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

$$= 0, \text{ otherwise} \qquad \ldots\ldots\ldots (16.1)$$

We may note the following important points in connection with binomial distribution:

(a) As n >0, p, q ≥ 0, it follows that f(x) ≥ 0 for every x

Also $\sum_{x} f(x) = f(0) + f(1) + f(2) + \ldots + f(n) = 1 \ldots\ldots\ldots(16.2)$

(b) Binomial distribution is known as biparametric distribution as it is characterised by two parameters n and p. This means that if the values of n and p are known, then the distribution is known completely.

(c) The mean of the binomial distribution is given by $\mu = np$ **…. (16.3)**

(d) Depending on the values of the two parameters, binomial distribution may be unimodal or bi- modal. μ_0**, the mode of binomial distribution, is given by** μ_0 **= the largest integer contained in (n+1)p if (n+1)p is a non-integer (n+1)p and (n+1)p - 1 if (n+1)p is an integer ….(16.4)**

(e) The variance of the binomial distribution is given by

$$\sigma^2 = npq \qquad \ldots\ldots\ldots\ldots (16.5)$$

Since p and q are numerically less than or equal to 1, npq < np

⇒ variance of a binomial variable is always less than its mean.

Also variance of X attains its maximum value at p = q = 0.5 and this maximum value is n/4.

(f) Additive property of binomial distribution.

If X and Y are two independent variables such that

$X \sim B(n_1, P)$

and $Y \sim B(n_{21} P)$

Then $(X+Y) \sim B(n_1 + n_2, P)$ **……………………………. (16.6)**

Applications of Binomial Distribution

Binomial distribution is applicable when the trials are independent and each trial has just two outcomes success and failure. It is applied in coin tossing experiments, sampling inspection plan, genetic experiments and so on.

Example 16.1: A coin is tossed 10 times. Assuming the coin to be unbiased, what is the probability of getting

(i) 4 heads?

(ii) at least 4 heads?

(iii) at most 3 heads?

Solution: We apply binomial distribution as the tossing are independent of each other. With every tossing, there are just two outcomes either a head, which we call a success or a tail, which we call a failure and the probability of a success (or failure) remains constant throughout.

Let X denotes the no. of heads. Then X follows binomial distribution with parameter n = 8 and p = 1/2 (since the coin is unbiased). Hence q = 1 – p = 1/2

The probability mass function of X is given by

$f(x) = {}^{n}c_{x}\ p^{x}\ q^{n-x}$

$= {}^{10}c_{x} \cdot (1/2)^{x} \cdot (1/2)^{10-x}$

$$= \frac{{}^{10}c_{x}}{2^{10}}$$

$= {}^{10}c_{x} / 1024$ for x = 0, 1, 2, ……….10

(i) probability of getting 4 heads

$= f(4)$

$= {}^{10}c_{4} / 1024$

$= 210 / 1024$

$= 105 / 512$

(ii) probability of getting at least 4 heads

$= P(X \geq 4)$

$= P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)$

$= {}^{10}c_{4} / 1024 + {}^{10}c_{5} / 1024 + {}^{10}c_{6} / 1024 + {}^{10}c_{7} / 1024 + {}^{10}c_{8} /1024 + {}^{10}c_{9} /1024 + {}^{10}c_{10} /1024$

$$= \frac{210 + 252 + 210 + 120 + 45 + 10 + 1}{1024}$$

= 848 / 1024

(iii) probability of getting at most 3 heads

= P (X ≤ 3)

= P (X = 0) + P (X = 1) + P (X = 2) + P (X = 3)

= f (0) + f (1) + f (2) + f (3)

$= {}^{10}c_0 / 1024 + {}^{10}c_1 / 1024 + {}^{10}c_2 / 1024 + {}^{10}c_3 / 1024$

$$= \frac{1 + 10 + 45 + 120}{1024}$$

= 176 / 1024

= 11/64

Example 16.2: If 15 dates are selected at random, what is the probability of getting two Sundays?

Solution: If X denotes the number at Sundays, then it is obvious that X follows binomial distribution with parameter n = 15 and p = probability of a Sunday in a week = 1/7 and q = 1 – p = 6 / 7.

Then $f(x) = {}^{15}c_x (1/7)^x . (6/7)^{15-x}$.

for x = 0, 1, 2,……….. 15.

Hence the probability of getting two Sundays

= f(2)

$= {}^{15}c_2 (1/7)^2 . (6/7)^{15-2}$

$$= \frac{105 \times 6^{13}}{7^{15}}$$

≅ 0.29

Example 16.3: The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is the probability that out of 5 workmen, 3 or more will contract the disease?

Solution: Let X denote the number of workmen in the sample. X follows binomial with parameters n = 5 and p = probability that a workman suffers from the occupational disease = 0.1

Hence q = 1 – 0.1 = 0.9.

Thus $f(x) = {}^5c_x \cdot (0.1)^x \cdot (0.9)^{5-x}$

For x = 0, 1, 2,…….,5.

The probability that 3 or more workmen will contract the disease

$= P(x \geq 3)$

$= f(3) + f(4) + f(5)$

$= {}^5c_3 (0.1)^3 (0.9)^{5-3} + {}^5c_4 (0.1)^4 . (0.9)^{5-4} + {}^5c_5 (0.1)^5$

$= 10 \times 0.001 \times 0.81 + 5 \times 0.0001 \times 0.9 + 1 \times 0.00001$

$= 0.0081 + 0.00045 + 0.00001$

$\cong 0.0086.$

Example 16.4: Find the probability of a success for the binomial distribution satisfying the following relation 4 P (x = 4) = P (x = 2) and having the parameter n as six.

Solution: We are given that n = 6. The probability mass function of x is given by

$f(x) = {}^nc_x \, p^x \, q^{n-x}$

$= {}^6c_x \, p^x \, q^{n-x}$

for x = 0, 1, …… ,6.

Thus $P(x = 4) = f(4)$:

$= {}^6c_4 \, p^4 \, q^{6-4}$

$= 15 \, p^4 \, q^2$

and $P(x = 2) = f(2)$

$= {}^6c_2 \, p^2 q^{6-2}$

$= 15p^2 \, q^4$

Hence $4P(x = 4) = P(x = 2)$

$\Rightarrow \quad 60 \, p^4 q^2 = 15 \, p^2 q^4$

$\Rightarrow \quad 15 \, p^2 q^2 (4p^2 - q^2) = 0$

$\Rightarrow \quad 4p^2 - q^2 = 0$ (as $p \neq 0, q \neq 0$)

$\Rightarrow \quad 4p^2 - (1 - p)^2 = 0$ (as $q = 1 - p$)

$\Rightarrow \quad (2p + 1 - p) = 0$ or $(2p - 1 + p) = 0$

$\Rightarrow \quad p = -1$ or $p = 1/3$

Thus $p = 1/3$ (as $p \neq -1$)

Example 16.5: Find the binomial distribution for which mean and standard deviation are 6 and 2 respectively.

Solution: Let x ~ B (n, p)

Given that mean of x = np = 6 … (1)

and SD of x = 2

$\Rightarrow$ variance of x = npq = 4 ….. (2)

Dividing (2) by (1), we get $q = \frac{2}{3}$

Hence $p = 1 - q = \frac{1}{3}$

Replacing p by $\frac{1}{3}$ in equation (1), we get $n \times \frac{1}{3} = 6$

$\Rightarrow n = 18$

Thus the probability mass function of x is given by

$f(x) = {}^{n}c_{x}\, p^{x} q^{n-x}$

$= {}^{18}c_{x}\, (1/3)^{x} \cdot (2/3)^{18-x}$

for x = 0, 1, 2,…… ,18

Example 16.6: Fit a binomial distribution to the following data:

x:	0	1	2	3	4	5
f:	3	6	10	8	3	2

Solution: In order to fit a theoretical probability distribution to an observed frequency distribution it is necessary to estimate the parameters of the probability distribution. There are several methods of estimating population parameters. One rather, convenient method is 'Method of Moments'. This comprises equating p moments of a probability distribution to p moments of the observed frequency distribution, where p is the number of parameters to be estimated. Since n = 5 is given, we need estimate only one parameter p. We equate the first moment about origin i.e. AM of the probability distribution to the AM of the given distribution and estimate p.

i.e. $n\hat{p} = \bar{x}$

$\Rightarrow \hat{p} = \frac{\bar{x}}{n}$ ($\hat{p}$ is read as p hat)

The fitted binomial distribution is then given by

$f(x) = {}^{n}c_{x}\, \hat{p}^{x} (1 - \hat{p})^{n-x}$

For x = 0, 1, 2, …… n

On the basis of the given data, we have

$$\bar{x} = \sum \frac{f_i x_i}{N}$$

$$= \frac{3\times0+6\times1+10\times2+8\times3+3\times4+2\times5}{3+6+10+8+3+2} = 2.25$$

Thus $\hat{p} = \bar{x}/n = \frac{2.25}{n} = 0.45$

and $\hat{q} = 1 - \hat{p} = 0.55$

The fitted binomial distribution is

$f(x) = {}^5c_x\,(0.45)^x\,(0.55)^{5-x}$

For x = 0, 1, 2, 3, 4, 5.

Table 16.1

Fitting Binomial Distribution to an Observed Distribution

X	f (x)	Expected frequency	Observed frequency
	$= {}^5c_x\,(0.4)^x\,(0.6)^{5-x}$	Nf (x) = 32 f (x)	
0	0.07776	2.49 ≅ 3	3
1	0.25920	8.29 ≅ 8	6
2	0.34560	11.06 ≅ 11	10
3	0.23040	7.37 ≅ 7	8
4	0.07680	2.46 ≅ 3	3
5	0.01024	0.33 ≅ 0	2
Total	1.000 00	32	32

A look at Table 16.1 suggests that the fitting of binomial distribution to the given frequency distribution is satisfactory.

Example 16.7: 6 coins are tossed 512 times. Find the expected frequencies of heads. Also, compute the mean and SD of the number of heads.

Solution: If x denotes the number of heads, then x follows binomial distribution with parameters

n = 6 and p = prob. of a head = ½, assuming the coins to be unbiased. The probability mass function of x is given by

$f(x) = {}^6c_x\,(1/2)^x.\,(1/2)^{6-x}$

$= {}^6c_x/2^6$

for x = 0, 1, …..6.

The expected frequencies are given by Nf (x).

Table 16.2

Finding Expected Frequencies when 6 coins are tossed 512 times

x	f (x)	Nf (x) Expected frequency	x f (x)	x^2f (x)
0	1/64	8	0	0
1	6/64	48	6/64	6/64
2	15/64	120	30/64	60/64
3	20/64	160	60/64	180/64
4	15/64	120	60/64	240/64
5	6/64	48	30/64	150/64
6	1/64	8	6/64	36/64
Total	1	512	3	10.50

Thus mean = $\mu = \sum_x x.f(x) = 3$

$$E(x^2) = \sum_x x^2.f(x) = 10.50$$

$$\text{Thus } \sigma^2 = \sum_x x^2.f(x) - \mu^2$$

$$= 10.50 - 3^2 = 1.50$$

$$\therefore \text{SD} = \sigma = \sqrt{1.50} \cong 1.22$$

Applying formula for mean and SD, we get

$\mu = np = 6 \times 1/2 = 3$

and $\sigma = \sqrt{npq} = \sqrt{6 \times \frac{1}{2} \times \frac{1}{2}} = \sqrt{1.50} \cong 1.22$

Example 16.8: An experiment succeeds thrice as after it fails. If the experiment is repeated 5 times, what is the probability of having no success at all ?

Solution: Denoting the probability of a success and failure by p and q respectively, we have,

$p = 3q$

$\Rightarrow p = 3(1 - p)$

$\Rightarrow p = 3/4$

$\therefore q = 1 - p = 1/4$

when n = 5 and p = 3/4, we have

$f(x) = {}^5c_x\,(3/4)^x\,(1/4)^{5-x}$

for n = 0, 1, ... , 5.

So probability of having no success

$= f(0)$

$= {}^5c_0\,(3/4)^0\,(1/4)^{5-0}$

$= 1/1024$

Example 16.9: What is the mode of the distribution for which mean and SD are 10 and $\sqrt{5}$ respectively.

Solution: As given np = 10 .... (1)

and $\sqrt{npq} = \sqrt{5}$

$\Rightarrow$ npq = 5 .... (2)

on solving (1) and (2), we get n = 20 and p = 1/2

Hence mode = Largest integer contained in (n+1)p

= Largest integer contained in (20+1) × 1/2

= Largest integer contained in 10.50

= 10.

Example 16.10: If x and y are 2 independent binomial variables with parameters 6 and 1/2 and 4 and 1/2 respectively, what is $P(x + y \geq 1)$?

Solution: Let z = x + y.

It follows that z also follows binomial distribution with parameters

(6 + 4) and 1/2

i.e. 10 and 1/2

Hence $P(z \geq 1)$

$= 1 - P(z < 1)$

$= 1 - P(z = 0)$

$= 1 - {}^{10}c_0\,(1/2)^0.\,(1/2)^{10-0}$

$= 1 - 1/2^{10}$

$= 1023/1024$

16.3 POISSON DISTRIBUTION

Poisson distribution is a theoretical discrete probability distribution which can describe many processes. Simon Denis Poisson of France introduced this distribution way back in the year 1837.

Poisson Model

Let us think of a random experiment under the following conditions:

I. The probability of finding success in a very small time interval (t, t + dt) is kt, where k (>0) is a constant.

II. The probability of having more than one success in this time interval is very low.

III. The probability of having success in this time interval is independent of t as well as earlier successes.

The above model is known as Poisson Model. The probability of getting x successes in a relatively long time interval T containing m small time intervals t i.e. T = mt. is given by

$$\frac{e^{-kt}.(kt)^x}{x!}$$

for x = 0, 1, 2, ... ∞ ...... (16.7)

Taking kT = m, the above form is reduced to

$$\frac{e^{-m}.m^x}{x!}$$

for x = 0, 1, 2, ... ∞ .... (16.8)

Definition of Poisson Distribution

A random variable X is defined to follow Poisson distribution with parameter λ, to be denoted by X ~ P (m) if the probability mass function of x is given by

$$f(x) = P(X = x) = \frac{e^{-m}.m^x}{x!} \text{ for } x = 0, 1, 2, \ldots \infty$$

$$= 0 \quad \text{otherwise} \qquad \ldots\ldots (16.9)$$

Here e is a transcendental quantity with an approximate value as 2.71828.

It is wiser to remember the following important points in connection with Poisson distribution:

(i) Since $e^{-m} = 1/e^m > 0$, whatever may be the value of m, m > 0, it follows that f (x) $\geq$ 0 for every x.

Also it can be established that $\sum_x$ f(x) = 1 i.e. f(0) + f(1) + f(2) +....... = 1.... (16.10)

(ii) Poisson distribution is known as a uniparametric distribution as it is characterised by only one parameter m.

(iii) The mean of Poisson distribution is given by m i,e μ = m. (16.11)

(iv) The variance of Poisson distribution is given by σ^2 = m (16.12)

(v) Like binomial distribution, Poisson distribution could be also unimodal or bimodal depending upon the value of the parameter m.

We have μ_0 = The largest integer contained in m if m is a non-integer

= m and m–1 if m is an integer .... (16.13)

(vi) Poisson approximation to Binomial distribution

If n, the number of independent trials of a binomial distribution, tends to infinity and p, the probability of a success, tends to zero, so that m = np remains finite, then a binomial distribution with parameters n and p can be approximated by a Poisson distribution with parameter m (= np).

In other words when n is rather large and p is rather small so that m = np is moderate then

β (n, p) $\cong$ P (m). .... (16.14)

(vii) Additive property of Poisson distribution

If X and y are two independent variables following Poisson distribution with parameters m_1 and m_2 respectively, then Z = X + Y also follows Poisson distribution with parameter $(m_1 + m_2)$.

i.e. if X ~ P (m_1)

and Y ~ P (m_2)

and X and Y are independent, then

Z = X + Y ~ P $(m_1 + m_2)$ .... (16.15)

Application of Poisson distribution

Poisson distribution is applied when the total number of events is pretty large but the probability of occurrence is very small. Thus we can apply Poisson distribution, rather profitably, for the following cases:

a) The distribution of the no. of printing mistakes per page of a large book.
b) The distribution of the no. of road accidents on a busy road per minute.
c) The distribution of the no. of radio-active elements per minute in a fusion process.
d) The distribution of the no. of demands per minute for health centre and so on.

Example 16.11: Find the mean and standard deviation of x where x is a Poisson variate satisfying the condition P (x = 2) = P (x = 3).

Solution: Let x be a Poisson variate with parameter m. The probability max function of x is then given by

$$f(x) = \frac{e^{-m} . m^x}{x!} \qquad \text{for } x = 0, 1, 2, \ldots\ldots \infty$$

now, P (x = 2) = P (x = 3)

$\Rightarrow$ f(2) = f(3)

$$\Rightarrow \frac{e^{-m}.m^2}{2!} = \frac{e^{-m}.m^3}{3!}$$

$$\Rightarrow \frac{e^{-m}.m^2}{2}(1 - m/3) = 0$$

$\Rightarrow 1 - m / 3 = 0$ (as $e^{-m} > 0, m > 0$)

$\Rightarrow m = 3$

Thus the mean of this distribution is m = 3 and standard deviation = $\sqrt{3} \cong 1.73$.

Example 16.12: The probability that a random variable x following Poisson distribution would assume a positive value is $(1 - e^{-2.7})$. What is the mode of the distribution?

Solution: If x ~ P (m), then its probability mass function is given by

$$f(x) = \frac{e^{-m}.m^2}{x!} \quad \text{for } x = 0, 1, 2, \ldots\ldots\ldots\ \infty$$

The probability that x assumes a positive value

$= P (x > 0)$

$= 1 - P (x \le 0)$

$= 1 - P (x = 0)$

$= 1 - f(0)$

$= 1 - e^{-m}$

As given,

$1 - e^{-m} = 1 - e^{-2.7}$

$\Rightarrow e^{-m} = e^{-2.7}$

$\Rightarrow m = 2.7$

Thus μ_0 = largest integer contained in 2.7

= 2

Example 16.13: The standard deviation of a Poisson variate is 1.732. What is the probability that the variate lies between –2.3 to 3.68?

Solution: Let x be a Poisson variate with parameter m.

Then SD of x is $\sqrt{m}$.

As given $\sqrt{m} = 1.732$

$\Rightarrow m = (1.732)^2 \cong 3.$

The probability that x lies between –2.3 and 3.68

$= P(-2.3 < x < 3.68)$

$= f(0) + f(1) + f(2) + f(3)$ (As x can assume 0, 1, 2, 3, 4 .....)

$$= \frac{e^{-3}.3^0}{0!} + \frac{e^{-3}.3^1}{1!} + \frac{e^{-3}.3^2}{2!} + \frac{e^{-3}.3^3}{3!}$$

$= e^{-3}(1 + 3 + 9/2 + 27/6)$

$= 13e^{-3}$

$$= \frac{13}{e^3}$$

$$= \frac{13}{(2.71828)^3} \text{ (as } e = 2.71828)$$

$\cong 0.65$

Example 16.14: X is a Poisson variate satisfying the following relation:

$P(X = 2) = 9P(X = 4) + 90P(X = 6)$.

What is the standard deviation of X?

Solution: Let X be a Poisson variate with parameter m. Then the probability mass function of X is

$$P(X = x) = f(x) = \frac{e^{-m}.m^x}{x!} \text{ for } x = 0, 1, 2, \ldots.. \infty$$

Thus $P(X = 2) = 9P(X = 4) + 90P(X = 6)$

$\Rightarrow f(2) = 9f(4) + 90f(6)$

$$\Rightarrow \frac{e^{-m}m^2}{2!} = \frac{9e^{-m}.m^4}{4!} + \frac{90.e^{-m}m^6}{6!}$$

$$\Rightarrow \frac{e^{-m}m^2}{2}\left(\frac{90m^4}{360} + \frac{9m^2}{12} - 1\right) = 0$$

$$\Rightarrow \frac{e^{-m}m^2}{8}(m^4 + 3m^2 - 4) = 0$$

$\Rightarrow e^{-m}.m^2(m^2 + 4)(m^2 - 1) = 0$

$\Rightarrow m^2 - 1 = 0$ (as $e^{-m} > 0$ $m > 0$ and $m^2 + 4 \neq 0$)

$\Rightarrow m = 1$ (as $m > 0$, $m \neq -1$)

Thus the standard deviation of X is $\sqrt{1} = 1$

Example 16.15: Between 9 and 10 AM, the average number of phone calls per minute coming into the switchboard of a company is 4. Find the probability that during one particular minute, there will be,

1. no phone calls
2. at most 3 phone calls (given $e^{-4} = 0.018316$)

Solution: Let X be the number of phone calls per minute coming into the switchboard of the company. We assume that X follows Poisson distribution with parameters m = average number of phone calls per minute = 4.

1. The probability that there will be no phone call during a particular minute

$= P(X = 0)$

$$= \frac{e^{-4}.4^0}{0!}$$

$= e^{-4}$

$= 0.018316$

2. The probability that there will be at most 3 phone calls

$= P(X \le 3)$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= \frac{e^{-4}.4^0}{0!} + \frac{e^{-4}.4^1}{1!} + \frac{e^{-4}.4^2}{2!} + \frac{e^{-4}.4^3}{3!}$$

$= e^{-4}(1 + 4 + 16/2 + 64/6)$

$= e^{-4} \times 71/3$

$= 0.018316 \times 71/3$

$\cong 0.43$

Example 16.16: If 2 per cent of electric bulbs manufactured by a company are known to be defectives, what is the probability that a sample of 150 electric bulbs taken from the production process of that company would contain

1. exactly one defective bulb?
2. more than 2 defective bulbs?

Solution: Let x be the number of bulbs produced by the company. Since the bulbs could be either defective or non-defective and the probability of bulb being defective remains the same, it follows that x is a binomial variate with parameters n = 150 and p = probability of a bulb being defective = 0.02. However since n is large and p is very small, we can approximate this binomial distribution with Poisson distribution with parameter m = np = 150 x 0.02 = 3.

1. The probability that exactly one bulb would be defective

$= P(X = 1)$

$= \dfrac{e^{-3}.3^{1}}{1!}$

$= e^{-3} \times 3$

$= \dfrac{3}{e^{3}}$

$= 3/(2.71828)^{3}$

$\cong 0.15$

2. The probability that there would be more than 2 defective bulbs

$= P(X > 2)$

$= 1 - P(X \le 2)$

$= 1 - [f(0) + f(1) + f(2)]$

$= 1 - \left(\dfrac{e^{-3} \times 3^{0}}{0!} + \dfrac{e^{-3} \times 3^{1}}{1!} + \dfrac{e^{-3} \times 3^{2}}{2!}\right)$

$= 1 - 8.5 \times e^{-3}$

$= 1 - 0.4232$

$= 0.5768 \cong 0.58$

Example 16.17: The manufacturer of a certain electronic component is certain that two per cent of his product is defective. He sells the components in boxes of 120 and guarantees that not more than two per cent in any box will be defective. Find the probability that a box, selected at random, would fail to meet the guarantee? Given that $e^{-2.40} = 0.0907$.

Solution: Let x denote the number of electric components. Then x follows binomial distribution with n = 120 and p = probability of a component being defective = 0.02. As before since n is quite large and p is rather small, we approximate the binomial distribution with parameters n and p by a Poisson distribution with parameter m = n.p = 120 × 0.02 = 2.40. Probability that a box, selected at random, would fail to meet the specification = probability that a sample of 120 items would contain more than 2.40 defective items.

$= P(X > 2.40)$

$= 1 - P(X \le 2.40)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$

$= 1 - [e^{-2.40} + e^{-2.40} \times 2.4 + e^{-2.40} \times \left(\dfrac{2.40}{2}\right)^{2}]$

$$= 1 - e^{-2.40}\left(1 + 2.40 + \frac{(2.40)^2}{2}\right)$$

$= 1 - 0.0907 \times 6.28$

$\cong 0.43$

Example 16.18: A discrete random variable x follows Poisson distribution. Find the values of

(i) P (X = at least 1)

(ii) $P(X \le 2 / X \ge 1)$

You are given E (x) = 2.20 and $e^{-2.20} = 0.1108$.

Solution: Since X follows Poisson distribution, its probability mass function is given by

$$f(x) = \frac{e^{-m}.m^x}{x!} \text{ for } x = 0, 1, 2, \ldots\ldots \infty$$

(i) P (X = at least 1)

$= P(X \ge 1)$

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - e^{-m}$

$= 1 - e^{-2.20}$ (as E (x) = m = 2.20, given)

$= 1 - 0.1108$ (as $e^{-2.20} = 0.1108$ as given)

$\cong 0.89$.

(ii) $P(x \le 2 / x \ge 1)$

$$= P\frac{[(X \le 2) \cap (X \ge 1)]}{P(X \ge 1)} \qquad \left(\text{as } P(A/B) = P\frac{(A \cap B)}{P(B)}\right.$$

$$= \frac{P(X=1) + P(X=2)}{1 - P(X<1)}$$

$$= \frac{f(1) + f(2)}{1 - f(0)}$$

$$= \frac{e^{-m}.m + e^{-m}.m^2/2}{1 - e^{-m}}$$

$$= \frac{e^{-2.20} \times 2.2 + e^{-2.20} \times (2.20)^2/2}{1 - e^{-2.20}} \qquad (\because m = 2.2)$$

$$= \frac{0.5119}{0.8892}$$

$$\cong 0.58$$

Fitting a Poisson distribution

As explained earlier, we can apply the method of moments to fit a Poisson distribution to an observed frequency distribution. Since Poisson distribution is uniparametric, we equate m, the parameter of Poisson distribution, to the arithmetic mean of the observed distribution and get the estimate of m.

i.e. $\hat{m} = \bar{x}$

The fitted Poisson distribution is then given by

$$\hat{f}(x) = \frac{e^{-\hat{m}} . (\hat{m})^x}{x!} \qquad \text{for } x = 0, 1, 2 ....\infty$$

Example 16.19: Fit a Poisson distribution to the following data :

Number of death:	0	1	2	3	4
Frequency:	122	46	23	8	1

(Given that $e^{-0.6} = 0.5488$)

Solution: The mean of the observed frequency distribution is

$$\bar{x} = \frac{\sum f_i x_i}{N}$$

$$= - \frac{122 \times 0 + 46 \times 1 + 23 \times 2 + 8 \times 3 + 1 \times 4}{122 + 46 + 23 + 8 + 1}$$

$$= \frac{120}{200}$$

$$= 0.6$$

Thus $\hat{m} = 0.6$

Hence $\hat{f}(0) = e^{-\hat{m}} = e^{-0.6} = 0.5488$

$$\hat{f}(1) = \frac{e^{-\hat{m}} \times m}{1!} = 0.6 \times e^{-0.6} = 0.3293$$

$$\frac{(0.6)^2}{2!} \times 0.5488 = 0.0988$$

$$\frac{(0.6)^3}{3!} \times 0.5488 = 0.0198$$

Lastly $P(X \geq 4) = 1 - P(X < 4)$.

Table 16.3

Fitting Poisson Distribution to an Observed Frequency Distribution of Deaths

X	f (x)	Expected frequency N × f (x)	Observed frequency
0	0.5488	109.76 = 110	122
1	0.6 x 0.5488 = 0.3293	65.86 = 65	46
2	$(0.6)^2/2$ x 0.5488 = 0.0988	19.76 = 20	23
3	$(0.6)^3/3$ x 0.5488 = 0.0198	3.96 = 4	8
4 or more	0.0033 (By subtraction)	0.66 = 1	1
Total	1	200	200

16.4 NORMAL OR GAUSSIAN DISTRIBUTION

The two distributions discussed so far, namely binomial and Poisson, are applicable when the random variable is discrete. In case of a continuous random variable like height or weight, it is impossible to distribute the total probability among different mass points because between any two unequal values, there remains an infinite number of values. Thus a continuous random variable is defined in term of its probability density function f (x), provided, of course, such a function really exists, f (x) satisfies the following condition:

$$f(x) \geq 0 \text{ for } x \in (-\infty, \infty)$$

and $$\int_{-\infty}^{+\infty} f(x) = 1.$$

The most important and universally accepted continuous probability distribution is known as normal distribution. Though many mathematicians like De-Moivre, Laplace etc. contributed towards the development of normal distribution, Karl Gauss was instrumental for deriving normal distribution and as such normal distribution is also referred to as Gaussian Distribution.

A continuous random variable x is defined to follow normal distribution with parameters μ and σ^2, to be denoted by

X ~ N (μ, σ ²) ………………. (16.16)

If the probability density function of the random variable x is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}.e^{-(\bar{x}-u)^2/2\sigma^2}$$

for $-\infty < x < \infty$ ……… (16.17)

where μ and σ are constants, and $\sigma > 0$

Some important points relating to normal distribution are listed below:

(a) The name Normal Distribution has its origin some two hundred years back as the then mathematician were in search for a normal model that can describe the probability distribution of most of the continuous random variables.

(b) If we plot the probability function y = f (x), then the curve, known as probability curve, takes the following shape:

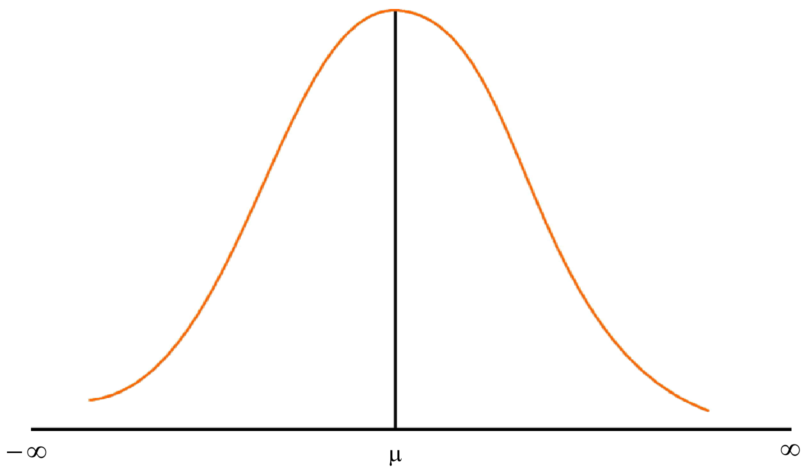


Fig. 16.1

Showing Normal Probability Curve

A quick look at figure 16.1 reveals that the normal curve is bell shaped and has one peak, which implies that the normal distribution has one unique mode. The line drawn through x = μ has divided the normal curve into two parts which are equal in all respect. Such a curve is known as symmetrical curve and the corresponding distribution is known as symmetrical distribution. Thus, we find that the normal distribution is symmetrical about x = μ. It may also be noted that the binomial distribution is also symmetrical about p = 0.5. We next note that the two tails of the normal curve extend indefinitely on both sides of the curve and both the left and right tails never touch the horizontal axis. The total area of the normal curve or for that any probability curve is taken to be unity i.e. one. Since the vertical line drawn through x = μ divides the curve into two equal halves, it automatically follows that,

The area between – ∞ to μ = the area between μ to ∞ = 0.5

When the mean is zero, we have

the area between $-\infty$ to 0 = the area between 0 to ∞ = 0.5

(c) If we take $\mu = 0$ and $\sigma = 1$ in (18.17), we have

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{for } -\infty < z < \infty \text{ .... (16.18)}$$

The random variable z is known as standard normal variate (or variable) or standard normal deviate. The probability that a standard normal variate X would take a value less than or equal to a particular value say X = x is given by

$\phi(x) = p(X \le x)$ .... (16.19)

$\phi(x)$ is known as the cumulative distribution function.

We also have $\phi(0) = P(X \le 0)$ = Area of the standard normal curve between $-\infty$ and 0 = 0.5 .... (16.20)

(d) **The normal distribution is known as biparametric distribution as it is characterised by two parameters μ and σ^2. Once the two parameters are known, the normal distribution is completely specified.**

Properties of Normal Distribution

1. Since $\pi = 22/7$, $e^{-\theta} = 1 / e^{\theta} > 0$, whatever θ may be,

   it follows that $f(x) \ge 0$ for every x.

   It can be shown that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

2. The mean of the normal distribution is given by μ. Further, since the distribution is symmetrical about $x = \mu$, it follows that the mean, median and mode of a normal distribution coincide, all being equal to μ.

3. The standard deviation of the normal distribution is given by σ.

   Mean deviation of normal distribution is $\sigma\sqrt{\frac{2}{\pi}}$

$$\sigma\sqrt{\frac{2}{\pi}} \cong 0.8\sigma \text{ .... (16.21)}$$

The first and third quartiles are given by

$Q_1 = \mu - 0.675\,\sigma$ .... (16.22)

and $Q_3 = \mu + 0.675\,\sigma$ .... (16.23)

so that, quartile deviation = $0.675\,\sigma$ .... (16.24)

4. The normal distribution is symmetrical about x = μ. As such, its skewness is zero i.e. the normal curve is neither inclined move towards the right (negatively skewed) nor towards the left (positively skewed).

5. The normal curve y = f (x) has two points of inflexion to be given by $x = \mu - \sigma$ and $x = \mu + \sigma$ i.e. at these two points, the normal curve changes its curvature from concave to convex and from convex to concave.

6. If $x \sim N(\mu, \sigma^2)$ then $z = x - \mu/\sigma \sim N(0, 1)$, z is known as standardised normal variate or normal deviate.

   We also have $P(z \le k) = \phi(k)$ .... (16.25)

   The values of ϕ(k) for different k are given in a table known as "Biometrika."

   Because of symmetry, we have

   $\phi(-k) = 1 - \phi(k)$ .... (16.26)

   We can evaluate the different probabilities in the following manner:

   $$P(x < a) = P\left[\frac{x-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right]$$

   $= P(z < k), (k = a - \mu/\sigma)$

   $= \phi(k)$ .... (16.27)

   Also $P(x \le a) = P(x < a)$ as x is continuous.

   $P(x > b) = 1 - P(x \le b)$

   $= 1 - \phi(b - \mu/\sigma)$ .... (16.28)

   and $P(a < x < b) = \phi(b - \mu/\sigma) - \phi(a - \mu/\sigma)$ .... (16.29)

   ordinate at x = a is given by

   $(1/\sigma)\, \phi(a - \mu/\sigma)$ .... (16.30)

   Also, $\phi(-k) = \phi(k)$ .... (16.31)

   The values of ϕ (k) for different k are also provided in the Biometrika Table.

7. Area under the normal curve is shown in the following figure :

$\mu - 3\sigma$	$\mu - 2\sigma$	$\mu - \sigma$	$x = \mu$	$\mu + \sigma$	$\mu + 2\sigma$	$\mu + 3\sigma$
(z = –3)	(z = –2)	(z = –1)	(z = 0)	(z = 1)	(z = 2)	(z = 3)

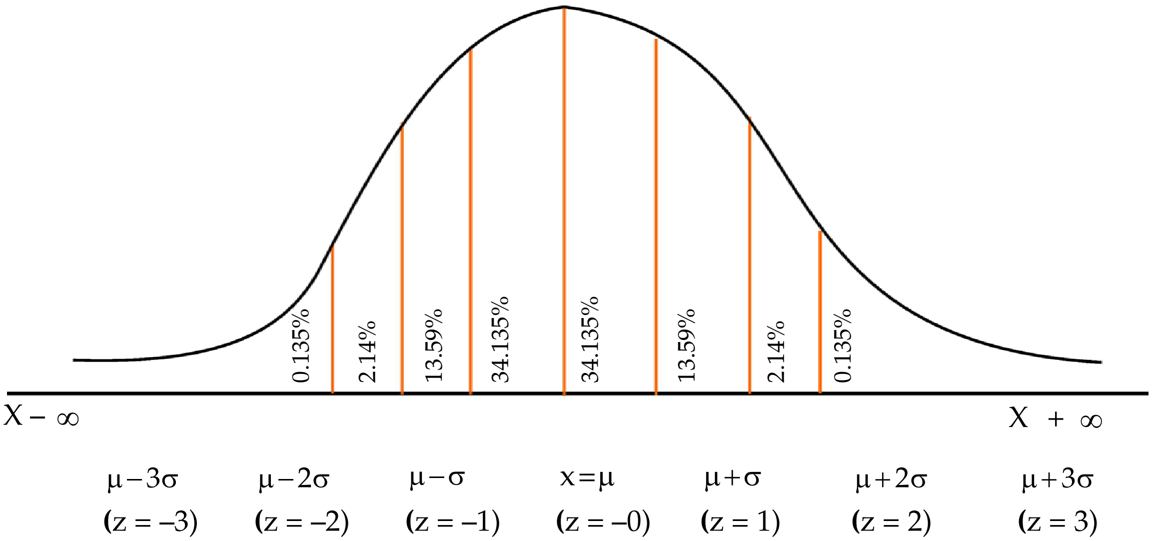


Fig. 16.2

Area Under Normal Curve

From this figure, we find that

$P(\mu - \sigma < x < \mu) = P(\mu < x < \mu + \sigma) = 0.34135$

or alternatively, $P(-1 < z < 0) = P(0 < z < 1) = 0.34135$

$P(\mu - 2\sigma < x < \mu) = P(\mu < x < \mu + 2\sigma) = 0.47725$

i.e. $P(-2 < z < 1) = P(1 < z < 2) = 0.47725$

$P(\mu - 3\sigma < x < \mu) = P(\mu < x < \mu + 3\sigma) = 0.49865$

i.e. $P(-3 < z < 0) = P(0 < z < 3) = 0.49865$

……. (16.32)

combining these results, we have

$P(\mu - \sigma < x < \mu + \sigma) = 0.6828$

$\Rightarrow P(-1 < z < 1) = 0.6828$

$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9546$

$\Rightarrow P(-2 < z < 2) = 0.9546$

and $P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$

$\Rightarrow P(-3 < z < 3) = 0.9973.$

………… (16.33)

We note that 99.73 per cent of the values of a normal variable lies between $(\mu - 3\sigma)$ and $(\mu + 3\sigma)$. Thus the probability that a value of x lies outside that limit is as low as 0.0027.

8. If x and y are independent normal variables with means and standard deviations as μ_1 and μ_2 and σ_1, and σ_2 respectively, then z = x + y also follows normal distribution with mean $(\mu_1 + \mu_2)$ and SD = $\sqrt{\sigma_1^2+\sigma_2^2}$ respectively.

i.e. If $x \sim N(\mu_1, \sigma_1^2)$

and $y \sim N(\mu_2, \sigma_2^2)$ and x and y are independent,

then $z = x + y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

…… (16.34)

Applications of Normal Distribution

The applications of normal distribution is not restricted to statistics only. Many science subjects, social science subjects, management, commerce etc. find many applications of normal distributions. Most of the continuous variables like height, weight, wage, profit etc. follow normal distribution. If the variable under study does not follow normal distribution, a simple transformation of the variable, in many a case, would lead to the normal distribution of the changed variable. When n, the number of trials of a binomial distribution, is large and p, the probability of a success, is moderate i.e. neither too large nor too small then the binomial distribution, also, tends to normal distribution. Poisson distribution, also for large value of m approaches normal distribution. Such transformations become necessary as it is easier to compute probabilities under the assumption of a normal distribution. Not only the distribution of discrete random variable, the probability distributions of t, chi-square and F also tend to normal distribution under certain specific conditions. In order to infer about the unknown universe, we take recourse to sampling and inferences regarding the universe is made possible only on the basis of normality assumption. Also the distributions of many a sample statistic approach normal distribution for large sample size.

Example 16.20: For a random variable x, the probability density function is given by

$$f(x) = \frac{e^{-(x-4)^2}}{\sqrt{\pi}}$$

for $-\infty < x < \infty$.

Identify the distribution and find its mean and variance.

Solution: The given probability density function may be written as

$$f(x) = \frac{1}{1/\sqrt{2}\times\sqrt{2\pi}} e^{-(x-4)^2/2\times1/2} \quad \text{for } -\infty < x < \infty$$

$$= \frac{1}{\sigma\times\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty$$

with $\mu = 4$ and $\sigma^2 = ½$

Thus the given probability density function is that of a normal distribution with $\mu = 4$ and variance $= \frac{1}{2}$.

Example 16.21: If the two quartiles of a normal distribution are 47.30 and 52.70 respectively, what is the mode of the distribution? Also find the mean deviation about median of this distribution.

Solution: The 1st and 3rd quartiles of $N(\mu, \sigma^2)$ are given by $(\mu - 0.675\sigma)$ and $(\mu + 0.675\sigma)$ respectively. As given,

$\mu - 0.675\sigma = 47.30$ …. (1)

$\mu + 0.675\sigma = 52.70$ …. (2)

Adding these two equations, we get

$2\mu = 100$ or $\mu = 50$

Thus Mode = Median = Mean = 50. Also $\sigma = 4$.

Also Mean deviation about median

= mean deviation about mode

= mean deviation about mean

$\cong 0.80\sigma$

= 3.20

Example 16.22: Find the points of inflexion of the normal curve

$$f(x) = \frac{1}{4\sqrt{2\pi}} . e^{-(x-10)^2/32}$$

for $-\infty < x < \infty$

Solution: Comparing f (x) to the probability densities function of a normal variable x , we find that $\mu = 10$ and $\sigma = 4$.

The points of inflexion are given by

$\mu - \sigma$ and $\mu + \sigma$

i.e. 10 – 4 and 10 + 4

i.e. 6 and 14.

Example 16.23: If x is a standard normal variable such that

$P(0 \le x \le b) = a$, what is the value of $P(|x| \ge b)$?

Solution: $P((x) \ge b)$

$= 1 - P(|x| \le b)$

$= 1 - P(-b \le x \le b)$

$= 1 - [P(0 \le x \le b) - P(-b \le x \le 0)]$

$= 1 - [\, P(0 \le x \le b) + P(0 \le x \le b)\,]$

$= 1 - 2a$

Example 16.24: X follows normal distribution with mean as 50 and variance as 100. What is $P(x \ge 60)$? Given $\phi(1) = 0.8413$

Solution: We are given that $x \sim N(\mu, \sigma^2)$ where

$\mu = 50$ and $\sigma^2 = 100 => \sigma = 10$

Thus $P(x \ge 60)$

$= 1 - P(x \le 60)$

$$= 1 - P\left(\frac{x-50}{10} \le \frac{60-50}{10}\right) = 1 - P(z \le 1)$$

$= 1 - \phi(1)$ (From 16.26)

$= 1 - 0.8413$

$\cong 0.16$

Example 16.25: If a random variable x follows normal distribution with mean as 120 and standard deviation as 40, what is the probability that $P(x \le 150 / x > 120)$?

Given that the area of the normal curve between $z = 0$ to $z = 0.75$ is 0.2734.

Solution: $P(x \le 150 / x > 120)$

$$= \frac{P(120 < x \le 150)}{P(x > 120)}$$

$$= \frac{P(120 < x \le 150)}{1 - P(x \le 120)}$$

$$= \frac{P\left(\frac{120-120}{40} \le \frac{x-120}{40} \le \frac{150-120}{40}\right)}{1 - P\left(\frac{x-120}{40} \le \frac{120-120}{40}\right)}$$

$$= \frac{P(0 < z \le 0.75)}{1 - P(z \le 0)}$$

$$= \frac{\phi(0.75) - \phi(0)}{1 - \phi(0)} \qquad \text{(From 16.29)}$$

$$= \frac{0.7734 - 0.50}{1 - 0.50}$$

$\cong 0.55$ (ϕ (0.75) = Area of the normal curve between $z = -\infty$ to $z = 0.75$ = area between $-\infty$ to 0 + Area between 0 to 0.75 = 0.50 + 0.2734 = 0.7734)

Example 16.26: X is a normal variable with mean = 25 and SD 10. Find the value of b such that the probability of the interval [2 5, b] is 0.4772 given ϕ (2) = 0.9772.

Solution: We are given that $x \sim N(\mu, \sigma^2)$ where $\mu = 25$ and $\sigma = 10$

and P [25 < x < b] = 0.4772

$$\Rightarrow \left[\frac{25-25}{10} < \frac{x-25}{10} < \frac{b-25}{10}\right] = 0.4772$$

$$\Rightarrow P\left[0 < z < \frac{b-25}{10}\right] = 0.4772$$

$$\Rightarrow \phi\left(\frac{b-25}{10}\right) - \phi(0) = 0.4772$$

$$\Rightarrow \phi\left(\frac{b-25}{10}\right) - 0.50 = 0.4772$$

$$\Rightarrow \phi\left(\frac{b-25}{10}\right) = 0.9772$$

$$\Rightarrow \phi\,\frac{b-25}{10} = \phi(2) \qquad \text{(as given)}$$

$$\Rightarrow \frac{b-25}{10} = 2$$

$\Rightarrow$ b = 25 + 2 × 10 = 45.

Example 16.27: In a sample of 500 workers of a factory, the mean wage and SD of wages are found to be ₹ 500 and ₹ 48 respectively. Find the number of workers having wages:

(i) more than ₹ 600

(ii) less than ₹ 450

(iii) between ₹ 548 and ₹ 600.

Solution: Let X denote the wage of the workers in the factory. We assume that X is normally distributed with mean wage as ₹ 500 and standard deviation of wages as ₹ 48 respectively.

(i) Probability that a worker selected at random would have wage more than ₹ 600

$= P (X > 600)$

$= 1 – P (X \le 600)$

$$= 1 - P\left(\frac{X-500}{48} \le \frac{600-500}{48}\right)$$

$= 1 – P (z \le 2.08)$

$= 1 – \phi (2.08)$

$= 1 – 0.9812$ (From Biometrika Table)

$= 0.0188$

Thus the number of workers having wages less than ₹ 600

$= 500 \times 0.0188$

$= 9.4$

$\cong 9$

(ii) Probability of a worker having wage less than ₹ 450

$= P (X < 450)$

$$= P\left(\frac{X-500}{48} < \frac{450-500}{48}\right)$$

$= P(z < – 1.04)$

$= \phi (– 1.04)$

$= 1 – \phi (1.04)$ (from 16.26)

$= 1 – 0.8508$ (from Biometrika Table)

$= 0.1492$

Hence the number of workers having wages less than ₹ 450

$= 500 \times 0.1492$

$\cong 75$

(iii) Probability of a worker having wage between ₹ 548 and ₹ 600.

$= P (548 < x < 600)$

$$= P\left(\frac{548-500}{48} < \frac{x-500}{48} < \frac{600-500}{48}\right)$$

$= P (1 < z < 2.08)$

$= \phi (2.08) - \phi (1)$

$= 0.9812 - 0.8413$ (consulting Biometrika)

$= 0.1399$

So the number of workers with wages between ₹ 548 and ₹ 600

$= 500 \times 0.1399$

$\cong 70.$

Example 16.28: The distribution of wages of a group of workers is known to be normal with mean ₹ 500 and SD ₹ 100. If the wages of 100 workers in the group are less than ₹ 430, what is the total number of workers in the group?

Solution: Let X denote the wage. It is given that X is normally distributed with mean as ₹ 500 and SD as ₹ 100 and P (X < 430) = 100/N, N being the total no. of workers in the group

$$\Rightarrow P\left(\frac{X-500}{100} < \frac{430-500}{100}\right) = \frac{100}{N}$$

$$\Rightarrow P(z < -0.70) = \frac{100}{N}$$

$$\Rightarrow \phi(-0.70) = \frac{100}{N}$$

$$\Rightarrow 1-\phi(0.70) = \frac{100}{N}$$

$$\Rightarrow 1-0.758 = \frac{100}{N}$$

$$\Rightarrow 0.242 = \frac{100}{N}$$

$$\Rightarrow N \cong 413.$$

Example 16.29: The mean height of 2000 students at a certain college is 165 cms and SD 9 cms. What is the probability that in a group of 5 students of that college, 3 or more students would have height more than 174 cm?

Solution: Let X denote the height of the students of the college. We assume that X is normally distributed with mean (μ) 165 cms and SD (σ) as 9 cms. If p denotes the probability that a student selected at random would have height more than 174 cms., then

$p = P(X > 174)$

$= 1 - P(X \leq 174)$

$= 1 - P\left(\frac{X-165}{9} \leq \frac{174-165}{9}\right)$

$= 1 - P(z \leq 1)$

$= 1 - \phi(1)$

$= 1 - 0.8413$

$= 0.1587$

If y denotes the number of students having height more than 174 cm. in a group of 5 students then $y \sim \beta(n, p)$ where n = 5 and p = 0.1587. Thus the probability that 3 or more students would be more than 174 cm.

$= p(y \geq 3)$

$= p(y = 3) + p(y = 4) + p(y = 5)$

$= 5_{C_3}(0.1587)^3.(0.8413)^2 + 5_{C_4}(0.1587)^4 \times (0.8413) + 5_{C_5}(0.1587)^5$

$= 0.02829 + 0.002668 + 0.000100$

$= 0.03106.$

Example 16.30: The mean of a normal distribution is 500 and 16 per cent of the values are greater than 600. What is the standard deviation of the distribution?

(Given that the area between $z = 0$ to $z = 1$ is 0.34)

Solution: Let σ denote the standard deviation of the distribution.

We are given that

$P(X > 600) = 0.16$

$\Rightarrow 1 - P(X \leq 600) = 0.16$

$\Rightarrow P(X \leq 600) = 0.84$

$\Rightarrow P\left(\frac{X-500}{\sigma} \leq \frac{600-500}{\sigma}\right) = 0.84$

$\Rightarrow P\left(z \leq \frac{100}{\sigma}\right) = 0.84$

$\Rightarrow \phi\left(\frac{100}{\sigma}\right) = \phi(1)$

$$\Rightarrow \frac{(100)}{\sigma} = 1$$

$$\Rightarrow \sigma = 100.$$

Example 16.31: In a business, it is assumed that the average daily sales expressed in Rupees follows normal distribution.

Find the coefficient of variation of sales given that the probability that the average daily sales is less than ₹ 124 is 0.0287 and the probability that the average daily sales is more than ₹ 270 is 0.4599.

Solution: Let us denote the average daily sales by x and the mean and SD of x by μ and σ respectively. As given,

P (x < 124) = 0.0287 ……..(1)

P (x > 270) = 0.4599 ……..(2)

From (1), we have

$$P\left(\frac{X-\mu}{\sigma} < \frac{124-\mu}{\sigma}\right) = 0.0287$$

$$\Rightarrow P\left(z < \frac{124-\mu}{\sigma}\right) = 0.0287$$

$$\Rightarrow \phi\left(\frac{124-\mu}{\sigma}\right) = 0.0287$$

$$\Rightarrow 1-\phi\left(\frac{\mu-124}{\sigma}\right) = 0.0287$$

$$\Rightarrow \phi\left(\frac{\mu-124}{\sigma}\right) = 0.9713$$

$$\Rightarrow \phi\left(\frac{\mu-124}{\sigma}\right) = \phi\,(2.085) \text{ (From Biometrika)}$$

$$\Rightarrow \left(\frac{\mu-124}{\sigma}\right) = 2.085 \quad \ldots\ldots.(3)$$

From (2) we have,

$$1 - P\,(x \leq 270) = 0.4599$$

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{270-\mu}{\sigma}\right) = 0.5401$$

$$\Rightarrow \phi\left(\frac{270-\mu}{\sigma}\right) = 0.5401$$

$$\Rightarrow \phi\left(\frac{270-\mu}{\sigma}\right) = \phi\,(0.1)$$

$$\Rightarrow \left(\frac{270-\mu}{\sigma}\right) = 0.1 \;\ldots\ldots(4)$$

Dividing (3) by (4), we get

$$\frac{\mu-124}{270-\mu} = 20.85$$

$$\Rightarrow \mu - 124 = 5629.50 - 20.85\,\mu$$

$$\Rightarrow \mu = 5753.50/21.85$$

$$= 263.32$$

Substituting this value of μ in (3), we get

$$\frac{263.32 - 124}{\sigma} = 2.085$$

$$\Rightarrow \sigma = 73$$

Thus the coefficient of variation of sales

$$= \sigma/\mu \times 100$$

$$= \frac{73}{263.32} \times 100$$

$$= 25.38$$

Example 16.32: x and y are independent normal variables with mean 100 and 80 respectively and standard deviation as 4 and 3 respectively. What is the distribution of (x + y)?

Solution: We know that if $x \sim N(\mu_1, \sigma_1^{\,2})$ and $y \sim N(\mu_2, \sigma_2^{\,2})$ and they are independent, then z = x + y follows normal with mean $(\mu_1 + \mu_2)$ and

$$SD = \sqrt{\sigma_1^2 + \sigma_2^2} \text{ respectively.}$$

Thus the distribution of (x + y) is normal with mean (100 + 80) or 180

and SD $\sqrt{4^2+3^2} = 5$

Standard Normal Distribution:

If a continuous random variable z follows standard normal distribution, to be denoted by $z \sim N(0, 1)$, then the probability density function of z is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \text{for } -\infty < z < \infty \;\ldots\ldots (16.35)$$

Some important properties of z are listed below :

(i) z has mean, median and mode all equal to zero.

(ii) The standard deviation of z is 1. Also the approximate values of mean deviation and quartile deviation are 0.8 and 0.675 respectively.

(iii) The standard normal distribution is symmetrical about $z = 0$.

(iv) The two points of inflexion of the probability curve of the standard normal distribution are –1 and 1.

(v) The two tails of the standard normal curve never touch the horizontal axis.

(vi) The upper and lower p per cent points of the standard normal variable z are given by

$$P(Z > z_p) = p \;\ldots\ldots (16.36)$$

$$\text{And } P(Z < z_{1-p}) = p$$

$$\text{i.e. } P(Z < -z_p) = p \text{ respectively} \;\ldots (16.37)$$

(since for a standard normal distribution $z_{1-p} = -z_p$)

Selecting P = 0.005, 0.025, 0.01 and 0.05 respectively,

We have $\mathbf{z_{0.005} = 2.58}$

$\mathbf{z_{0.025} = 1.96}$

$\mathbf{z_{0.01} = 2.33}$

$\mathbf{z_{0.05} = 1.645}$ ………… (16.38)

These are shown in fig 16.3.

(vii) If $\bar{x}$ denotes the arithmetic mean of a random sample of size n drawn from a normal population then,

$$Z = \frac{\sqrt{n}\,(\bar{x} - \mu)}{\sigma} \sim N(0, 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(16.39)}$$

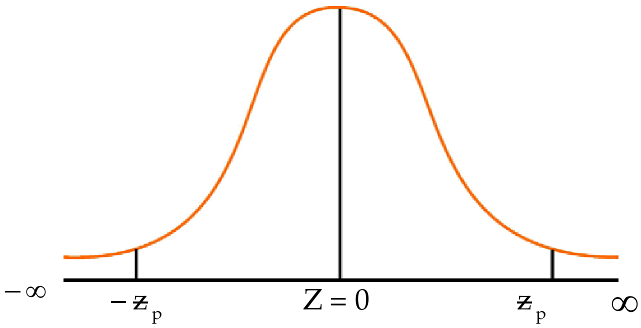


Fig. 16.3

Showing upper and lower p % points of the standard normal variable.

SUMMARY

- A probability distribution also possesses all the characteristics of an observed distribution. We define population mean (μ), population median $(\tilde{\mu})$, population mode (μ_0), population standard deviation (σ) etc. exactly same way we have done earlier. These characteristics are known as population parameters.
- Probability distribution or a Continuous probability distribution depending on the random variable under study.
- Two important discrete probability distributions are (a) Binomial Distribution and (b) Poisson distribution.
- Normal Distribution is a important continuous probability distribution
- A discrete random variable x is defined to follow binomial distribution with parameters n and p, to be denoted by x ~ B (n, p), if the probability mass function of x is given by

  $f(x) = p(X = x) = {}^{n}c_x\, p^x q^{n-x}$ for x = 0, 1, 2, ..., n

  = 0, otherwise

- Additive property of binomial distribution.

  If X and Y are two independent variables such that

  $X \sim \beta\ (n_1, P)$

  and $Y \sim \beta\ (n_2, P)$

  Then $(X+Y) \sim \beta\ (n_1 + n_2,\ P)$

- **Definition of Poisson Distribution**

  A random variable X is defined to follow Poisson distribution with parameter λ, to be denoted by X ~ P (m) if the probability mass function of x is given by

  $$f(x) = P(X = x) = \frac{e^{-m}.m^x}{x!} \text{ for } x = 0, 1, 2, \ldots \infty$$

  $$= 0 \qquad \text{otherwise}$$

  (i) Since $e^{-m} = 1/e^m > 0$, whatever may be the value of m, m > 0, it follows that f (x) $\geq$ 0 for every x.

  Also it can be established that $\sum_x f(x) = 1$ i.e. f(0) + f(1) + f(2) +....... = 1

  (ii) Poisson distribution is known as a uniparametric distribution as it is characterised by only one parameter m.

  (iii) The mean of Poisson distribution is given by m i.e $\mu = m$.

  (iv) The variance of Poisson distribution is given by $\sigma^2 = m$

  (v) Like binomial distribution, Poisson distribution could be also unimodal or bimodal depending upon the value of the parameter m.

  (vi) Poisson approximation to Binomial distribution

  (vii) Additive property of Poisson distribution

- A continuous random variable x is defined to follow normal distribution with parameters μ and σ^2, to be denoted by

  $X \sim N(\mu, \sigma^2)$

  If the probability density function of the random variable x is given by

  $$f(x) = \frac{1}{\sigma\sqrt{2\pi}}.e^{-(\bar{x}-u)^2/2\sigma^2}$$

  for $-\infty < x < \infty$

  where μ and σ are constants, and $\sigma > 0$

- **Properties of Normal Distribution**

1. Since $\pi = 22/7$, $e^{-\theta} = 1 / e^{\theta} > 0$, whatever θ may be,

   it follows that f (x) $\geq$ 0 for every x.

   It can be shown that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

2. The mean of the normal distribution is given by μ. Further, since the distribution is symmetrical about x = μ, it follows that the mean, median and mode of a normal distribution coincide, all being equal to μ.

3. The standard deviation of the normal distribution is given by σ.

   Mean deviation of normal distribution is

$$\sigma\sqrt{\frac{2}{\pi}} \cong 0.8\sigma$$

   The first and third quartiles are given by

   $Q_1 = \mu - 0.675\,\sigma$

   and $Q_3 = \mu + 0.675\,\sigma$

   so that, quartile deviation = 0.675 σ

4. The normal distribution is symmetrical about x = μ. As such, its skewness is zero i.e. the normal curve is neither inclined move towards the right (negatively skewed) nor towards the left (positively skewed).

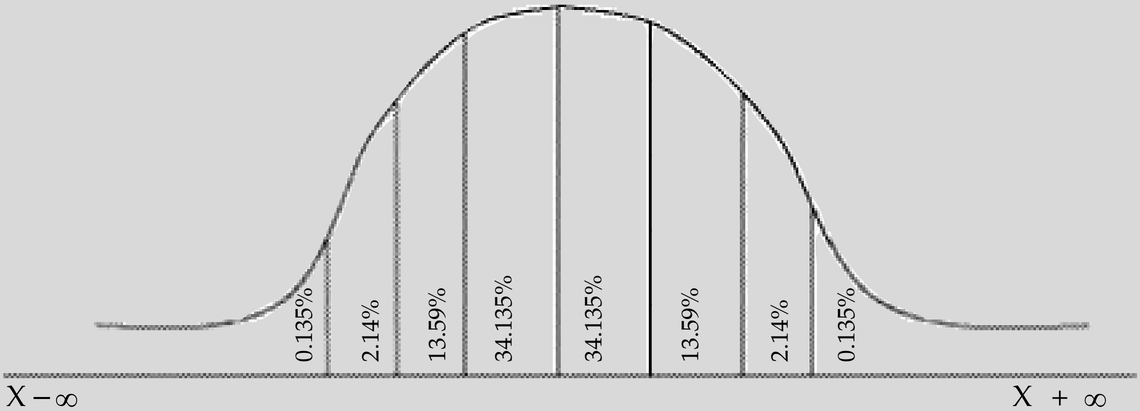
5. The normal curve y = f (x) has two points of inflexion to be given by x = μ − σ and x = μ + σ i.e. at these two points, the normal curve changes its curvature from concave to convex and from convex to concave.

6. If x ~ N (μ, σ^2) then z = x – μ/σ ~ N (0, 1), z is known as standardised normal variate or normal deviate.

   We also have P (z ≤ k) = ϕ (k)

7. Area under the normal curve is shown in the following figure :

μ – 3σ	μ – 2σ	μ – σ	x = μ	μ + σ	μ + 2σ	μ + 3σ
(z = –3)	(z = –2)	(z = –1)	(z = 0)	(z = 1)	(z = 2)	(z = 3)



$P(\mu - \sigma < x < \mu + \sigma) = 0.6828$

$\Rightarrow P(-1 < z < 1) = 0.6828$

$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9546$

$\Rightarrow P(-2 < z < 2) = 0.9546$

and $P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$

$\Rightarrow P(-3 < z < 3) = 0.9973$.

8. We note that 99.73 per cent of the values of a normal variable lies between $(\mu - 3\sigma)$ and $(\mu + 3\sigma)$. Thus the probability that a value of x lies outside that limit is as low as 0.0027.

9. If x and y are independent normal variables with means and standard deviations as μ_1 and μ_2 and σ_1, and σ_2 respectively, then $z = x + y$ also follows normal distribution with mean $(\mu_1 + \mu_2)$ and SD = $\sqrt{\sigma_1^2 + \sigma_2^2}$ respectively.

- **Standard Normal Distribution**

If a continuous random variable z follows standard normal distribution, to be denoted by $z \sim N(0, 1)$, then the probability density function of z is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \text{for } -\infty < z < \infty$$

Some important properties of z are listed below :

(i) z has mean, median and mode all equal to zero.

(ii) The standard deviation of z is 1. Also the approximate values of mean deviation and quartile deviation are 0.8 and 0.675 respectively.

(iii) The standard normal distribution is symmetrical about $z = 0$.

(iv) The two points of inflexion of the probability curve of the standard normal distribution are –1 and 1.

(v) The two tails of the standard normal curve never touch the horizontal axis.

(vi) The upper and lower p per cent points of the standard normal variable z are given by

$P(Z > z_p) = p$

And $P(Z < z_{1-p}) = p$

i.e. $P(Z < -z_p) = p$ respectively

(since for a standard normal distribution $z_{1-p} = -z_p$)

Selecting P = 0.005, 0.025, 0.01 and 0.05 respectively,

We have $z_{0.005} = 2.58$

$z_{0.025} = 1.96$

$z_{0.01} = 2.33$

$z_{0.05} = 1.645$

These are shown in fig 13.3.

(vii) If $\bar{x}$ denotes the arithmetic mean of a random sample of size n drawn from a normal population then,

$$Z = \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \sim N(0, 1)$$

EXERCISE

Set : A

Write down the correct answers. Each question carries 1 mark.

1. A theoretical probability distribution.
   (a) does not exist. (b) exists in theory.
   (c) exists in real life. (d) both (b) and (c).
2. Probability distribution may be
   (a) discrete. (b) continuous. (c) infinite. (d) (a) or (b).
3. An important discrete probability distribution is
   (a) Poisson distribution. (b) Normal distribution.
   (c) Cauchy distribution. (d) Log normal distribution.
4. An important continuous probability distribution
   (a) Binomial distribution. (b) Poisson distribution.
   (c) Geometric distribution. (d) Normal distribution.

5. Parameter is a characteristic of
   (a) population. (b) sample. (c) probability distribution. (d) both (a) and (b).
6. An example of a parameter is
   (a) sample mean. (b) population mean.
   (c) binomial distribution. (d) sample size.
7. A trial is an attempt to
   (a) make something possible. (b) make something impossible.
   (c) prosecute an offender in a court of law.
   (d) produce an outcome which is neither certain nor impossible.
8. The important characteristic(s) of Bernoulli trials
   (a) each trial is associated with just two possible outcomes.
   (b) trials are independent. (c) trials are infinite.
   (d) both (a) and (b).
9. The probability mass function of binomial distribution is given by
   (a) $f(x) = p^x q^{n-x}$. (b) $f(x) = {}^nc_x p^x q^{n-x}$.
   (c) $f(x) = {}^nc_x q^x p^{n-x}$. (d) $f(x) = {}^nc_x p^{n-x} q^x$.
10. If x is a binomial variable with parameters n and p, then x can assume
    (a) any value between 0 and n.
    (b) any value between 0 and n, both inclusive.
    (c) any whole number between 0 and n, both inclusive.
    (d) any number between 0 and infinity.
11. A binomial distribution is
    (a) never symmetrical. (b) never positively skewed.
    (c) never negatively skewed. (d) symmetrical when p = 0.5.
12. The mean of a binomial distribution with parameter n and p is
    (a) n (1– p). (b) np (1 – p). (c) np. (d) $\sqrt{np(1-p)}$.
13. The variance of a binomial distribution with parameters n and p is
    (a) $np^2 (1 - p)$. (b) $\sqrt{np(1-p)}$. (c) nq (1 – q). (d) $n^2p^2 (1-p)^2$.
14. An example of a bi-parametric discrete probability distribution is
    (a) binomial distribution. (b) poisson distribution.
    (c) normal distribution. (d) both (a) and (b).

15. For a binomial distribution, mean and mode

(a) are never equal. (b) are always equal.

(c) are equal when q = 0.50. (d) do not always exist.

16. The mean of binomial distribution is

(a) always more than its variance. (b) always equal to its variance.

(c) always less than its variance. (d) always equal to its standard deviation.

17. For a binomial distribution, there may be

(a) one mode. (b) two modes. (c) multi modes (d) (a) or (b).

18. The maximum value of the variance of a binomial distribution with parameters n and p is

(a) n/2. (b) n/4. (c) np (1 – p). (d) 2n.

19. The method usually applied for fitting a binomial distribution is known as

(a) method of least square. (b) method of moments.

(c) method of probability distribution. (d) method of deviations.

20. Which one is not a condition of Poisson model?

(a) the probability of having success in a small time interval is constant.

(b) the probability of having success more than one in a small time interval is very small.

(c) the probability of having success in a small interval is independent of time and also of earlier success.

(d) the probability of having success in a small time interval (t, t + dt) is kt for a positive constant k.

21. Which one is uniparametric distribution?

(a) Binomial. (b) Poisson. (c) Normal. (d) Hyper geometric.

22. For a Poisson distribution,

(a) mean and standard deviation are equal. (b) mean and variance are equal.

(c) standard deviation and variance are equal. (d) both (a) and (b).

23. Poisson distribution may be

(a) unimodal. (b) bimodal. (c) Multi-modal. (d) (a) or (b).

24. Poisson distribution is

(a) always symmetric. (b) always positively skewed.

(c) always negatively skewed. (d) symmetric only when m = 2.

25. A binomial distribution with parameters n and p can be approximated by a Poisson distribution with parameter m = np is

(a) $n \to \infty$. (b) $p \to 0$.

(c) $n \to \infty$ and $p \to 0$. (d) $n \to \infty$ and $p \to 0$ so that np remains finite..

26. For Poisson fitting to an observed frequency distribution,

(a) we equate the Poisson parameter to the mean of the frequency distribution.

(b) we equate the Poisson parameter to the median of the distribution.

(c) we equate the Poisson parameter to the mode of the distribution.

(d) none of these.

27. The most important continuous probability distribution is known as

(a) Binomial distribution. (b) Normal distribution.

(c) Chi-square distribution. (d) Sampling distribution.

28. The probability density function of a normal variable x is given by

(a) $f(x) = \frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$ for $-\propto < x < \propto$.

(b) $f(x) = \frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{-(x-\mu)^2}{2\sigma^2}}$ for $0 < x < \propto$.

(c) $f(x) = \frac{1}{\sqrt{2\pi\sigma}}.e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\propto < x < \propto$.

(d) none of these.

29. The total area of the normal curve is

(a) one. (b) 50 per cent.

(c) 0.50. (d) any value between 0 and 1.

30. The normal curve is

(a) Bell-shaped. (b) U- shaped.

(c) J-shaped. (d) Inverted J-shaped.

31. The normal curve is

(a) positively skewed. (b) negatively skewed.

(c) symmetrical. (d) all these.

32. Area of the normal curve

(a) between $-\propto$ to μ is 0.50. (b) between μ to $\propto$ is 0.50.

(c) between $-\propto$ to $\propto$ is 0.50. (d) both (a) and (b).

33. The cumulative distribution function of a random variable X is given by

(a) $F(x) = P(X \leq x)$.
(b) $F(X) = P(X \leq x)$.
(c) $F(x) = P(X \geq x)$.
(d) $F(x) = P(X = x)$.

34. The mean and mode of a normal distribution

(a) may be equal.
(b) may be different.
(c) are always equal.
(d) (a) or (b).

35. The mean deviation about median of a standard normal variate is

(a) $0.675\ \sigma$.
(b) 0.675.
(c) $0.80\ \sigma$.
(d) 0.80.

36. The quartile deviation of a normal distribution with mean 10 and SD 4 is

(a) 0.675.
(b) 67.50.
(c) 2.70.
(d) 3.20.

37. For a standard normal distribution, the points of inflexion are given by

(a) $\mu - \sigma$ and $\mu + \sigma$.
(b) $-\sigma$ and σ.
(c) -1 and 1.
(d) 0 and 1.

38. The symbol ϕ (a) indicates the area of the standard normal curve between

(a) 0 to a.
(b) a to ∞.
(c) $-\propto$ to a.
(d) $-\propto$ to $\propto$.

39. The interval $(\mu - 3\sigma, \mu + 3\sigma)$ covers

(a) 95% area of a normal distribution.
(b) 96% area of a normal distribution.
(c) 99% area of a normal distribution.
(d) all but 0.27% area of a normal distribution.

40. Number of misprints per page of a thick book follows

(a) Normal distribution.
(b) Poisson distribution.
(c) Binomial distribution.
(d) Standard normal distribution.

41. The results of ODI matches between India and Pakistan follows

(a) Binomial distribution.
(b) Poisson distribution.
(c) Normal distribution.
(d) (b) or (c).

42. The wage of workers of a factory follow

(a) Binomial distribution.
(b) Poisson distribution.
(c) Normal distribution.
(d) Chi-square distribution.

43. If X and Y are two independent normal random variables, then the distribution of (X+Y) is

(a) normal.
(b) standard normal.
(c) T.
(d) chi-square.

Set B :

Write down the correct answers. Each question carries 2 marks.

1. What is the standard deviation of the number of recoveries among 48 patients when the probability of recovering is 0.75?

(a) 36. (b) 81. (c) 9. (d) 3.

2. X is a binomial variable with n = 20. What is the mean of X if it is known that x is symmetric?

(a) 5. (b) 10. (c) 2. (d) 8.

3. If X ~ B (n, p), what would be the greatest value of the variance of x when n = 16?

(a) 2. (b) 4. (c) 8. (d) $\sqrt{5}$.

4. If x is a binomial variate with parameter 15 and 1/3, what is the value of mode of the distribution?

(a) 5 and 6. (b) 5. (c) 5.50. (d) 6.

5. What is the number of trials of a binomial distribution having mean and SD as 3 and 1.5 respectively?

(a) 2. (b) 4. (c) 8. (d) 12.

6. What is the probability of getting 3 heads if 6 unbiased coins are tossed simultaneously?

(a) 0.50. (b) 0.25. (c) 0.3125. (d) 0.6875.

7. If the overall percentage of success in an exam is 60, what is the probability that out of a group of 4 students, at least one has passed?

(a) 0.6525. (b) 0.9744. (c) 0.8704. (d) 0.0256.

8. What is the probability of making 3 correct guesses in 5 True – False answer type questions?

(a) 0.3125. (b) 0.5676. (c) 0.6875. (d) 0.4325

9. If the standard deviation of a Poisson variate X is 2, what is P (1.5 < X < 2.9)?

(a) 0.231. (b) 0.158. (c) 0.15. (d) 0.144.

10. If the mean of a Poisson variable X is 1, what is P (X = takes the value at least 1)?

(a) 0.456. (b) 0.821. (c) 0.632. (d) 0.254.

11. If X ~ P (m) and its coefficient of variation is 50, what is the probability that X would assume only non-zero values?

(a) 0.018. (b) 0.982. (c) 0.989. (d) 0.976.

12. If 1.5 per cent of items produced by a manufacturing units are known to be defective, what is the probability that a sample of 200 items would contain no defective item?

(a) 0.05. (b) 0.15. (c) 0.20. (d) 0.22.

13. For a Poisson variate X, P (X = 1) = P (X = 2). What is the mean of X?

(a) 1.00. (b) 1.50. (c) 2.00. (d) 2.50.

14. If 1 per cent of an airline's flights suffer a minor equipment failure in an aircraft, what is the probability that there will be exactly two such failures in the next 100 such flights?

(a) 0.50. (b) 0.184. (c) 0.265. (d) 0.256.

15. If for a Poisson variable X, f(2) = 3 f(4), what is the variance of X?

(a) 2. (b) 4. (c) $\sqrt{2}$. (d) 3.

16. What is the coefficient of variation of x, characterised by the following probability density function: $f(x) = \frac{1}{4\sqrt{2\pi}} e^{-(x-10)^2/32}$ for $-\infty < x < \infty$

(a) 50. (b) 60. (c) 40. (d) 30.

17. What is the first quartile of X having the following probability density function?

$$f(x) = \frac{1}{\sqrt{72\pi}} e^{-(x-10)^2/72} \quad \text{for } -\infty < x < \infty$$

(a) 4. (b) 5. (c) 5.95. (d) 6.75.

18. If the two quartiles of N (μ, σ^2) are 14.6 and 25.4 respectively, what is the standard deviation of the distribution?

(a) 9. (b) 6. (c) 10. (d) 8.

19. If the mean deviation of a normal variable is 16, what is its quartile deviation?

(a) 10.00. (b) 13.50. (c) 15.00. (d) 12.05.

20. If the points of inflexion of a normal curve are 40 and 60 respectively, then its mean deviation is

(a) 40. (b) 45. (c) 50. (d) 60.

21. If the quartile deviation of a normal curve is 4.05, then its mean deviation is

(a) 5.26. (b) 6.24. (c) 4.24. (d) 4.80.

22. If the Ist quartile and mean deviation about median of a normal distribution are 13.25 and 8 respectively, then the mode of the distribution is

(a) 20. (b) 10. (c) 15. (d) 12.

23. If the area of standard normal curve between z = 0 to z = 1 is 0.3413, then the value of ϕ (1) is

(a) 0.5000. (b) 0.8413. (c) –0.5000. (d) 1.

24. If X and Y are 2 independent normal variables with mean as 10 and 12 and SD as 3 and 4, then (X+Y) is normally distributed with

(a) mean = 22 and SD = 7. (b) mean = 22 and SD = 25.

(c) mean = 22 and SD = 5. (d) mean = 22 and SD = 49.

Set : C

Answer the following questions. Each question carries 5 marks.

1. If it is known that the probability of a missile hitting a target is 1/8, what is the probability that out of 10 missiles fired, at least 2 will hit the target?

   (a) 0.4258. (b) 0.3968. (c) 0.5238. (d) 0.3611.

2. X is a binomial variable such that 2 P(X = 2) = P(X = 3) and mean of X is known to be 10/3. What would be the probability that X assumes at most the value 2?

   (a) 16/81. (b) 17/81. (c) 47/243. (d) 46/243.

3. Assuming that one-third of the population is tea drinkers and each of 1000 enumerators takes a sample of 8 individuals to find out whether they are tea drinkers or not, how many enumerators are expected to report that five or more people are tea drinkers?

   (a) 100. (b) 95. (c) 88. (d) 90.

4. If a random variable X follows binomial distribution with mean as 5 and satisfying the condition 10 P (X = 0) = P (X = 1), what is the value of $P(x \geq 1 / x > 0)$?

   (a) 0.67. (b) 0.56. (c) 0.99. (d) 0.82.

5. Out of 128 families with 4 children each, how many are expected to have at least one boy and one girl?

   (a) 100. (b) 105. (c) 108. (d) 112.

6. In 10 independent rollings of a biased die, the probability that an even number will appear 5 times is twice the probability that an even number will appear 4 times. What is the probability that an even number will appear twice when the die is rolled 8 times?

   (a) 0.0304. (b) 0.1243. (c) 0.2315. (d) 0.1926.

7. If a binomial distribution is fitted to the following data:

x:	0	1	2	3	4
f:	16	25	32	17	10

   then the sum of the expected frequencies for x = 2, 3 and 4 would be

   (a) 58. (b) 59. (c) 60. (d) 61.

8. If X follows normal distribution with $\mu = 50$ and $\sigma = 10$, what is the value of $P(x \leq 60 / x > 50)$?

   (a) 0.8413. (b) 0.6828. (c) 0.1587. (d) 0.7256.

9. X is a Poisson variate satisfying the following condition 9 P (X = 4) + 90 P (X = 6) = P (X = 2). What is the value of $P(X \leq 1)$?

   (a) 0.5655 (b) 0.6559 (c) 0.7358 (d) 0.8201

10. A random variable x follows Poisson distribution and its coefficient of variation is 50. What is the value of P (x > 1 / x > 0)?

(a) 0.1876 (b) 0.2341 (c) 0.9254 (d) 0.8756

11. A renowned hospital usually admits 200 patients every day. One per cent patients, on an average, require special room facilities. On one particular morning, it was found that only one special room is available. What is the probability that more than 3 patients would require special room facilities?

(a) 0.1428 (b) 0.1732 (c) 0.2235 (d) 0.3450

12. A car hire firm has 2 cars which is hired out everyday. The number of demands per day for a car follows Poisson distribution with mean 1.20. What is the proportion of days on which some demand is refused? (Given $e^{1.20} = 3.32$).

(a) 0.25 (b) 0.3012 (c) 0.12 (d) 0.03

13. If a Poisson distribution is fitted to the following data:

Mistake per page	0	1	2	3	4	5
Number of pages	76	74	29	17	3	1

Then the sum of the expected frequencies for x = 0, 1 and 2 is

(a) 150. (b) 184. (c) 165. (d) 148.

14. The number of accidents in a year attributed to taxi drivers in a locality follows Poisson distribution with an average 2. Out of 500 taxi drivers of that area, what is the number of drivers with at least 3 accidents in a year?

(a) 162 (b) 180 (c) 201 (d) 190

15. In a sample of 800 students, the mean weight and standard deviation of weight are found to be 50 kg and 20 kg respectively. On the assumption of normality, what is the number of students weighing between 46 Kg and 62 Kg? Given area of the standard normal curve between $z = 0$ to $z = 0.20 = 0.0793$ and area between $z = 0$ to $z = 0.60 = 0.2257$.

(a) 250 (b) 244 (c) 240 (d) 260

16. The salary of workers of a factory is known to follow normal distribution with an average salary of ₹ 10,000 and standard deviation of salary as ₹ 2,000. If 50 workers receive salary more than ₹ 14,000, then the total no. of workers in the factory is

(a) 2,193 (b) 2,000 (c) 2,200 (d) 2,500

17. For a normal distribution with mean as 500 and SD as 120, what is the value of k so that the interval [500, k] covers 40.32 per cent area of the normal curve? Given ϕ (1.30) = 0.9032.

(a) 740 (b) 750 (c) 656 (d) 800

18. The average weekly food expenditure of a group of families has a normal distribution with mean ₹ 1,800 and standard deviation ₹ 300. What is the probability that out of 5 families belonging to this group, at least one family has weekly food expenditure in excess of ₹ 2,100? Given ϕ (1) = 0.84.

(a) 0.418 (b) 0.582 (c) 0.386 (d) 0.614

19. If the weekly wages of 5000 workers in a factory follows normal distribution with mean and SD as ₹ 700 and ₹ 50 respectively, what is the expected number of workers with wages between ₹ 660 and ₹ 720?

    (a) 2,050 (b) 2,200 (c) 2,218 (d) 2,300

20. 50 per cent of a certain product have weight 60 kg or more whereas 10 per cent have weight 55 kg or less. On the assumption of normality, what is the variance of weight?

    Given ϕ (1.28) = 0.90.

    (a) 15.21 (b) 9.00 (c) 16.00 (d) 22.68

ANSWERS

Set : A

1.	(d)	**2.**	(d)	**3.**	(a)	**4.**	(d)	**5.**	(a)	**6.**	(b)	**7.**	(d)	**8.**	(d)
9.	(b)	**10.**	(c)	**11.**	(d)	**12.**	(c)	**13.**	(c)	**14.**	(a)	**15.**	(c)	**16.**	(a)
17.	(c)	**18.**	(b)	**19.**	(b)	**20.**	(a)	**21.**	(b)	**22.**	(b)	**23.**	(d)	**24.**	(b)
25.	(d)	**26.**	(a)	**27.**	(b)	**28.**	(a)	**29.**	(a)	**30.**	(a)	**31.**	(c)	**32**	(d)
33.	(a)	**34.**	(c)	**35.**	(d)	**36.**	(c)	**37.**	(c)	**38.**	(c)	**39.**	(d)	**40.**	(b)
41.	(a)	**42.**	(c)	**43.**	(a)										

Set : B

1.	(d)	**2.**	(b)	**3.**	(b)	**4.**	(b)	**5.**	(d)	**6.**	(c)	**7.**	(b)	**8.**	(a)
9.	(d)	**10.**	(c)	**11.**	(b)	**12.**	(a)	**13.**	(c)	**14.**	(b)	**15.**	(a)	**16.**	(c)
17.	(c)	**18.**	(d)	**19.**	(b)	**20.**	(a)	**21.**	(d)	**22.**	(a)	**23.**	(b)	**24.**	(c)

Set : C

1.	(d)	**2.**	(b)	**3.**	(c)	**4.**	(c)	**5.**	(d)	**6.**	(a)	**7.**	(d)	**8.**	(b)
9.	(c)	**10.**	(c)	**11.**	(a)	**12.**	(d)	**13.**	(b)	**14.**	(a)	**15.**	(b)	**16.**	(a)
17.	(c)	**18.**	(b)	**19.**	(c)	**20.**	(a)								

ADDITIONAL QUESTION BANK

1. When a coin is tossed 10 times then we use

(a) Normal Distribution (b) Poisson Distribution
(c) Binomial Distribution (d) None

2. In Binomial Distribution 'n' means

(a) Number of trials of the experiment (b) the probability of getting success
(c) Number of success (d) none

3. Binomial probability Distribution is a

(a) Continuous (b) discrete
(c) both (d) none

4. When there are a fixed number of repeated trial of any experiments under identical conditions for which only one of two mutually exclusive outcomes, success or failure can result in each trial then, we use

(a) Normal Distribution (b) Binomial Distribution

(c) Poisson Distribution (d) None

5. In Binomial Distribution 'p' denotes Probability of

(a) Success (b) Failure (c) Both (d) None

6. When p = 0.5, the binomial distribution is

(a) asymmetrical (b) symmetrical (c) Both (d) None

7. When 'p' is larger than 0. 5, the binomial distribution is

(a) asymmetrical (b) symmetrical (c) Both (d) None

8. Mean of Binomial distribution is

(a) npq (b) np (c) both (d) none

9. Variance of Binomial distribution is

(a) npq (b) np (c) both (d) none

10. When p = 0.1 the binomial distribution is skewed to the

(a) left (b) right (c) both (d) none

11. If in Binomial distribution np = 9 and npq = 2. 25 then q is equal to

(a) 0.25 (b) 0.75 (c) 1 (d) none

12. In Binomial Distribution

(a) mean is greater than variance (b) mean is less than variance
(c) mean is equal to variance (d) none

13. Standard deviation of binomial distribution is

(a) $(npq)^2$ (b) $\sqrt{npq}$

(c) $(np)^2$ (d) $\sqrt{np}$

14. _________ distribution is a limiting case of Binomial distribution

(a) Normal (b) Poisson (c) Both (d) none

15. When the number of trials is large and probability of success is small then we use the distribution

(a) Normal (b) Poisson

(c) Binomial (d) none

16. In Poisson Distribution, probability of success is very close to

(a) 1 (b) – 1 (c) 0 (d) none

17. In Poisson Distribution np is

(a) finite (b) infinite (c) 0 (d) none

18. In _______________ distribution, mean = variance

(a) Normal (b) Binomial (c) Poisson (d) none

19. In Poisson distribution mean is equal to

(a) (λ) (b) np (c) square root mp (d) square root mpq

20. In Binomial distribution standard deviation is equal to

(a) $\sqrt{np}$ (b) $(np)^2$ (c) $\sqrt{npq}$ (d) $(npq)^2$

21. For continuous events ________________ distribution is used.

(a) Normal (b) Poisson (c) Binomial (d) none

22. Probability density function is associated with

(a) discrete random variable (b) continuous random variables

(c) both (d) none

23. Probability density function is always

(a) greater than 0 (b) greater than equal to 0

(c) less than 0 (d) less than equal to 0

24. For continuous random variables probability of the entire space is

(a) 0 (b) –1 (c) 1 (d) none

25. For discrete random variables the probability of the entire space is

(a) 0 (b) 1 (c) –1 (d) none

26. Binomial distribution is symmetrical if

(a) $p > q$ (b) $p < q$ (c) $p = q$ (d) none

27. The Poisson distribution tends to be symmetrical if the mean value is

(a) high (b) low (c) zero (d) none

28. The curve of ____________ distribution has single peak

(a) Poisson (b) Binomial (c) Normal (d) none

29. The curve of _________ distribution is unimodal and bell shaped with the highest point over the mean

(a) Poisson (b) Normal (c) Binomial (d) none

30. Because of the symmetry of Normal distribution the median and the mode have the ______ value as that of the mean

(a) greater (b) smaller (c) same (d) none

31. For a Normal distribution, the total area under the normal curve is

(a) 0 (b) 1 (c) 2 (d) –1

32. In Normal distribution the probability has the maximum value at the

(a) mode (b) mean (c) median (d) All

33. In Normal distribution the probability decreases gradually on either side of the mean but never touches the axis.

(a) True (b) false (c) both (d) none

34. Whatever may be the parameter of __________ distribution, it has same shape.

(a) Normal (b) Binomial (c) Poisson (d) none

35. In Standard Normal distribution

(a) mean=1, S.D=0 (b) mean=1, S.D=1
(c) mean = 0, S.D = 1 (d) mean=0, S.D=0

36. The Number of methods for fitting the normal curve is

(a) 1 (b) 2 (c) 3 (d) 4

37. ____________ distribution is symmetrical around t = 0

(a) Normal (b) Poisson (c) Binomial (d) t

38. As the degree of freedom increases, the ________ distribution approaches the Standard Normal distribution

(a) t (b) Binomial (c) Poisson (d) Normal

39. _________ distribution is asymptotic to the horizontal axis.

(a) Binomial (b) Normal (c) Poisson (d) t

40. ________ distribution has a greater spread than Normal distribution curve

(a) t (b) Binomial (c) Poisson (d) none

41. In Binomial Distribution if n is infinitely large, the probability p of occurrence of event' is close to _______ and q is close to _________

(a) 0, 1 (b) 1, 0 (c) 1, 1 (d) none

42. Poisson distribution approaches a Normal distribution as n

(a) increase infinitely (b) decrease (c) increases moderately (d) none

43. If neither p nor q is very small but n sufficiently large, the Binomial distribution is very closely approximated by _________ distribution

(a) Poisson (b) Normal (c) t (d) none

44. For discrete random variable x, Expected value of x (i.e E(x)) is defined as the sum of products of the different values and the corresponding probabilities.

(a) True (b) false (c) both (d) none

45. For a probability distribution, ———————— is the expected value of x.

(a) median (b) mode (c) mean (d) none

46. _________ is the expected value of $(x - m)^2$, where m is the mean.

(a) median (b) variance (c) standard deviation (d) mode

47. The probability distribution of x is given below :

value of x :	1	0	Total
probability :	p	1–p	1

Mean is equal to

(a) p (b) 1–p (c) 0 (d) 1

48. For n independent trials in Binomial distribution the sum of the powers of p and q is always n , whatever be the no. of success.

(a) True (b) false (c) both (d) none

49. In Binomial distribution parameters are

(a) n and q (b) n and p (c) p and q (d) none

50. In Binomial distribution if n = 4 and p = 1/3 then the value of variance is

(a) 8/3 (b) 8/9 (c) 4/3 (d) none

51. In Binomial distribution if mean = 20, S.D.= 4 then q is equal to

(a) 2/5 (b) 3/8 (c) 1/5 (d) 4/5

52. If in a Binomial distribution mean = 20 , S.D.= 4 then p is equal to

(a) 2/5 (b) 3/5 (c) 1/5 (d) 4/5

53. If is a Binomial distribution mean = 20 , S.D.= 4 then n is equal to

(a) 80 (b) 100 (c) 90 (d) none

54. Poisson distribution is a ___________ probability distribution .

(a) discrete (b) continuous (c) both (d) none

55. Number of radio-active atoms decaying in a given interval of time is an example of

(a) Binomial distribution (b) Normal distribution
(c) Poisson distribution (d) None

56. __________ distribution is sometimes known as the "distribution of rare events".

(a) Poisson (b) Normal (c) Binomial (d) none

57. The probability that x assumes a specified value in continuous probability distribution is

(a) 1 (b) 0 (c) –1 (d) none

58. In Normal distribution mean, median and mode are

(a) equal (b) not equal (c) zero (d) none

59. In Normal distribution the quartiles are equidistant from

(a) median (b) mode (c) mean (d) none

60. In Normal distribution as the distance from the ___________ increases, the curve comes closer and closer to the horizontal axis.

(a) median (b) mean (c) mode (d) none

61. The probability density function of a continuous random variable is defined as follows :

$f(x) = c$ when $-1 \leq x \leq 1 = 0$, otherwise the value of c is

(a) 1 (b) –1 (c) 1/2 (d) 0

62. A continuous random variable x has the probability density fn.$f(x) = \frac{1}{2} - ax$, $0 \leq x \leq 4$
When 'a' is a constant. The value of ' a' is

(a) 7/8 (b) 1/8 (c) 3/16 (d) none

63. An unbiased die is tossed 500 times.The mean of the number of 'Sixes' in these 500 tosses is

(a) 50/6 (b) 500/6 (c) 5/6 (d) none

64. An unbiased die is tossed 500 times. The Standard deviation of the number of 'sixes' in these 500 tossed is

(a) 50/6 (b) 500/6 (c) 5/6 (d) none

65. A random variable x follows Binomial distribution with mean 2 and variance 1.2. then the value of n is

(a) 8 (b) 2 (c) 5 (d) none

66. A random variable x follows Binomial distribution with mean 2 and variance 1.6 then the value of p is

(a) 1/5 (b) 4/5 (c) 3/5 (d) none

67. "The mean of a Binomial distribution is 5 and standard deviation is 3".

(a) True (b) false (c) both (d) none

68. The expected value of a constant k is the constant

(a) k (b) k–1 (c) k+1 (d) none

69. The probability distribution whose frequency function f(x)= 1/n($x = x_1, x_2, \ldots, x_n$) is known as

(a) Binomial distribution (b) Poisson distribution
(c) Uniform distribution(d) Normal distribution

70. Theoretical distribution is a

(a) Random distribution (b) Standard distribution
(c) Probability distribution (d) None

71. Probability function is known as

(a) frequency function (b) continuous function
(c) discrete function (d) none

72. The number of points obtained in a single throw of an unbiased die follows :

(a) Binomial distribution (b) Poisson distribution
(c) Uniform distribution(d) None

73. The Number of points in a single throw of an unbiased die has frequency function

(a) f(x)=1/4 (b) f(x)= 1/5 (c) f(x) = 1/6 (d) none

74. In uniform distribution random variable x assumes n values with

(a) equal probability (b) unequal probability (c) zero (d) none

75. In a discrete random variable x follows uniform distribution and assumes only the values 8 , 9, 11, 15, 18, 20. Then P(x = 9) is

(a) 2/6 (b) 1/7 (c) 1/5 (d) 1/6

76. In a discrete random variable x follows uniform distribution and assumes only the values 8 , 9, 11, 15, 18, 20. Then P(x = 12) is

(a) 1/6 (b) 0 (c) 1/7 (d) none

77. In a discrete random variable x follows uniform distribution and assumes only the values 8, 9, 11, 15, 18, 20. Then $P(x < 15)$ is

(a) 1/2 (b) 2/3 (c) 1 (d) none

78. In a discrete random variable x follows uniform distribution and assumes only the values 8 , 9, 11, 15, 18, 20. Then $P(x \leq 15)$ is

(a) 2/3 (b) 1/3 (c) 1 (d) none

79. In a discrete random variable x follows uniform distribution and assumes only the values 8, 9, 11, 15, 18, 20. Then $P(x > 15)$ is

(a) 2/3 (b) 1/3 (c) 1 (d) none

80. In a discrete random variable x follows uniform distribution and assumes only the values 8, 9, 11, 15, 18, 20. Then $P(|x - 14| < 5)$ is

(a) 1/3 (b) 2/3 (c) 1/2 (d) 1

81. When $f(x)= 1/n$ then mean is

(a) $(n-1)/2$ (b) $(n+1)/2$ (c) $n/2$ (d) none

82. In continuous probability distribution $P(x \leq t)$ means

(a) Area under the probability curve to the left of the vertical line at t .

(b) Area under the probability curve to the right of the vertical line at t .

(c) both (d) none

83. In continuous probability distribution F(x) is called.

(a) frequency distribution function (b) cumulative distribution function
(c) probability density function (d) none

84. The probability density function of a continuous random variable is $y = k(x-1)$, $(1 \leq x \leq 2)$ then the value of the constant k is

(a) –1 (b) 1 (c) 2 (d) 0

ANSWERS

1.	(c)	**2.**	(a)	**3.**	(b)	**4.**	(b)	**5.**	(a)
6.	(b)	**7.**	(a)	**8.**	(b)	**9.**	(a)	**10.**	(b)
11.	(a)	**12.**	(a)	**13.**	(b)	**14.**	(b)	**15.**	(b)
16.	(c)	**17.**	(a)	**18.**	(c)	**19.**	(a)	**20.**	(c)
21.	(a)	**22.**	(b)	**23.**	(b)	**24.**	(c)	**25.**	(b)
26.	(c)	**27.**	(a)	**28.**	(c)	**29.**	(b)	**30.**	(c)
31.	(b)	**32.**	(b)	**33.**	(a)	**34.**	(a)	**35.**	(c)
36.	(b)	**37.**	(d)	**38.**	(a)	**39.**	(d)	**40.**	(a)
41.	(a)	**42.**	(a)	**43.**	(b)	**44.**	(a)	**45.**	(c)
46.	(b)	**47.**	(a)	**48.**	(a)	**49.**	(b)	**50.**	(b)
51.	(d)	**52.**	(c)	**53.**	(b)	**54.**	(a)	**55.**	(c)
56.	(a)	**57.**	(b)	**58.**	(a)	**59.**	(c)	**60.**	(b)
61.	(c)	**62.**	(b)	**63.**	(b)	**64.**	(a)	**65.**	(c)
66.	(a)	**67.**	(b)	**68.**	(a)	**69.**	(c)	**70.**	(c)
71.	(a)	**72.**	(c)	**73.**	(c)	**74.**	(a)	**75.**	(d)
76.	(b)	**77.**	(a)	**78.**	(a)	**79.**	(b)	**80.**	(c)
81.	(b)	**82.**	(a)	**83.**	(b)	**84.**	(c)		

NOTES

CHAPTER BL

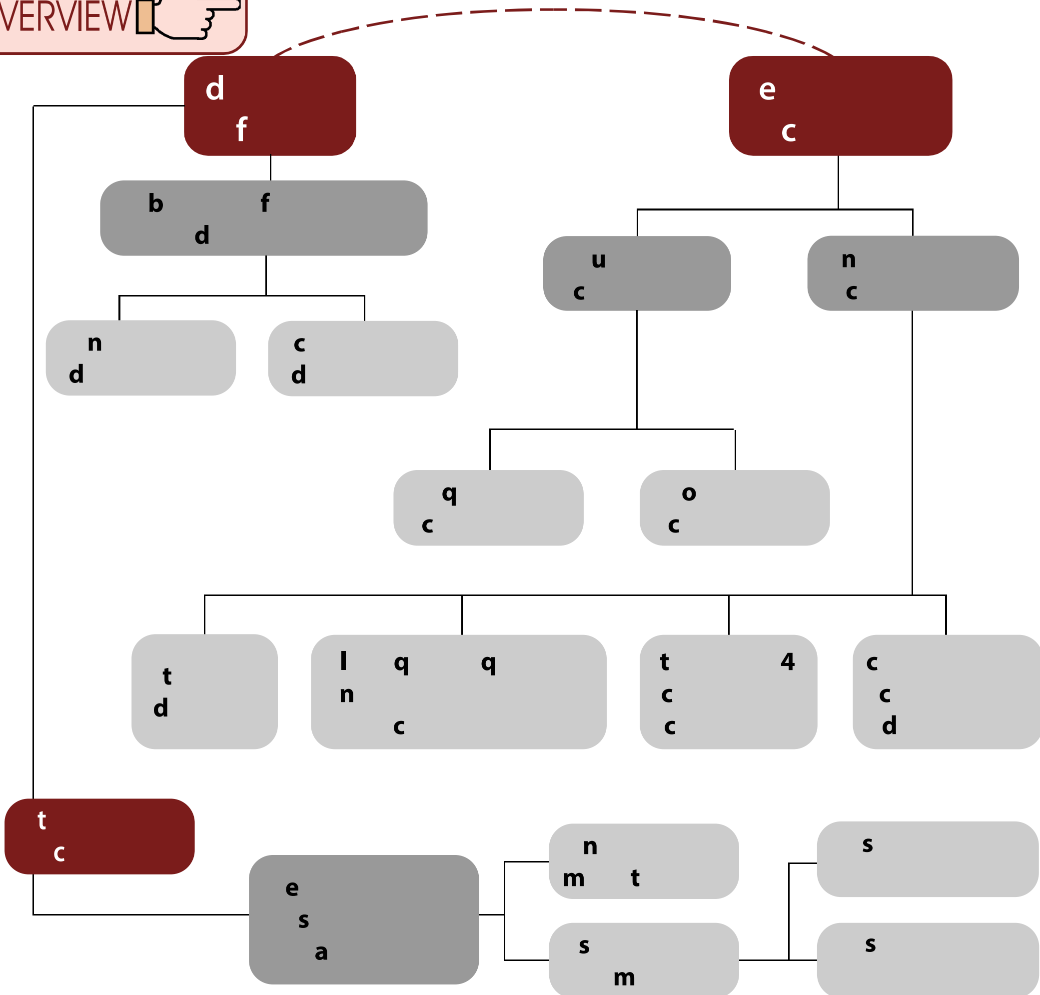
CORRELATION AND REGRESSION

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- The meaning of bivariate data and techniques of preparation of bivariate distribution;
- The concept of correlation between two variables and quantitative measurement of correlation including the interpretation of positive, negative and zero correlation;
- Concept of regression and its application in estimation of a variable from known set of data.

CHAPTER OVERVIEW



17.1 INTRODUCTION

In the previous chapter, we discussed many a statistical measure relating to Univariate distribution i.e. distribution of one variable like height, weight, mark, profit, wage and so on. However, there are situations that demand study of more than one variable simultaneously. A businessman may be keen to know what amount of investment would yield a desired level of profit or a student may want to know whether performing better in the selection test would enhance his or her chance of doing well in the final examination. With a view to answering this series of questions, we need to study more than one variable at the same time. Correlation Analysis and Regression Analysis are the two analyses that are made from a multivariate distribution i.e. a distribution of more than one variable. In particular when there are two variables, say x and y, we study bivariate distribution. We restrict our discussion to bivariate distribution only.

Correlation analysis, it may be noted, helps us to find an association or the lack of it between the two variables x and y. Thus if x and y stand for profit and investment of a firm or the marks in Statistics and Mathematics for a group of students, then we may be interested to know whether x and y are associated or independent of each other. The extent or amount of correlation between x and y is provided by different measures of Correlation namely Product Moment Correlation Coefficient or Rank Correlation Coefficient or Coefficient of Concurrent Deviations. In Correlation analysis, we must be careful about a cause and effect relation between the variables under consideration because there may be situations where x and y are related due to the influence of a third variable although no causal relationship exists between the two variables.

Regression analysis, on the other hand, is concerned with predicting the value of the dependent variable corresponding to a known value of the independent variable on the assumption of a mathematical relationship between the two variables and also an average relationship between them.

17.2 BIVARIATE DATA

When data are collected on two variables simultaneously, they are known as bivariate data and the corresponding frequency distribution, derived from it, is known as Bivariate Frequency Distribution. If x and y denote marks in Maths and Stats for a group of 30 students, then the corresponding bivariate data would be (x_i, y_i) for i = 1, 2, …. 30 where (x_1, y_1) denotes the marks in Mathematics and Statistics for the student with serial number or Roll Number 1, (x_2, y_2), that for the student with Roll Number 2 and so on and lastly (x_{30}, y_{30}) denotes the pair of marks for the student bearing Roll Number 30.

As in the case of a Univariate Distribution, we need to construct the frequency distribution for bivariate data. Such a distribution takes into account the classification in respect of both the variables simultaneously. Usually, we make horizontal classification in respect of x and vertical classification in respect of the other variable y. Such a distribution is known as Bivariate Frequency Distribution or Joint Frequency Distribution or Two way classification of the two variables x and y.

ILLUSTRATIONS:

Example 17.1: Prepare a Bivariate Frequency table for the following data relating to the marks in Statistics (x) and Mathematics (y):

(15, 13),	(1, 3),	(2, 6),	(8, 3),	(15, 10),	(3, 9),	(13, 19),
(10, 11),	(6, 4),	(18, 14),	(10, 19),	(12, 8),	(11, 14),	(13, 16),
(17, 15),	(18, 18),	(11, 7),	(10, 14),	(14, 16),	(16, 15),	(7, 11),
(5, 1),	(11, 15),	(9, 4),	(10, 15),	(13, 12)	(14, 17),	(10, 11),
(6, 9),	(13, 17),	(16, 15),	(6, 4),	(4, 8),	(8, 11),	(9, 12),
(14, 11),	(16, 15),	(9, 10),	(4, 6),	(5, 7),	(3, 11),	(4, 16),
(5, 8),	(6, 9),	(7, 12),	(15, 6),	(18, 11),	(18, 19),	(17, 16)
(10, 14)						

Take mutually exclusive classification for both the variables, the first class interval being 0-4 for both.

Solution:

From the given data, we find that

Range for x = 19–1 = 18

Range for y = 19–1 = 18

We take the class intervals 0-4, 4-8, 8-12, 12-16, 16-20 for both the variables. Since the first pair of marks is (15, 13) and 15 belongs to the fourth class interval (12-16) for x and 13 belongs to the fourth class interval for y, we put a stroke in the (4, 4)-th cell. We carry on giving tally marks till the list is exhausted.

Table 17.1

Bivariate Frequency Distribution of Marks in Statistics and Mathematics.

		MARKS IN MATHS					
X \ Y		0-4	4-8	8-12	12-16	16-20	Total
MARKS IN STATS	0–4	I (1)	I (1)	II (2)			4
	4–8	I (1)	IIII (4)	~~IIII~~ (5)	I (1)	I (1)	12
	8–12	I (1)	II (2)	IIII (4)	~~IIII~~ I (6)	I (1)	14
	12–16		I (1)	III (3)	II (2)	~~IIII~~ (5)	11
	16–20			I (1)	~~IIII~~ (5)	III (3)	9
	Total	3	8	15	14	10	50

We note, from the above table, that some of the cell frequencies (f_{ij}) are zero. Starting from the above Bivariate Frequency Distribution, we can obtain two types of univariate distributions which are known as:

(a) Marginal distribution.

(b) Conditional distribution.

If we consider the distribution of Statistics marks along with the marginal totals presented in the last column of Table 17.1, we get the marginal distribution of marks in Statistics. Similarly, we can obtain one more marginal distribution of Mathematics marks. The following table shows the marginal distribution of marks of Statistics.

Table 17.2

Marginal Distribution of Marks in Statistics

Marks	No. of Students
0-4	4
4-8	12
8-12	14
12-16	11
16-20	9
Total	50

We can find the mean and standard deviation of marks in Statistics from Table 17.2. They would be known as marginal mean and marginal SD of Statistics marks. Similarly, we can obtain the marginal mean and marginal SD of Mathematics marks. Any other statistical measure in respect of x or y can be computed in a similar manner.

If we want to study the distribution of Statistics Marks for a particular group of students, say for those students who got marks between 8 to 12 in Mathematics, we come across another univariate distribution known as conditional distribution.

Table 17.3

Conditional Distribution of Marks in Statistics for Students having Mathematics Marks between 8 to 12

Marks	No. of Students
0-4	2
4-8	5
8-12	4
12-16	3
16-20	1
Total	15

We may obtain the mean and SD from the above table. They would be known as conditional mean and conditional SD of marks of Statistics. The same result holds for marks in Mathematics. In particular, if there are m classifications for x and n classifications for y, then there would be altogether (m + n) conditional distribution.

17.3 CORRELATION ANALYSIS

While studying two variables at the same time, if it is found that the change in one variable is reciprocated by a corresponding change in the other variable either directly or inversely, then the two variables are known to be associated or correlated. Otherwise, the two variables are known to be dissociated or uncorrelated or independent. There are two types of correlation.

(i) Positive correlation

(ii) Negative correlation

If two variables move in the same direction i.e. an increase (or decrease) on the part of one variable introduces an increase (or decrease) on the part of the other variable, then the two variables are known to be positively correlated. As for example, height and weight yield and rainfall, profit and investment etc. are positively correlated.

On the other hand, if the two variables move in the opposite directions i.e. an increase (or a decrease) on the part of one variable results a decrease (or an increase) on the part of the other variable, then the two variables are known to have a negative correlation. The price and demand of an item, the profits of Insurance Company and the number of claims it has to meet etc. are examples of variables having a negative correlation.

The two variables are known to be uncorrelated if the movement on the part of one variable does not produce any movement of the other variable in a particular direction. As for example, Shoe-size and intelligence are uncorrelated.

17.4 MEASURES OF CORRELATION

We consider the following measures of correlation:

(a) Scatter diagram

(b) Karl Pearson's Product moment correlation coefficient

(c) Spearman's rank correlation co-efficient

(d) Co-efficient of concurrent deviations

(a) SCATTER DIAGRAM

This is a simple diagrammatic method to establish correlation between a pair of variables. Unlike product moment correlation co-efficient, which can measure correlation only when the variables are having a linear relationship, scatter diagram can be applied for any type of correlation – linear as well as non-linear i.e. curvilinear. Scatter diagram can distinguish between different types of correlation although it fails to measure the extent of relationship between the variables.

Each data point, which in this case a pair of values (x_i, y_i) is represented by a point in the rectangular axes of cordinates. The totality of all the plotted points forms the scatter diagram. The pattern of the plotted points reveals the nature of correlation. In case of a positive correlation, the plotted points lie from lower left corner to upper right corner, in case of a negative correlation the plotted points concentrate from upper left to lower right and in case of zero correlation, the plotted points would be equally distributed without depicting any particular pattern. The following figures show different types of correlation and the one to one correspondence between scatter diagram and product moment correlation coefficient.

FIGURE 17.1
Showing Positive Correlation

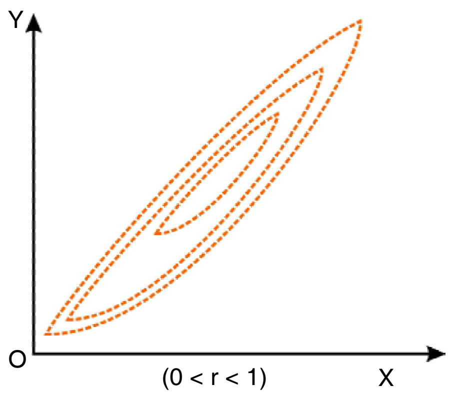
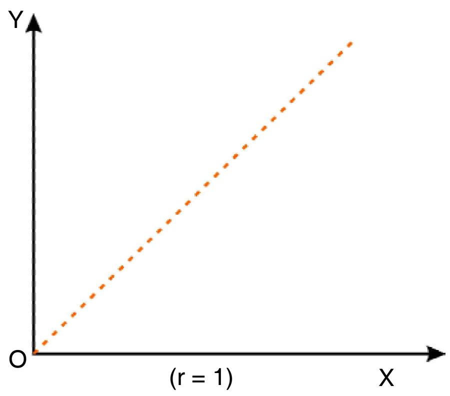
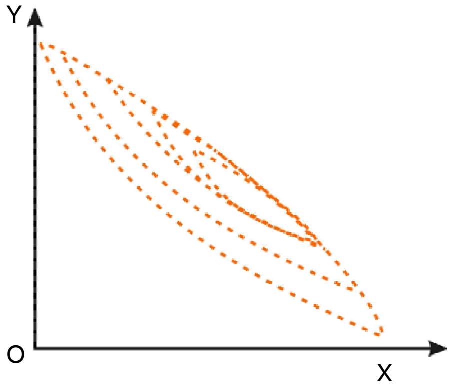


FIGURE 17.2
Showing Perfect Correlation



(0 < r <1)

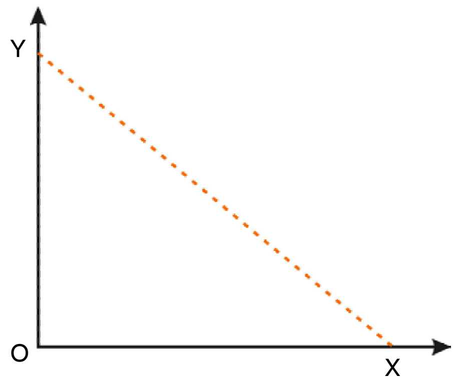
FIGURE 17.3
Showing Negative Correlation



(–1 < r <0)

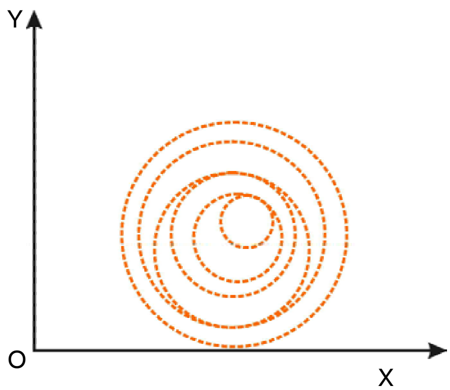
(r = 1)

FIGURE 17.4
Showing Perfect Negative Correlation



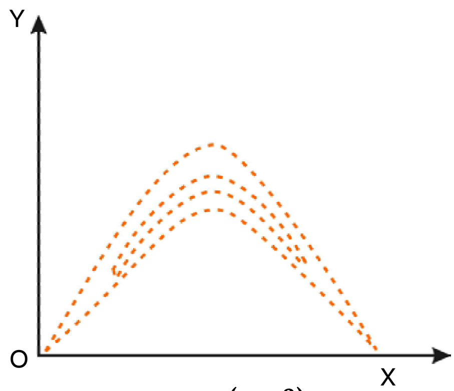
(r = –1)

FIGURE 17.5
Showing No Correlation



(r = 0)

FIGURE 17.6
Showing Curvilinear Correlation



(r = 0)

(b) KARL PEARSON'S PRODUCT MOMENT CORRELATION COEFFICIENT

This is by for the best method for finding correlation between two variables provided the relationship between the two variables is linear. Pearson's correlation coefficient may be defined as the ratio of covariance between the two variables to the product of the standard deviations of the two variables. If the two variables are denoted by x and y and if the corresponding bivariate data are (x_i, y_i) for i = 1, 2, 3, ….., n, then the coefficient of correlation between x and y, due to Karl Pearson, in given by :

$$r = r_{xy} = \frac{\text{Cov}(x, y)}{S_x \times S_y} \quad \text{......(17.1)}$$

where

$$\text{cov}(x, y) = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{\sum x_i y_i}{n} - \bar{x}\,\bar{y} \quad \text{......(17.2)}$$

$$S_x = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}} = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2} \quad \text{......(17.3)}$$

and $$S_y = \sqrt{\frac{\sum(y_i - \bar{y})^2}{n}} = \sqrt{\frac{\sum y_i^2}{n} - \bar{y}^2} \quad \text{......(17.4)}$$

A single formula for computing correlation coefficient is given by

$$r = \frac{n\sum x_i y_i - \sum x_i \times \sum y_i}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}} \quad \text{......(17.5)}$$

In case of a bivariate frequency distribution, we have

$$\text{Cov}(x,y) = \frac{\sum_{i,j} x_i y_i f_{ij}}{N} - \bar{x} \times \bar{y} \quad \text{......(17.6)}$$

$$S_x = \sqrt{\frac{\sum_i f_{io} x_i^2}{N} - \bar{x}^2} \quad \text{......(17.7)}$$

and $$S_y = \sqrt{\frac{\sum_j f_{oj} y_j^2}{N} - \bar{y}^2} \quad \text{......(17.8)}$$

where x_i = Mid-value of the i^{th} class interval of x.

y_j = Mid-value of the j^{th} class interval of y

f_{io} = Marginal frequency of x

f_{oj} = Marginal frequency of y

f_{ij} = frequency of the $(i, j)^{th}$ cell

$$N = \sum_{i,j} f_{ij} = \sum_i f_{io} = \sum_j f_{oj} = \text{Total frequency} \quad \text{............... (17.9)}$$

PROPERTIES OF CORRELATION COEFFICIENT

(i) **The Coefficient of Correlation is a unit-free measure.**

This means that if x denotes height of a group of students expressed in cm and y denotes their weight expressed in kg, then the correlation coefficient between height and weight would be free from any unit.

(ii) **The coefficient of correlation remains invariant under a change of origin and/or scale of the variables under consideration depending on the sign of scale factors.**

This property states that if the original pair of variables x and y is changed to a new pair of variables u and v by effecting a change of origin and scale for both x and y i.e.

$$u = \frac{x-a}{b} \text{ and } v = \frac{y-c}{d}$$

where a and c are the origins of x and y and b and d are the respective scales and then we have

$$r_{xy} = \frac{bd}{|b||d|} r_{uv} \quad \text{...(17.10)}$$

r_{xy} and r_{uv} being the coefficient of correlation between x and y and u and v respectively, (17.10) established, numerically, the two correlation coefficients remain equal and they would have opposite signs only when b and d, the two scales, differ in sign.

(iii) **The coefficient of correlation always lies between –1 and 1, including both the limiting values i.e.**

$$-1 \le r \le 1 \quad \text{..(17.11)}$$

Example 17.2: Compute the correlation coefficient between x and y from the following data n = 10, $\sum xy = 220$, $\sum x^2 = 200$, $\sum y^2 = 262$

$\sum x = 40$ and $\sum y = 50$

Solution:

From the given data, we have by applying (17.5),

$$r = \frac{n\sum xy - \sum x \times \sum y}{\sqrt{n\sum x^2 - (\sum x)^2} \times \sqrt{n\sum y^2 - (\sum y)^2}}$$

$$= \frac{10 \times 220 - 40 \times 50}{\sqrt{10 \times 200 - (40)^2} \times \sqrt{10 \times 262 - (50)^2}}$$

$$= \frac{2200 - 2000}{\sqrt{2000 - 1600} \times \sqrt{2620 - 2500}}$$

$$= \frac{200}{20 \times 10.9545}$$

$$= 0.91$$

Thus there is a good amount of positive correlation between the two variables x and y.

Alternately

As given, $\bar{x} = \frac{\sum x}{n} = \frac{40}{10} = 4$

$$\bar{y} = \frac{\sum y}{n} = \frac{50}{10} = 5$$

$$\text{Cov}(x, y) = \frac{\sum xy}{n} - \bar{x}.\bar{y}$$

$$= \frac{220}{10} - 4.5 = 2$$

$$S_x = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2}$$

$$= \sqrt{\frac{200}{10} - 4^2} = 2$$

$$S_y = \sqrt{\frac{\Sigma y_i^2}{n} - \bar{y}^2}$$

$$= \sqrt{\frac{262}{10} - 5^2}$$

$$= \sqrt{26.20 - 25} = 1.0954$$

Thus applying formula (17.1), we get

$$r = \frac{\text{cov}(x, y)}{S_x . S_y}$$

$$= \frac{2}{2 \times 1.0954} = 0.91$$

As before, we draw the same conclusion.

Example 17.3: Find product moment correlation coefficient from the following information:

x :	2	3	5	5	6	8
y :	9	8	8	6	5	3

Solution:

In order to find the covariance and the two standard deviation, we prepare the following table:

Table 17.3

Computation of Correlation Coefficient

x_i (1)	y_i (2)	x_iy_i (3)= (1) x (2)	x_i^2 (4)= $(1)^2$	y_i^2 (5)= $(2)^2$
2	9	18	4	81
3	8	24	9	64
5	8	40	25	64
5	6	30	25	36
6	5	30	36	25
8	3	24	64	9
29	39	166	163	279

We have

$$\bar{x} = \frac{29}{6} = 4.8333 \quad \bar{y} = \frac{39}{6} = 6.50$$

$$\text{cov}(x, y) = \frac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}$$

$$= 166/6 - 4.8333 \times 6.50 = -3.7498$$

$$= \sqrt{\frac{\Sigma x_i^2}{n} - (\bar{x})^2}$$

$$= \sqrt{\frac{163}{6} - (4.8333)^2}$$

$$= \sqrt{27.1667 - 23.3608} = 1.95$$

$$S_y = \sqrt{\frac{\Sigma y_i^2}{n} - (\bar{y})^2}$$

$$= \sqrt{\frac{279}{6} - (6.50)^2}$$

$$= \sqrt{46.50 - 42.25} = 2.0616$$

Thus the correlation coefficient between x and y in given by

$$r = \frac{\text{cov}(x, y)}{S_x \times S_y}$$

$$= \frac{-3.7498}{1.9509 \times 2.0616}$$

$$= -0.93$$

We find a high degree of negative correlation between x and y. Also, we could have applied formula (17.5) as we have done for the first problem of computing correlation coefficient.

Sometimes, a change of origin reduces the computational labor to a great extent. This we are going to do in the next problem.

Example 17.4: The following data relate to the test scores obtained by eight salesmen in an aptitude test and their daily sales in thousands of rupees:

Salesman :	1	2	3	4	5	6	7	8
scores :	60	55	62	56	62	64	70	54
Sales :	31	28	26	24	30	35	28	24

Solution:

Let the scores and sales be denoted by x and y respectively. We take a, origin of x as the average of the two extreme values i.e. 54 and 70. Hence a = 62 similarly, the origin of y is taken as $b = \frac{24+35}{2} \cong 30$

Table 17.4

Computation of Correlation Coefficient Between Test Scores and Sales.

Scores (x_i) (1)	Sales in ₹ 1000 (y_i) (2)	$u_i = x_i - 62$ (3)	$v_i = y_i - 30$ (4)	$u_i v_i$ (5)=(3)x(4)	u_i^2 (6)=(3)^2	v_i^2 (7)=(4)^2
60	31	–2	1	–2	4	1
55	28	–7	–2	14	49	4
62	26	0	–4	0	0	16
56	24	–6	–6	36	36	36
62	30	0	0	0	0	0
64	35	2	5	10	4	25
70	28	8	–2	–16	64	4
54	24	–8	–6	48	64	36
Total	—	–13	–14	90	221	122

Since correlation coefficient remains unchanged due to change of origin, we have

$$r = r_{xy} = r_{uv} = \frac{n\sum u_i v_i - \sum u_i \times \sum v_i}{\sqrt{n\sum u_i^2 - \left(\sum u_i\right)^2} \times \sqrt{n\sum v_i^2 - \left(\sum v_i\right)^2}}$$

$$= \frac{8\times 90 - (-13)\times(-14)}{\sqrt{8\times 221 - (-13)^2} \times \sqrt{8\times 122 - (-14)^2}}$$

$$= \frac{538}{\sqrt{1768 - 169} \times \sqrt{976 - 196}}$$

$$= 0.48$$

In some cases, there may be some confusion about selecting the pair of variables for which correlation is wanted. This is explained in the following problem.

Example 17.5: Examine whether there is any correlation between age and blindness on the basis of the following data:

Age in years :	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of Persons (in thousands) :	90	120	140	100	80	60	40	20
No. of blind Persons :	10	15	18	20	15	12	10	06

Solution:

Let us denote the mid-value of age in years as x and the number of blind persons per lakh as y. Then as before, we compute correlation coefficient between x and y.

Table 17.5

Computation of correlation between age and blindness

Age in years (1)	Mid-value x (2)	No. of Persons ('000) P (3)	No. of blind B (4)	No. of blind per lakh y=B/P × 1 lakh (5)	xy (2)×(5) (6)	x^2 $(2)^2$ (7)	y^2 $(5)^2$ (8)
0-10	5	90	10	11	55	25	121
10-20	15	120	15	12	180	225	144
20-30	25	140	18	13	325	625	169
30-40	35	100	20	20	700	1225	400
40-50	45	80	15	19	855	2025	361
50-60	55	60	12	20	1100	3025	400
60-70	65	40	10	25	1625	4225	625
70-80	75	20	6	30	2250	5625	900
Total	320	—	—	150	7090	17000	3120

The correlation coefficient between age and blindness is given by

$$r = \frac{n\sum xy - \sum x.\sum y}{\sqrt{n\sum x^2 - (\sum x)^2} \times \sqrt{n\sum y^2 - (\sum y)^2}}$$

$$= \frac{8 \times 7090 - 320 \times 150}{\sqrt{8 \times 17000 - (320)^2} \times \sqrt{8 \times 3120 - (150)^2}}$$

$$= \frac{8720}{183.3030.49.5984}$$

$$= 0.96$$

which exhibits a very high degree of positive correlation between age and blindness.

Example 17.6: Coefficient of correlation between x and y for 20 items is 0.4. The AM's and SD's of x and y are known to be 12 and 15 and 3 and 4 respectively. Later on, it was found that the pair (20, 15) was wrongly taken as (15, 20). Find the correct value of the correlation coefficient.

Solution:

We are given that n = 20 and the original r = 0.4, $\bar{x}$ = 12, $\bar{y}$ = 15, S_x = 3 and S_y = 4

$$r = \frac{\text{cov}(x, y)}{S_x \times S_y} \quad = 0.4 = \frac{\text{cov}(x, y)}{3 \times 4}$$

$$= \text{Cov}(x, y) = 4.8$$

$$= \frac{\sum xy}{n} - \bar{x}\,\bar{y} = 4.8$$

$$= \frac{\sum xy}{20} - 12 \times 15 = 4.8$$

$$= \sum xy = 3696$$

Hence, corrected $\sum xy$= 3696 – 20 × 15 + 15 × 20 = 3696

Also, $S_x^2 = 9$

$= (\sum x^2 / 20) - 12^2 = 9$

$\sum x^2 = 3060$

Similarly, $S_y^2 = 16$

$$S_y^2 = \frac{\sum y^2}{20} - 15^2 = 16$$

$\sum y^2 = 4820$

Thus corrected $\sum x = n\bar{x}$ – wrong value + correct value.

$= 20 \times 12 - 15 + 20$

$= 245$

Similarly corrected $\sum y = 20 \times 15 - 20 + 15 = 295$

Corrected $\sum x^2 = 3060 - 15^2 + 20^2 = 3235$

Corrected $\sum y^2 = 4820 - 20^2 + 15^2 = 4645$

Thus corrected value of the correlation coefficient by applying formula (17.5)

$$= \frac{20 \times 3696 - 245 \times 295}{\sqrt{20 \times 3235 - (245)^2} \times \sqrt{20 \times 4645 - (295)^2}}$$

$$= \frac{73920 - 72275}{68.3740 \times 76.6480}$$

$= 0.31$

Example 17.7: Compute the coefficient of correlation between marks in Statistics and Mathematics for the bivariate frequency distribution shown in Table 17.6

Solution:

For the sake of computational advantage, we effect a change of origin and scale for both the variable x and y.

$$\text{Define } u_i = \frac{x_i - a}{b} = \frac{x_i - 10}{4}$$

$$\text{And } v_j = \frac{y_i - c}{d} = \frac{y_i - 10}{4}$$

Where x_i and y_j denote respectively the mid-values of the x-class interval and y-class interval respectively. The following table shows the necessary calculation on the right top corner of each cell, the product of the cell frequency, corresponding u value and the respective v value has been shown. They add up in a particular row or column to provide the value of $f_{ij}u_iv_j$ for that particular row or column.

Table 17.6

Computation of Correlation Coefficient Between Marks of Mathematics and Statistics

Class Interval / Mid-value			0-4	4-8	8-12	12-16	16-20				
			2	6	10	14	18				
Class Interval	Mid -value	V_j / u_i	–2	–1	0	1	2	f_{io}	$f_{io}u_i$	$f_{io}u_i^2$	$f_{ij}u_iv_j$
0-4	2	–2	1 [4]	1 [2]	2 [0]			4	–8	16	6
4-8	6	–1	2 [4]	4 [4]	5 [0]	1 [–1]	1 [–2]	13	–13	13	5
8-12	10	0		2 [0]	4 [0]	6 [0]	1 [0]	13	0	0	0
12-16	14	1		1 [–1]	3 [0]	2 [2]	5 [10]	11	11	11	11
16-20	18	2			1 [0]	5 [10]	3 [12]	9	18	36	22
		f_{oj}	3	8	15	14	10	50	5	76	44
		$f_{oj}v_j$	–6	–8	0	14	20	20			
		$f_{oj}v_j^2$	12	8	0	14	40	74			
		$f_{ij}u_iv_j$	8	5	0	11	20	44		CHECK	

A single formula for computing correlation coefficient from bivariate frequency distribution is given by

$$r = \frac{N\sum_{i,j} f_{ij}u_iv_j - \sum f_{io}u_i \times \sum f_{oj}v_j}{\sqrt{N\sum f_{io}u_i^2 - \left(\sum f_{io}u_i\right)^2 \times \sum f_{oj}v_j^2 - \left(\sum f_{oj}v_j\right)^2}} \quad \text{..........................(17.10)}$$

$$= \frac{50 \times 44 - 8 \times 20}{\sqrt{50 \times 76 - 8^2}\sqrt{50 \times 74 - 20^2}}$$

$$= \frac{2040}{61.1228 \times 57.4456}$$

$$= 0.58$$

The value of r shown a good amount of positive correlation between the marks in Statistics and Mathematics on the basis of the given data.

Example 17.8: Given that the correlation coefficient between x and y is 0.8, write down the correlation coefficient between u and v where

(i) 2u + 3x + 4 = 0 and 4v + 16y + 11 = 0

(ii) 2u – 3x + 4 = 0 and 4v + 16y + 11 = 0

(iii) 2u – 3x + 4 = 0 and 4v – 16y + 11 = 0

(iv) 2u + 3x + 4 = 0 and 4v – 16y + 11 = 0

Solution:

Using (17.10), we find that

$$r_{xy} = \frac{bd}{|b||d|} r_{uv}$$

i.e. $r_{xy} = r_{uv}$ if b and d are of same sign and $r_{uv} = -r_{xy}$ when b and d are of opposite signs, b and d being the scales of x and y respectively. In (i), u = (–2) + (-3/2) x and v = (–11/4) + (–4)y.

Since b = –3/2 and d = –4 are of same sign, the correlation coefficient between u and v would be the same as that between x and y i.e. $r_{xy} = 0.8 = r_{uv}$

In (ii), u = (–2) + (3/2)x and v = (–11/4) + (–4)y Hence b = 3/2 and d = –4 are of opposite signs and we have $r_{uv} = -r_{xy} = -0.8$

Proceeding in a similar manner, we have r_{uv} = 0.8 and – 0.8 in (iii) and (iv).

(c) SPEARMAN'S RANK CORRELATION COEFFICIENT

When we need finding correlation between two qualitative characteristics, say, beauty and intelligence, we take recourse to using rank correlation coefficient. Rank correlation can also be applied to find the level of agreement (or disagreement) between two judges so far as assessing a qualitative characteristic is concerned. As compared to product moment correlation coefficient, rank correlation coefficient is easier to compute, it can also be advocated to get a first hand impression about the correlation between a pair of variables.

Spearman's rank correlation coefficient is given by

$$r_R = 1 - \frac{6\sum d_i^2}{n(n^2-1)} \quad \text{...} (17.11)$$

where r_R denotes rank correlation coefficient and it lies between –1 and 1 inclusive of these two values.

$d_i = x_i - y_i$ represents the difference in ranks for the i-th individual and n denotes the number of individuals.

In case u individuals receive the same rank, we describe it as a tied rank of length u. In case of a tied rank, formula (17.11) is changed to

$$r_R = 1 - \frac{6\left[\sum_i d_i^2 + \sum_j \frac{\left(tj^3 - t_j\right)}{12}\right]}{n\left(n^2 - 1\right)} \quad \text{.. (17.12)}$$

In this formula, t_j represents the j^{th} tie length and the summation $\sum_j (t_j^3 - t_j)$ extends over the lengths of all the ties for both the series.

Example 17.9: compute the coefficient of rank correlation between sales and advertisement expressed in thousands of rupees from the following data:

Sales :	90	85	68	75	82	80	95	70
Advertisement :	7	6	2	3	4	5	8	1

Solution:

Let the rank given to sales be denoted by x and rank of advertisement be denoted by y. We note that since the highest sales as given in the data, is 95, it is to be given rank 1, the second highest sales 90 is to be given rank 2 and finally rank 8 goes to the lowest sales, namely 68. We have given rank to the other variable advertisement in a similar manner. Since there are no ties, we apply formula (17.11).

Table 17.7

Computation of Rank correlation between Sales and Advertisement.

Sales (x_i)	Advertisement (y_i)	Rank for Sales (x_i)	Rank for Advertisement (y_i)	$d_i = x_i - y_i$	d_i^2
90	7	2	2	0	0
85	6	3	3	0	0
68	2	8	7	1	1
75	3	6	6	0	0
82	4	4	5	–1	1
80	5	5	4	1	1
95	8	1	1	0	0
70	1	7	8	–1	1
Total	—	—	—	0	4

Since n = 8 and $\sum d_i^2 = 4$, applying formula (17.11), we get.

$$r_R = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 4}{8(8^2-1)}$$

$$= 1 - 0.0476$$

$$= 0.95$$

The high positive value of the rank correlation coefficient indicates that there is a very good amount of agreement between sales and advertisement.

Example 17.10: Compute rank correlation from the following data relating to ranks given by two judges in a contest:

Serial No. of Candidate :	1	2	3	4	5	6	7	8	9	10
Rank by Judge A :	10	5	6	1	2	3	4	7	9	8
Rank by Judge B :	5	6	9	2	8	7	3	4	10	1

Solution:

We directly apply formula (17.11) as ranks are already given.

Table 17.8

Computation of Rank Correlation Coefficient between the ranks given by 2 Judges

Serial No.	Rank by A (x_i)	Rank by B (y_i)	$d_i = x_i - y_i$	d_i^2
1	10	5	5	25
2	5	6	–1	1
3	6	9	–3	9
4	1	2	–1	1
5	2	8	–6	36
6	3	7	–4	16
7	4	3	1	1
8	7	4	3	9
9	8	10	–2	4
10	9	1	8	64
Total	—	—	0	166

The rank correlation coefficient is given by

$$r_R = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 166}{10(10^2-1)}$$

$$= -0.006$$

The very low value (almost 0) indicates that there is hardly any agreement between the ranks given by the two Judges in the contest.

Example 17.11: Compute the coefficient of rank correlation between Eco. marks and stats. Marks as given below:

Eco Marks :	80	56	50	48	50	62	60
Stats Marks :	90	75	75	65	65	50	65

Solution:

This is a case of tied ranks as more than one student share the same mark both for Economics and Statistics. For Eco. the student receiving 80 marks gets rank 1 one getting 62 marks receives rank 2, the student with 60 receives rank 3, student with 56 marks gets rank 4 and since there are two students, each getting 50 marks, each would be receiving a common rank, the average of the next two ranks 5 and 6 i.e. $\frac{5+6}{2}$ i.e. 5.50 and lastly the last rank..

7 goes to the student getting the lowest Eco marks. In a similar manner, we award ranks to the students with stats marks.

Table 17.9

Computation of Rank Correlation Between Eco Marks and Stats Marks with Tied Marks

Eco Mark (x_i)	Stats Mark (y_i)	Rank for Eco (x_i)	Rank for Stats (y_i)	$d_i = x_i - y_i$	d_i^2
80	90	1	1	0	0
56	75	4	2.50	1.50	2.25
50	75	5.50	2.50	3	9
48	65	7	5	2	4
50	65	5.50	5	0.50	0.25
62	50	2	7	–5	25
60	65	3	5	–2	4
Total	—	—	—	0	44.50

For Economics mark there is one tie of length 2 and for stats mark, there are two ties of lengths 2 and 3 respectively.

$$\text{Thus } \frac{\sum\left(t_j^3 - t_j\right)}{12} = \frac{\left(2^3 - 2\right) + \left(2^3 - 2\right) + \left(3^3 - 3\right)}{12} = 3$$

$$\text{Thus } r_R = 1 - \frac{6\left[\sum_i d_i^2 + \sum_j \frac{\left(tj^3 - t_j\right)}{12}\right]}{n\left(n^2 - 1\right)}$$

$$= 1 - \frac{6 \times (44.50 + 3)}{7(7^2 - 1)}$$

$$= 0.15$$

Example 17.12: For a group of 8 students, the sum of squares of differences in ranks for Mathematics and Statistics marks was found to be 50 what is the value of rank correlation coefficient?

Solution:

As given $n = 8$ and $\sum d_i^2 = 50$. Hence the rank correlation coefficient between marks in Mathematics and Statistics is given by

$$r_R = 1 - \frac{6\sum d_i^2}{n\left(n^2 - 1\right)}$$

$$= 1 - \frac{6 \times 50}{8(8^2 - 1)}$$

$$= 0.40$$

Example 17.13: For a number of towns, the coefficient of rank correlation between the people living below the poverty line and increase of population is 0.50. If the sum of squares of the differences in ranks awarded to these factors is 82.50, find the number of towns.

Solution:

As given $r_R = 0.50$, $\sum d_i^2 = 82.50$.

$$\text{Thus } r_R = 1 - \frac{6\sum d_i^2}{n\left(n^2 - 1\right)}$$

$$0.50 = 1 - \frac{6 \times 82.50}{n\left(n^2 - 1\right)}$$

$= n\,(n^2 - 1) = 990$

$= n\,(n^2 - 1) = 10(10^2 - 1)$

$\therefore$ n = 10 as n must be a positive integer.

Example 17.14: While computing rank correlation coefficient between profits and investment for 10 years of a firm, the difference in rank for a year was taken as 7 instead of 5 by mistake and the value of rank correlation coefficient was computed as 0.80. What would be the correct value of rank correlation coefficient after rectifying the mistake?

Solution:

We are given that n = 10,

r_R = 0.80 and the wrong $d_i = 7$ should be replaced by 5.

$$r_R = 1 - \frac{6\sum d_i^2}{n\left(n^2 - 1\right)}$$

$$0.80 = 1 - \frac{6\sum d_i^2}{10\left(10^2 - 1\right)}$$

$$\sum d_i^2 = 33$$

Corrected $\sum d_i^2 = 33 - 7^2 + 5^2 = 9$

Hence rectified value of rank correlation coefficient

$$= 1 - \frac{6 \times 9}{10 \times \left(10^2 - 1\right)}$$

$= 0.95$

(d) COEFFICIENT OF CONCURRENT DEVIATIONS

A very simple and casual method of finding correlation when we are not serious about the magnitude of the two variables is the application of concurrent deviations. This method involves in attaching a positive sign for a x-value (except the first) if this value is more than the previous value and assigning a negative value if this value is less than the previous value. This is done for the y-series as well. The deviation in the x-value and the corresponding y-value is known to be concurrent if both the deviations have the same sign.

Denoting the number of concurrent deviation by c and total number of deviations as m (which must be one less than the number of pairs of x and y values), the coefficient of concurrent deviation is given by

$$r_C = \pm\sqrt{\pm\frac{(2c-m)}{m}} \quad \text{...(17.13)}$$

If (2c–m) >0, then we take the positive sign both inside and outside the radical sign and if (2c–m) <0, we are to consider the negative sign both inside and outside the radical sign.

Like Pearson's correlation coefficient and Spearman's rank correlation coefficient, the coefficient of concurrent deviations also lies between –1 and 1, both inclusive.

Example 17.15: Find the coefficient of concurrent deviations from the following data.

Year :	1990	1991	1992	1993	1994	1995	1996	1997
Price :	25	28	30	23	35	38	39	42
Demand :	35	34	35	30	29	28	26	23

Solution:

Table 17.10

Computation of Coefficient of Concurrent Deviations.

Year	Price	Sign of deviation from the previous figure (a)	Demand	Sign of deviation from the previous figure (b)	Product of deviation (ab)
1990	25		35		
1991	28	+	34	–	–
1992	30	+	35	+	+
1993	23	–	30	–	+
1994	35	+	29	–	–
1995	38	+	28	–	–
1996	39	+	26	–	–
1997	42	+	23	–	–

In this case, m = number of pairs of deviations = 7

c = No. of positive signs in the product of deviation column = Number of concurrent deviations = 2

Thus $r_C = \pm\sqrt{\pm\frac{(2c-m)}{m}}$

$$= \pm\sqrt{\pm\frac{(4-7)}{7}}$$

$$= \pm\sqrt{\pm\frac{(-3)}{7}}$$

$$= -\sqrt{\frac{3}{7}} = -0.65$$

(Since $\frac{2c-m}{m} = \frac{-3}{7}$ we take negative sign both inside and outside of the radical sign)

Thus there is a negative correlation between price and demand.

17.5 REGRESSION ANALYSIS

In regression analysis, we are concerned with the estimation of one variable for a given value of another variable (or for a given set of values of a number of variables) on the basis of an average mathematical relationship between the two variables (or a number of variables). Regression analysis plays a very important role in the field of every human activity. A businessman may be keen to know what would be his estimated profit for a given level of investment on the basis of the past records. Similarly, an outgoing student may like to know her chance of getting a first class in the final University Examination on the basis of her performance in the college selection test.

When there are two variables x and y and if y is influenced by x i.e. if y depends on x, then we get a simple linear regression or simple regression. y is known as dependent variable or regression or explained variable and x is known as independent variable or predictor or explanator. In the previous examples since profit depends on investment or performance in the University Examination is dependent on the performance in the college selection test, profit or performance in the University Examination is the dependent variable and investment or performance in the selection test is the Independent variable.

In case of a simple regression model if y depends on x, then the regression line of y on x in given by

$y = a + bx$ …………………… (17.14)

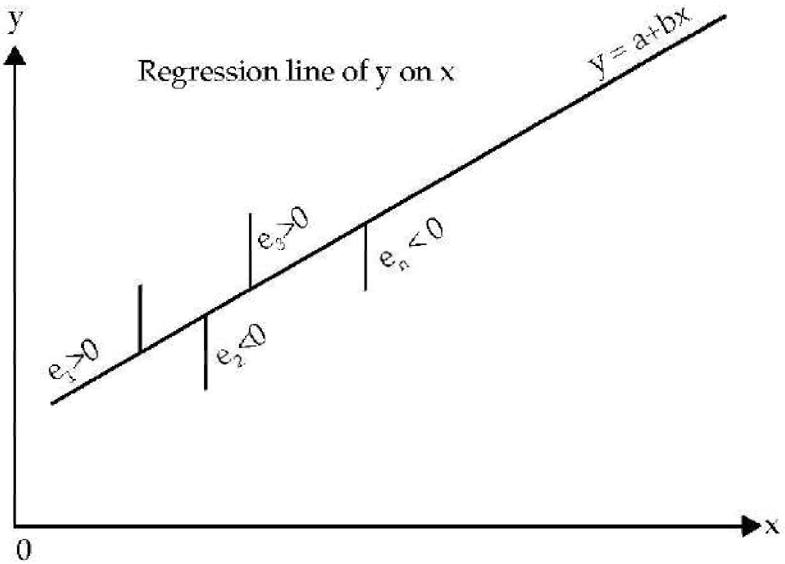
Here a and b are two constants and they are also known as regression parameters. Furthermore, b is also known as the regression coefficient of y on x and is also denoted by b_{yx}. We may define the regression line of y on x as the line of best fit obtained by the method of least squares and used for estimating the value of the dependent variable y for a known value of the independent variable x.

The method of least squares involves in minimizing

$$\sum e_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum (y_i - a - bx_i)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17.15)$$

where y_i demotes the actual or observed value and $\hat{y}_i = a + b_{xi}$, the estimated value of y_i for a given value of x_i, e_i is the difference between the observed value and the estimated value and e_i is technically known as error or residue. This summation intends over n pairs of observations of (x_i, y_i). The line of regression of y or x and the errors of estimation are shown in the following figure.

FIGURE **17.7**



SHOWING REGRESSION LINE OF y on x

AND ERRORS OF ESTIMATION

Minimisation of (17.15) yields the following equations known as 'Normal Equations'

$$\sum y_i = na + b\sum x_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (17.16)$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (17.17)$$

Solving there two equations for b and a, we have the "least squares" estimates of b and a as

$$b = \frac{Cov(x,y)}{S_x^{\,2}}$$

$$= \frac{r.S_x.S_y}{S_x^2}$$

$$= \frac{r.S_y}{S_x} \quad ......(17.18)$$

After estimating b, estimate of a is given by

$$a = \bar{y} - b\bar{x} \quad ......(17.19)$$

Substituting the estimates of b and a in (17.14), we get

$$\frac{(y - \bar{y})}{S_y} = \frac{r(x - \bar{x})}{S_x} \quad ......(17.20)$$

There may be cases when the variable x depends on y and we may take the regression line of x on y as

$x = a\hat{} + b\hat{}y$

Unlike the minimization of vertical distances in the scatter diagram as shown in figure (17.7) for obtaining the estimates of a and b, in this case we minimize the horizontal distances and get the following normal equation in a^ and b^, the two regression parameters :

$\sum x_i = na\hat{} + b\hat{}\sum y_i$ ...... (17.21)

$\sum x_i y_i = a\hat{}\sum y_i + b\hat{}\sum y_i^2$ ...... (17.22)

or solving these equations, we get

$$b\hat{} = b_{xy} = \frac{cov(x,y)}{S_y^2} = \frac{r.S_x}{S_y} \quad ......(17.23)$$

and $a\hat{} = \bar{x} - b\hat{}\,\bar{y}$ ......(17.24)

A single formula for estimating b is given by

$$b\hat{} = byx = \frac{n\Sigma xy - \Sigma x.\Sigma y}{n(\Sigma x^2) - (\Sigma x)} \quad ......(17.25)$$

Similarly, $b\hat{} = bxy = \dfrac{n\Sigma xy - \Sigma x.\Sigma y}{n\Sigma y^2 - (\Sigma y)^2}$ ......(17.26)

The standardized form of the regression equation of x on y, as in (17.20), is given by

$$\frac{x-\bar{x}}{S_x}=r\frac{(y-\bar{y})}{S_y} \qquad \text{.................................... (17.27)}$$

Example 17.15: Find the two regression equations from the following data:

x:	2	4	5	5	8	10
y:	6	7	9	10	12	12

Hence estimate y when x is 13 and estimate also x when y is 15.

Solution:

Table 17.11

Computation of Regression Equations

x_i	y_i	$x_i\, y_i$	x_i^2	y_i^2
2	6	12	4	36
4	7	28	16	49
5	9	45	25	81
5	10	50	25	100
8	12	96	64	144
10	12	120	100	144
34	56	351	234	554

On the basis of the above table, we have

$$\bar{x}=\frac{\Sigma x_i}{n}=\frac{34}{6}=5.6667$$

$$\bar{y}=\frac{\Sigma y_i}{n}=\frac{56}{6}=9.3333$$

$$\text{cov}(x, y) = \frac{\Sigma x_i y_i}{n}-\bar{x}\,\bar{y}$$

$$= \frac{351}{6}-5.6667\times 9.3333$$

$$= 58.50–52.8890$$

$$= 5.6110$$

$$S_x^{\,2} = \frac{\Sigma x_i^2}{n}-(\bar{x})^2$$

$$= \frac{234}{6} - (5.6667)^2$$

$$= 39 - 32.1115$$

$$= 6.8885$$

$$S_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2$$

$$= \frac{554}{6} - (9.3333)^2$$

$$= 92.3333 - 87.1105$$

$$= 5.2228$$

The regression line of y on x is given by

$y = a + bx$

Where $\hat{b} = \frac{\text{cov}(x, y)}{S_x^2}$

$$= \frac{5.6110}{6.8885}$$

$$= 0.8145$$

and $\hat{a} = \bar{y} - b\bar{x}$

$$= 9.3333 - 0.8145 \times 5.6667$$

$$= 4.7178$$

Thus the estimated regression equation of y on x is

$y = 4.7178 + 0.8145x$

When x = 13, the estimated value of y is given by $\hat{y} = 4.7178 + 0.8145 \times 13 = 15.3063$

The regression line of x on y is given by

$x = \hat{a} + \hat{b}\,y$

Where $\hat{b} = \frac{\text{cov}(x, y)}{S_y^2}$

$$= \frac{5.6110}{5.2228}$$

$= 1.0743$

and $\hat{a} = \bar{x} - \hat{b}\,\bar{y}$

$= 5.6667 - 1.0743 \times 9.3333$

$= -4.3601$

Thus the estimated regression line of x on y is

$x = -4.3601 + 1.0743y$

When y = 15, the estimate value of x is given by

$\hat{x} = -4.3601 + 1.0743 \times 15$

$= 11.75$

Example 17.16: Marks of 8 students in Mathematics and statistics are given as:

Mathematics:	80	75	76	69	70	85	72	68
Statistics:	85	65	72	68	67	88	80	70

Find the regression lines. When marks of a student in Mathematics are 90, what are his most likely marks in statistics?

Solution:

We denote the marks in Mathematics and Statistics by x and y respectively. We are to find the regression equation of y on x and also of x or y. Lastly, we are to estimate y when x = 90. For computation advantage, we shift origins of both x and y.

Table 17.12

Computation of regression lines

Maths mark (x_i)	Stats mark (y_i)	$u_i = x_i - 74$	$v_i = y_i - 76$	$u_i v_i$	u_i^2	v_i^2
80	85	6	9	54	36	81
75	65	1	–11	–11	1	121
76	72	2	–4	–8	4	16
69	68	–5	–8	40	25	64
70	67	–4	–9	36	16	81
85	88	11	12	132	121	144
72	80	–2	4	–8	4	16
68	70	–6	–6	36	36	36
595	595	3	–13	271	243	559

The regression coefficients b (or b_{yx}) and b′ (or b_{xy}) remain unchanged due to a shift of origin.

Applying (17.25) and (17.26), we get

$$b = b_{yx} = b_{vu} = \frac{n\sum u_i v_i - \sum u_i . \sum v_i}{n\sum u_i^2 - (\sum u_i)^2}$$

$$= \frac{8.(271) - (3).(-13)}{8.(243) - (3)^2}$$

$$= \frac{2168 + 39}{1944 - 9}$$

$$= 1.1406$$

and $$b\hat{} = b_{xy} = b_{uv} = \frac{n\sum u_i v_i - \sum u_i . \sum v_i}{n\sum v_i^2 - (\sum v_i)^2}$$

$$= \frac{8.(271) - (3).(-13)}{8.(559) - (-13)^2}$$

$$= \frac{2168 + 39}{4472 - 169}$$

$$= 0.5129$$

Also $$a\hat{} = \bar{y} - b\hat{}\,\bar{x}$$

$$= \frac{(595)}{8} - 1.1406\frac{(595)}{8}$$

$$= 74.375 - 1.1406 \times 74.375$$

$$= -10.4571$$

and $$a\hat{} = \bar{x} - b\hat{}\,\bar{y}$$

$$= 74.375 - 0.5129 \times 74.375$$

$$= 36.2280$$

The regression line of y on x is

$y = -10.4571 + 1.1406x$

and the regression line of x on y is

$x = 36.2281 + 0.5129y$

For x = 90, the most likely value of y is

$$\hat{y} = -10.4571 + 1.1406 \times 90$$

$$= 92.1969$$

$$\cong 92$$

Example 17.17: The following data relate to the mean and SD of the prices of two shares in a stock Exchange:

Share	Mean (in ₹)	SD (in ₹)
Company A	44	5.60
Company B	58	6.30

Coefficient of correlation between the share prices = 0.48

Find the most likely price of share A corresponding to a price of ₹ 60 of share B and also the most likely price of share B for a price of ₹ 50 of share A.

Solution:

Denoting the share prices of Company A and B respectively by x and y, we are given that

$\bar{x}$ = ₹ 44, $\bar{y}$ = ₹ 58

S_x = ₹ 5.60, S_y = ₹ 6.30

and r = 0.48

The regression line of y on x is given by

$$y = a + bx$$

Where $$b = r \times \frac{S_y}{S_x}$$

$$= 0.48 \times \frac{6.30}{5.60}$$

$$= 0.54$$

$$a = \bar{y} - b\bar{x}$$

= ₹ (58 – 0.54 × 44)

= ₹ 34.24

Thus the regression line of y on x i.e. the regression line of price of share B on that of share A is given by

y = ₹ (34.24 + 0.54x)

When x = ₹ 50, = ₹ (34.24 + 0.54 × 50)

= ₹ 61.24

= The estimated price of share B for a price of ₹ 50 of share A is ₹ 61.24

Again the regression line of x on y is given by

x = a^ + b^y

Where b^ = $r \times \frac{S_x}{S_y}$

= $0.48 \times \frac{5.60}{6.30}$

= 0.4267

a^ = $\bar{x} - b\hat{}\ \bar{y}$

= ₹ (44 – 0.4267 × 58)

= ₹ 19.25

Hence the regression line of x on y i.e. the regression line of price of share A on that of share B in given by

x = ₹ (19.25 + 0.4267y)

When y = ₹ 60, $\hat{x}$ = ₹ (19.25 + 0.4267 × 60)

= ₹ 44.85

Example 17.18: The following data relate the expenditure or advertisement in thousands of rupees and the corresponding sales in lakhs of rupees.

Expenditure on Ad :	8	10	10	12	15
Sales :	18	20	22	25	28

Find an appropriate regression equation.

Solution:

Since sales (y) depend on advertisement (x), the appropriate regression equation is of y on x i.e. of sales on advertisement. We have, on the basis of the given data,

n = 5, $\sum x$ = 8+10+10+12+15 = 55

$\sum y$ = 18+20+22+25+28 = 113

$\sum xy$ = 8×18+10×20+10×22+12×25+15×28 = 1284

$\sum x^2$ = $8^2+10^2+10^2+12^2+15^2$ = 633

$$\therefore b = \frac{n\sum xy - \sum x \times \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{5 \times 1284 - 55 \times 113}{5 \times 633 - (55)^2}$$

$$= \frac{205}{140}$$

$$= 1.4643$$

$$a = \bar{y} - b\bar{x}$$

$$= \frac{113}{5} - 1.4643 \times \frac{55}{5}$$

$$= 22.60 - 16.1073$$

$$= 6.4927$$

Thus, the regression line of y or x i.e. the regression line of sales on advertisement is given by

$$y = 6.4927 + 1.4643x$$

17.6 PROPERTIES OF REGRESSION LINES

We consider the following important properties of regression lines:

(i) **The regression coefficients remain unchanged due to a shift of origin but change due to a shift of scale.**

This property states that if the original pair of variables is (x, y) and if they are changed to the pair (u, v) where

$$u = \frac{x-a}{p} \text{ and } v = \frac{y-c}{q}$$

$$b_{yx} = \frac{q}{p} \times b_{vu} \quad \text{.........................} \quad (17.28)$$

$$\text{and } bxy = \frac{p}{q} \times b_{uv} \quad \text{........................} \quad (17.29)$$

(ii) **The two lines of regression intersect at the point $(\bar{x}, \bar{y})$, where x and y are the variables under consideration.**

According to this property, the point of intersection of the regression line of y on x and the regression line of x on y is $(\bar{x}, \bar{y})$ i.e. the solution of the simultaneous equations in x and y.

(iii) **The coefficient of correlation between two variables x and y is the simple geometric mean of the two regression coefficients. The sign of the correlation coefficient would be the common sign of the two regression coefficients.**

This property says that if the two regression coefficients are denoted by b_{yx} (=b) and b_{xy} (=b′) then the coefficient of correlation is given by

$$r = \pm\sqrt{b_{yx} \times b_{xy}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (17.30)$$

If both the regression coefficients are negative, r would be negative and if both are positive, r would assume a positive value.

Example 17.19: If the relationship between two variables x and u is u + 3x = 10 and between two other variables y and v is 2y + 5v = 25, and the regression coefficient of y on x is known as 0.80, what would be the regression coefficient of v on u?

Solution:

$u + 3x = 10$

$$u = \frac{(x - 10/3)}{-1/3}$$

and $2y + 5v = 25$

$$\Rightarrow \qquad v = \frac{(y - 25/2)}{-5/2}$$

From (17.28), we have

$$b_{yx} = \frac{q}{p} \times b_{vu}$$

$$\text{or,} \qquad 0.80 = \frac{-5/2}{-1/3} \times b_{vu}$$

$$\Rightarrow \qquad 0.80 = \frac{15}{2} \times b_{vu}$$

$$\Rightarrow \qquad b_{vu} = \frac{2}{15} \times 0.80 = \frac{8}{75}$$

Example 17.20: For the variables x and y, the regression equations are given as 7x – 3y – 18 = 0 and 4x – y – 11 = 0

(i) Find the arithmetic means of x and y.

(ii) Identify the regression equation of y on x.

(iii) Compute the correlation coefficient between x and y.

(iv) Given the variance of x is 9, find the SD of y.

Solution:

(i) Since the two lines of regression intersect at the point $(\bar{x}, \bar{y})$, replacing x and y by $\bar{x}$ and $\bar{y}$ respectively in the given regression equations, we get

$$7\bar{x} - 3\bar{y} - 18 = 0$$

and $$4\bar{x} - \bar{y} - 11 = 0$$

Solving these two equations, we get $\bar{x} = 3$ and $\bar{y} = 1$

Thus the arithmetic means of x and y are given by 3 and 1 respectively.

(ii) Let us assume that $7x - 3y - 18 = 0$ represents the regression line of y on x and $4x - y - 11 = 0$ represents the regression line of x on y.

Now $7x - 3y - 18 = 0$

$$\Rightarrow \quad y = (-6) + \frac{(7)}{3}x$$

$$\therefore \quad b_{yx} = \frac{7}{3}$$

Again $4x - y - 11 = 0$

$$\Rightarrow \quad x = \frac{(11)}{4} + \frac{(1)}{4}y \qquad \therefore b_{xy} = \frac{1}{4}$$

Thus $r^2 = b_{yx} \times b_{xy}$

$$= \frac{7}{3} \times \frac{1}{4}$$

$$= \frac{7}{12} < 1$$

Since $|r| \leq 1 \Rightarrow r^2 \leq 1$, our assumptions are correct. Thus, $7x - 3y - 18 = 0$ truly represents the regression line of y on x.

(iii) Since $r^2 = \frac{7}{12}$

$\therefore \quad r = \sqrt{\frac{7}{12}}$ (We take the sign of r as positive since both the regression coefficients are positive)

$= 0.7638$

(iv) $b_{yx} = r \times \frac{S_y}{S_x}$

$\Rightarrow \frac{7}{3} = 0.7638 \times \frac{S_y}{3}$ $(\therefore S_x^2 = 9$ as given)

$\Rightarrow S_y = \frac{7}{0.7638}$

$= 9.1647$

17.7 REVIEW OF CORRELATION AND REGRESSION ANALYSIS

So far we have discussed the different measures of correlation and also how to fit regression lines applying the method of 'Least Squares'. It is obvious that we take recourse to correlation analysis when we are keen to know whether two variables under study are associated or correlated and if correlated, what is the strength of correlation. The best measure of correlation is provided by Pearson's correlation coefficient. However, one severe limitation of this correlation coefficient, as we have already discussed, is that it is applicable only in case of a linear relationship between the two variables.

If two variables x and y are independent or uncorrelated then obviously the correlation coefficient between x and y is zero. However, the converse of this statement is not necessarily true i.e. if the correlation coefficient, due to Pearson, between two variables comes out to be zero, then we cannot conclude that the two variables are independent. All that we can conclude is that no linear relationship exists between the two variables. This, however, does not rule out the existence of some non linear relationship between the two variables. For example, if we consider the following pairs of values on two variables x and y.

(–2, 4), (–1, 1), (0, 0), (1, 1) and (2, 4), then cov (x, y) = (–2+ 4) + (–1+1) + (0×0) + (1×1) + (2×4) = 0

as $\bar{x} = 0$

Thus $r_{xy} = 0$

This does not mean that x and y are independent. In fact the relationship between x and y is $y = x^2$. Thus it is always wiser to draw a scatter diagram before reaching conclusion about the existence of correlation between a pair of variables.

There are some cases when we may find a correlation between two variables although the two variables are not causally related. This is due to the existence of a third variable which is related to both the variables under consideration. Such a correlation is known as spurious correlation or non-sense correlation. As an example, there could be a positive correlation between production of rice and that of iron in India for the last twenty years due to the effect of a third variable time on both these variables. It is necessary to eliminate the influence of the third variable before computing correlation between the two original variables.

Correlation coefficient measuring a linear relationship between the two variables indicates the amount of variation of one variable accounted for by the other variable. A better measure for this purpose is provided by the square of the correlation coefficient, Known as 'coefficient of determination'. This can be interpreted as the ratio between the explained variance to total variance i.e.

$$r^2 = \frac{\text{Explained variance}}{\text{Total variance}}$$

Thus a value of 0.6 for r indicates that $(0.6)^2 \times 100\%$ or 36 per cent of the variation has been accounted for by the factor under consideration and the remaining 64 per cent variation is due to other factors. The 'coefficient of non-determination' is given by $(1-r^2)$ and can be interpreted as the ratio of unexplained variance to the total variance.

Coefficient of non-determination = $(1-r^2)$

Regression analysis, as we have already seen, is concerned with establishing a functional relationship between two variables and using this relationship for making future projection. This can be applied, unlike correlation for any type of relationship linear as well as curvilinear. **The two lines of regression coincide i.e. become identical when r = –1 or 1 or in other words, there is a perfect negative or positive correlation between the two variables under discussion. If r = 0 Regression lines are perpendicular to each other.**

SUMMARY

- The change in one variable is reciprocated by a corresponding change in the other variable either directly or inversely, then the two variables are known to be associated or correlated.

  There are two types of correlation.

  (i) Positive correlation

  (ii) Negative correlation

- We consider the following measures of correlation:

  (a) Scatter diagram: This is a simple diagrammatic method to establish correlation between a pair of variables.

  (b) Karl Pearson's Product moment correlation coefficient:

$$r = r_{xy} = \frac{Cov(x, y)}{S_x \times S_y}$$

A single formula for computing correlation coefficient is given by

$$r = \frac{n\sum x_i y_i - \sum x_i \times \sum y_i}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}}$$

(i) The Coefficient of Correlation is a unit-free measure.

(ii) The coefficient of correlation remains invariant under a change of origin and/or scale of the variables under consideration depending on the sign of scale factors.

(iii) The coefficient of correlation always lies between –1 and 1, including both the limiting values i.e. $-1 \leq r \leq +1$

(c) Spearman's rank correlation co-efficient: Spearman's rank correlation coefficient is given by

$\gamma_R = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$, where γ_R denotes rank correlation coefficient and it lies between – 1 and 1 inclusive of these two values. $d_i = x_i - y_i$ represents the difference in ranks for the i-th individual and n denotes the number of individuals.

In case u individuals receive the same rank, we describe it as a tied rank of length u. In case of a tied rank,

$$\gamma_R = 1 - \frac{6\left[\sum_i d_i + \sum_j \frac{\left(tj^3 - t_j\right)}{12}\right]}{n(n^2-1)}$$

In this formula, t_j represents the j^{th} tie length and the summation extends over the lengths of all the ties for both the series.

(d) Co-efficient of concurrent deviations: The coefficient of concurrent deviation is given by

$$\gamma_C = \pm\sqrt{\pm\frac{(2c-m)}{m}}$$

If (2c–m) >0, then we take the positive sign both inside and outside the radical sign and if (2c–m) <0, we are to consider the negative sign both inside and outside the radical sign.

- In regression analysis, we are concerned with the estimation of one variable for given value of another variable (or for a given set of values of a number of variables) on the basis of an average mathematical relationship between the two variables (or a number of variables).
- In case of a simple regression model if y depends on x, then the regression line of y on x is given by y = a + bx, here a and b are two constants and they are also known as regression parameters. Furthermore, b is also known as the regression coefficient of y on x and is also denoted by b_{yx}
- The method of least squares is solving the equations of regression lines

  The normal equations are

  $\Sigma y_i = na + b\Sigma x_i$

  $\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2$

  Solving the normal equations

  $$b_{yx} = \frac{\text{cov}(x_i y_i)}{S_x^2} = \frac{r.S_x.S_y}{S_x^2} = r.\frac{S_y}{S_x}$$

- The regression coefficients remain unchanged due to a shift of origin but change due to a shift of scale.

  This property states that if the original pair of variables is (x, y) and if they are changed to the pair (u, v) where

  $$u = \frac{x-a}{p} \text{ and } v = \frac{y-c}{q}$$

  $$b_{yx} = \frac{p}{q} \times b_{vu} \text{ and } bxy = \frac{q}{p} \times b_{uv}$$

- The two lines of regression intersect at the point $(\bar{x}, \bar{y})$, where x and y are the variables under consideration.

  According to this property, the point of intersection of the regression line of y on x and the regression line of x on y is $(\bar{x}, \bar{y})$ i.e. the solution of the simultaneous equations in x and y.

- The coefficient of correlation between two variables x and y is the simple geometric mean of the two regression coefficients. The sign of the correlation coefficient would be the common sign of the two regression coefficients.

  $$r = \pm\sqrt{b_{yx} \times b_{xy}}$$

- Correlation coefficient measuring a linear relationship between the two variables indicates the amount of variation of one variable accounted for by the other variable. A better measure for this purpose is provided by the square of the correlation coefficient, known.

as 'coefficient of determination'. This can be interpreted as the ratio between the explained variance to total variance i.e.

$$r^2 = \frac{\text{Explained variance}}{\text{Total variance}}$$

- The 'coefficient of non-determination' is given by $(1–r^2)$ and can be interpreted as the ratio of unexplained variance to the total variance.
- The two lines of regression coincide i.e. become identical when r = –1 or 1 or in other words, there is a perfect negative or positive correlation between the two variables under discussion. If r = 0, Regression lines are perpendicular to each other.

EXERCISE

SET A

Write the correct answers. Each question carries 1 mark.

1. Bivariate Data are the data collected for
   (a) Two variables irrespective of time
   (b) More than two variables
   (c) Two variables at the same point of time
   (d) Two variables at different points of time.
2. For a bivariate frequency table having (p + q) classification the total number of cells is
   (a) p (b) p + q
   (c) q (d) pq
3. Some of the cell frequencies in a bivariate frequency table may be
   (a) Negative (b) Zero
   (c) a or b (d) Non of these
4. For a p x q bivariate frequency table, the maximum number of marginal distributions is
   (a) p (b) p + q
   (c) 1 (d) 2
5. For a p x q classification of bivariate data, the maximum number of conditional distributions is
   (a) p (b) p + q
   (c) pq (d) p or q
6. Correlation analysis aims at
   (a) Predicting one variable for a given value of the other variable
   (b) Establishing relation between two variables

(c) Measuring the extent of relation between two variables

(d) Both (b) and (c).

7. Regression analysis is concerned with

(a) Establishing a mathematical relationship between two variables

(b) Measuring the extent of association between two variables

(c) Predicting the value of the dependent variable for a given value of the independent variable

(d) Both (a) and (c).

8. What is spurious correlation?

(a) It is a bad relation between two variables.

(b) It is very low correlation between two variables.

(c) It is the correlation between two variables having no causal relation.

(d) It is a negative correlation.

9. Scatter diagram is considered for measuring

(a) Linear relationship between two variables

(b) Curvilinear relationship between two variables

(c) Neither (a) nor (b)

(d) Both (a) and (b).

10. If the plotted points in a scatter diagram lie from upper left to lower right, then the correlation is

(a) Positive (b) Zero

(c) Negative (d) None of these.

11. If the plotted points in a scatter diagram are evenly distributed, then the correlation is

(a) Zero (b) Negative

(c) Positive (d) (a) or (b).

11. If all the plotted points in a scatter diagram lie on a single line, then the correlation is

(a) Perfect positive (b) Perfect negative

(c) Both (a) and (b) (d) Either (a) or (b).

13. The correlation between shoe-size and intelligence is

(a) Zero (b) Positive

(c) Negative (d) None of these.

14. The correlation between the speed of an automobile and the distance travelled by it after applying the brakes is

(a) Negative (b) Zero

(c) Positive (d) None of these.

15. Scatter diagram helps us to
    (a) Find the nature of correlation between two variables
    (b) Compute the extent of correlation between two variables
    (c) Obtain the mathematical relationship between two variables
    (d) Both (a) and (c).
16. Pearson's correlation coefficient is used for finding
    (a) Correlation for any type of relation
    (b) Correlation for linear relation only
    (c) Correlation for curvilinear relation only
    (d) Both (b) and (c).
17. Product moment correlation coefficient is considered for
    (a) Finding the nature of correlation
    (b) Finding the amount of correlation
    (c) Both (a) and (b)
    (d) Either (a) and (b).
18. If the value of correlation coefficient is positive, then the points in a scatter diagram tend to cluster
    (a) From lower left corner to upper right corner
    (b) From lower left corner to lower right corner
    (c) From lower right corner to upper left corner
    (d) From lower right corner to upper right corner.
19. When r = 1, all the points in a scatter diagram would lie
    (a) On a straight line directed from lower left to upper right
    (b) On a straight line directed from upper left to lower right
    (c) On a straight line
    (d) Both (a) and (b).
20. Product moment correlation coefficient may be defined as the ratio of
    (a) The product of standard deviations of the two variables to the covariance between them
    (b) The covariance between the variables to the product of the variances of them
    (c) The covariance between the variables to the product of their standard deviations
    (d) Either (b) or (c).
21. The covariance between two variables is
    (a) Strictly positive
    (b) Strictly negative
    (c) Always 0
    (d) Either positive or negative or zero.
22. The coefficient of correlation between two variables

(a) Can have any unit.

(b) Is expressed as the product of units of the two variables

(c) Is a unit free measure

(d) None of these.

23. What are the limits of the correlation coefficient?

(a) No limit (b) –1 and 1, excluding the limits

(c) 0 and 1, including the limits (d) –1 and 1, including the limits

24. In case the correlation coefficient between two variables is 1, the relationship between the two variables would be

(a) y = a + bx (b) y = a + bx, b > 0

(c) y = a + bx, b < 0 (d) y = a + bx, both a and b being positive.

25. If the relationship between two variables x and y is given by 2x + 3y + 4 = 0, then the value of the correlation coefficient between x and y is

(a) 0 (b) 1

(c) –1 (d) negative.

26. For finding correlation between two attributes, we consider

(a) Pearson's correlation coefficient

(b) Scatter diagram

(c) Spearman's rank correlation coefficient

(d) Coefficient of concurrent deviations.

27. For finding the degree of agreement about beauty between two Judges in a Beauty Contest, we use

(a) Scatter diagram (b) Coefficient of rank correlation

(c) Coefficient of correlation (d) Coefficient of concurrent deviation.

28. If there is a perfect disagreement between the marks in Geography and Statistics, then what would be the value of rank correlation coefficient?

(a) Any value (b) Only 1

(c) Only –1 (d) (b) or (c)

29. When we are not concerned with the magnitude of the two variables under discussion, we consider

(a) Rank correlation coefficient (b) Product moment correlation coefficient

(c) Coefficient of concurrent deviation (d) (a) or (b) but not (c).

30. What is the quickest method to find correlation between two variables?

(a) Scatter diagram (b) Method of concurrent deviation

(c) Method of rank correlation (d) Method of product moment correlation

31. What are the limits of the coefficient of concurrent deviations?
    (a) No limit
    (b) Between –1 and 0, including the limiting values
    (c) Between 0 and 1, including the limiting values
    (d) Between –1 and 1, the limiting values inclusive
32. If there are two variables x and y, then the number of regression equations could be
    (a) 1 (b) 2
    (c) Any number (d) 3.
33. Since Blood Pressure of a person depends on age, we need to consider
    (a) The regression equation of Blood Pressure on age
    (b) The regression equation of age on Blood Pressure
    (c) Both (a) and (b)
    (d) Either (a) or (b).
34. The method applied for deriving the regression equations is known as
    (a) Least squares (b) Concurrent deviation
    (c) Product moment (d) Normal equation.
35. The difference between the observed value and the estimated value in regression analysis is known as
    (a) Error (b) Residue
    (c) Deviation (d) (a) or (b).
36. The errors in case of regression equations are
    (a) Positive (b) Negative
    (c) Zero (d) All these.
37. The regression line of y on x is derived by
    (a) The minimisation of vertical distances in the scatter diagram
    (b) The minimisation of horizontal distances in the scatter diagram
    (c) Both (a) and (b)
    (d) (a) or (b).
38. The two lines of regression become identical when
    (a) r = 1 (b) r = –1
    (c) r = 0 (d) (a) or (b).
39. What are the limits of the two regression coefficients?
    (a) No limit (b) Must be positive
    (c) One positive and the other negative
    (d) Product of the regression coefficient must be numerically less than unity.

40. The regression coefficients remain unchanged due to a

(a) Shift of origin (b) Shift of scale

(c) Both (a) and (b) (d) (a) or (b).

41. If the coefficient of correlation between two variables is –0.9, then the coefficient of determination is

(a) 0.9 (b) 0.81

(c) 0.1 (d) 0.19.

42. If the coefficient of correlation between two variables is 0.7 then the percentage of variation unaccounted for is

(a) 70% (b) 30%

(c) 51% (d) 49%

SET B

Answer the following questions by writing the correct answers. Each question carries 2 marks.

1. If for two variable x and y, the covariance, variance of x and variance of y are 40, 16 and 256 respectively, what is the value of the correlation coefficient?

(a) 0.01 (b) 0.625

(c) 0.4 (d) 0.5

2. If cov(x, y) = 15, what restrictions should be put for the standard deviations of x and y?

(a) No restriction.

(b) The product of the standard deviations should be more than 15.

(c) The product of the standard deviations should be less than 15.

(d) The sum of the standard deviations should be less than 15.

3. If the covariance between two variables is 20 and the variance of one of the variables is 16, what would be the variance of the other variable?

(a) $S_y^2 \geq 25$ (b) More than 10

(c) Less than 10 (d) More than 1.25

4. If y = a + bx, then what is the coefficient of correlation between x and y?

(a) 1 (b) –1

(c) 1 or –1 according as b > 0 or b < 0 (d) none of these.

5. If r = 0.6 then the coefficient of non-determination is

(a) 0.4 (b) –0.6

(c) 0.36 (d) 0.64

6. If u + 5x = 6 and 3y – 7v = 20 and the correlation coefficient between x and y is 0.58 then what would be the correlation coefficient between u and v?

(a) 0.58 (b) –0.58

(c) –0.84 (d) 0.84

7. If the relation between x and u is $3x + 4u + 7 = 0$ and the correlation coefficient between x and y is –0.6, then what is the correlation coefficient between u and y?
   (a) –0.6 (b) 0.8
   (c) 0.6 (d) –0.8

8. From the following data

x:	2	3	5	4	7
y:	4	6	7	8	10

The coefficient of correlation was found to be 0.93. What is the correlation between u and v as given below?

u:	–3	–2	0	–1	2
v:	–4	–2	–1	0	2

(a) –0.93 (b) 0.93 (c) 0.57 (d) –0.57

9. Referring to the data presented in Q. No. 8, what would be the correlation between u and v?

u:	10	15	25	20	35
v:	–24	–36	–42	–48	–60

(a) –0.6 (b) 0.6 (c) –0.93 (d) 0.93

10. If the sum of squares of difference of ranks, given by two judges A and B, of 8 students is 21, what is the value of rank correlation coefficient?
    (a) 0.7 (b) 0.65 (c) 0.75 (d) 0.8

11. If the rank correlation coefficient between marks in management and mathematics for a group of student is 0.6 and the sum of squares of the differences in ranks is 66, what is the number of students in the group?
    (a) 10 (b) 9 (c) 8 (d) 11

12. While computing rank correlation coefficient between profit and investment for the last 6 years of a company the difference in rank for a year was taken 3 instead of 4. What is the rectified rank correlation coefficient if it is known that the original value of rank correlation coefficient was 0.4?
    (a) 0.3 (b) 0.2 (c) 0.25 (d) 0.28

13. For 10 pairs of observations, No. of concurrent deviations was found to be 4. What is the value of the coefficient of concurrent deviation?
    (a) $\sqrt{0.2}$ (b) $-\sqrt{0.2}$ (c) 1/3 (d) –1/3

14. The coefficient of concurrent deviation for p pairs of observations was found to be $1/\sqrt{3}$. If the number of concurrent deviations was found to be 6, then the value of p is.
    (a) 10 (b) 9 (c) 8 (d) none of these

15. What is the value of correlation coefficient due to Pearson on the basis of the following data:

x:	–5	–4	–3	–2	–1	0	1	2	3	4	5
y:	27	18	11	6	3	2	3	6	11	18	27

(a) 1 (b) –1 (c) 0 (d) –0.5

16. Following are the two normal equations obtained for deriving the regression line of y and x:

    5a + 10b = 40

    10a + 25b = 95

    The regression line of y on x is given by

    (a) 2x + 3y = 5 (b) 2y + 3x = 5 (c) y = 2 + 3x (d) y = 3 + 5x

17. If the regression line of y on x and of x on y are given by 2x + 3y = –1 and 5x + 6y = –1 then the arithmetic means of x and y are given by

    (a) (1, –1) (b) (–1, 1) (c) (–1, –1) (d) (2, 3)

18. Given the regression equations as 3x + y = 13 and 2x + 5y = 20, which one is the regression equation of y on x?

    (a) 1st equation (b) 2nd equation (c) both (a) and (b) (d) none of these.

19. Given the following equations: 2x – 3y = 10 and 3x + 4y = 15, which one is the regression equation of x on y ?

    (a) 1st equation (b) 2nd equation (c) both the equations (d) none of these

20. If u = 2x + 5 and v = –3y – 6 and regression coefficient of y on x is 2.4, what is the regression coefficient of v on u?

    (a) 3.6 (b) –3.6 (c) 2.4 (d) –2.4

21. If 4y – 5x = 15 is the regression line of y on x and the coefficient of correlation between x and y is 0.75, what is the value of the regression coefficient of x on y?

    (a) 0.45 (b) 0.9375 (c) 0.6 (d) none of these

22. If the regression line of y on x and that of x on y are given by y = –2x + 3 and 8x = –y + 3 respectively, what is the coefficient of correlation between x and y?

    (a) 0.5 (b) $-1/\sqrt{2}$ (c) –0.5 (d) none of these

23. If the regression coefficient of y on x, the coefficient of correlation between x and y and variance of y are –3/4, $\frac{\sqrt{3}}{2}$ and 4 respectively, what is the variance of x?

    (a) $2/\sqrt{3/2}$ (b) 16/3 (c) 4/3 (d) 4

24. If y = 3x + 4 is the regression line of y on x and the arithmetic mean of x is –1, what is the arithmetic mean of y?

    (a) 1 (b) –1 (c) 7 (d) none of these

SET C

Write down the correct answers. Each question carries 5 marks.

1. What is the coefficient of correlation from the following data?

x:	1	2	3	4	5
y:	8	6	7	5	5

    (a) 0.75 (b) –0.75 (c) –0.85 (d) 0.82

2. The coefficient of correlation between x and y where

x:	64	60	67	59	69
y:	57	60	73	62	68

is

(a) 0.655 (b) 0.68 (c) 0.73 (d) 0.758

3. What is the coefficient of correlation between the ages of husbands and wives from the following data?

Age of husband (year):	46	45	42	40	38	35	32	30	27	25
Age of wife (year):	37	35	31	28	30	25	23	19	19	18

(a) 0.58 (b) 0.98 (c) 0.89 (d) 0.92

4. The following results relate to bivariate data on (x, y):

$\sum xy = 414$, $\sum x = 120$, $\sum y = 90$, $\sum x^2 = 600$, $\sum y^2 = 300$, n = 30. Later or, it was known that two pairs of observations (12, 11) and (6, 8) were wrongly taken, the correct pairs of observations being (10, 9) and (8, 10). The corrected value of the correlation coefficient is

(a) 0.752 (b) 0.768 (c) 0.846 (d) 0.953

5. The following table provides the distribution of items according to size groups and also the number of defectives:

Size group:	9-11	11-13	13-15	15-17	17-19
No. of items:	250	350	400	300	150
No. of defective items:	25	70	60	45	20

The correlation coefficient between size and defectives is

(a) 0.25 (b) 0.12 (c) 0.14 (d) 0.07

6. For two variables x and y, it is known that cov (x, y) = 8, r = 0.4, variance of x is 16 and sum of squares of deviation of y from its mean is 250. The number of observations for this bivariate data is

(a) 7 (b) 8 (c) 9 (d) 10

7. Eight contestants in a musical contest were ranked by two judges A and B in the following manner:

Serial Number of the contestants:	1	2	3	4	5	6	7	8
Rank by Judge A:	7	6	2	4	5	3	1	8
Rank by Judge B:	5	4	6	3	8	2	1	7

The rank correlation coefficient is

(a) 0.65 (b) 0.63 (c) 0.60 (d) 0.57

8. Following are the marks of 10 students in Botany and Zoology:

Serial No.:	1	2	3	4	5	6	7	8	9	10

Marks in

Botany:	58	43	50	19	28	24	77	34	29	75
Marks in Zoology:	62	63	79	56	65	54	70	59	55	69

The coefficient of rank correlation between marks in Botany and Zoology is

(a) 0.65 (b) 0.70 (c) 0.72 (d) 0.75

9. What is the value of Rank correlation coefficient between the following marks in Physics and Chemistry:

Roll No.:	1	2	3	4	5	6
Marks in Physics:	25	30	46	30	55	80
Marks in Chemistry:	30	25	50	40	50	78

(a) 0.782 (b) 0.696 (c) 0.932 (d) 0.857

10. What is the coefficient of concurrent deviations for the following data:

Supply:	68	43	38	78	66	83	38	23	83	63	53
Demand:	65	60	55	61	35	75	45	40	85	80	85

(a) 0.82 (b) 0.85 (c) 0.89 (d) –0.81

11. What is the coefficient of concurrent deviations for the following data:

Year:	1996	1997	1998	1999	2000	2001	2002	2003
Price:	35	38	40	33	45	48	49	52
Demand:	36	35	31	36	30	29	27	24

(a) –1 (b) 0.43 (c) 0.5 (d) $\sqrt{2}$

12. The regression equation of y on x for the following data:

x	41	82	62	37	58	96	127	74	123	100
y	28	56	35	17	42	85	105	61	98	73

Is given by

(a) y = 1.2x – 15 (b) y = 1.2x + 15 (c) y = 0.93x – 14.68 (d) y = 1.5x – 10.89

13. The following data relate to the heights of 10 pairs of fathers and sons:

(175, 173), (172, 172), (167, 171), (168, 171), (172, 173), (171, 170), (174, 173), (176, 175) (169, 170), (170, 173)

The regression equation of height of son on that of father is given by

(a) y = 100 + 5x (b) y = 102.60 + 0.405x (c) y = 89.653 + 0.582x (d) y = 88.758 + 0.562x

14. The two regression coefficients for the following data:

x:	38	23	43	33	28
y:	28	23	43	38	8

are

(a) 1.2 and 0.4 (b) 1.6 and 0.8 (c) 1.7 and 0.8 (d) 1.8 and 0.3

15. For y = 25, what is the estimated value of x, from the following data:

X:	11	12	15	16	18	19	21
Y:	21	15	13	12	11	10	9

(a) 15 (b) 13.926 (c) 6.07 (d) 14.986

16. Given the following data:

Variable:	x	y
Mean:	80	98
Variance:	4	9

Coefficient of correlation = 0.6

What is the most likely value of y when x = 90 ?

(a) 90 (b) 103 (c) 104 (d) 107

17. The two lines of regression are given by

8x + 10y = 25 and 16x + 5y = 12 respectively.

If the variance of x is 25, what is the standard deviation of y?

(a) 16 (b) 8 (c) 64 (d) 4

18. Given below the information about the capital employed and profit earned by a company over the last twenty five years:

	Mean	SD
Capital employed (0000 ₹)	62	5
Profit earned (000 ₹)	25	6

Correlation coefficient between capital employed and profit = 0.92. The sum of the Regression coefficients for the above data would be:

(a) 1.871 (b) 2.358 (c) 1.968 (d) 2.346

19. The coefficient of correlation between cost of advertisement and sales of a product on the basis of the following data:

Ad cost (000 ₹):	75	81	85	105	93	113	121	125
Sales (000 000 ₹):	35	45	59	75	43	79	87	95

is

(a) 0.85 (b) 0.89 (c) 0.95 (d) 0.98

ANSWERS

Set A

1.	(c)	**2.**	(d)	**3.**	(b)	**4.**	(d)	**5.**	(b)	**6.**	(d)
7.	(d)	**8.**	(c)	**9.**	(d)	**10.**	(c)	**11.**	(a)	**12.**	(d)
13.	(a)	**14.**	(a)	**15.**	(a)	**16.**	(b)	**17.**	(c)	**18.**	(a)
19.	(a)	**20.**	(c)	**21.**	(d)	**22.**	(c)	**23.**	(d)	**24.**	(b)
25.	(c)	**26.**	(c)	**27.**	(b)	**28.**	(c)	**29.**	(c)	**30.**	(b)
31.	(d)	**32.**	(b)	**33.**	(a)	**34.**	(a)	**35.**	(d)	**36.**	(d)
37.	(a)	**38.**	(d)	**39.**	(d)	**40.**	(a)	**41.**	(b)	**42.**	(c)

Set B

1.	(b)	**2.**	(b)	**3.**	(a)	**4.**	(c)	**5.**	(d)	**6.**	(b)
7.	(c)	**8.**	(b)	**9.**	(c)	**10.**	(c)	**11.**	(a)	**12.**	(b)
13.	(d)	**14.**	(a)	**15.**	(c)	**16.**	(c)	**17.**	(a)	**18.**	(b)
19.	(d)	**20.**	(b)	**21.**	(a)	**22.**	(c)	**23.**	(b)	**24.**	(a)

Set C

1.	(c)	**2.**	(a)	**3.**	(b)	**4.**	(c)	**5.**	(d)	**6.**	(d)
7.	(d)	**8.**	(c)	**9.**	(d)	**10.**	(c)	**11.**	(a)	**12.**	(c)
13.	(b)	**14.**	(a)	**15.**	(c)	**16.**	(d)	**17.**	(b)	**18.**	(a)
19.	(c)										

ADDITIONAL QUESTION BANK

1. ______________ is concerned with the measurement of the "strength of association" between variables.

   (a) correlation (b) regression (c) both (d) none

2. ______________ gives the mathematical relationship of the variables.

   (a) correlation (b) regression (c) both (d) none

3. When high values of one variable are associated with high values of the other & low values of one variable are associated with low values of another, then they are said to be

   (a) positively correlated (b) directly correlated
   (c) both (d) none

4. If high values of one tend to low values of the other, they are said to be

   (a) negatively correlated (b) inversely correlated
   (c) both (d) none

5. Correlation coefficient between two variables is a measure of their linear relationship .

   (a) true (b) false (c) both (d) none

6. Correlation coefficient is dependent of the choice of both origin & the scale of observations.

   (a) True (b) false (c) both (d) none

7. Correlation coefficient is a pure number.

   (a) true (b) false (c) both (d) none

8. Correlation coefficient is ______________ of the units of measurement.

   (a) dependent (b) independent (c) both (d) none

9. The value of correlation coefficient lies between

   (a) –1 and +1 (b) –1 and 0
   (c) 0 and 1 Inclusive of these two values (d) none.

10. Correlation coefficient can be found out by

    (a) Scatter Diagram (b) Rank Method (c) both (d) none.

11. Covariance measures _________ variations of two variables.

    (a) joint (b) single (c) both (d) none

12. In calculating the Karl Pearson's coefficient of correlation it is necessary that the data should be of numerical measurements. The statement is

    (a) valid (b) not valid (c) both (d) none

13. Rank correlation coefficient lies between

    (a) 0 to 1 (b) –1 to +1 inclusive of these value
    (c) –1 to 0 (d) both

14. A coefficient near +1 indicates tendency for the larger values of one variable to be associated with the larger values of the other.

(a) true (b) false (c) both (d) none

15. In rank correlation coefficient the association need not be linear.

(a) true (b) false (c) both (d) none

16. In rank correlation coefficient only an increasing/decreasing relationship is required.

(a) false (b) true (c) both (d) none

17. Great advantage of ____________ is that it can be used to rank attributes which can not be expressed by way of numerical value .

(a) concurrent correlation (b) regression
(c) rank correlation (d) none

18. The sum of the difference of rank is

(a) 1 (b) –1 (c) 0 (d) none.

19. Karl Pearson's coefficient is defined from

(a) ungrouped data (b) grouped data (c) both (d) none.

20. Correlation methods are used to study the relationship between two time series of data which are recorded annually, monthly, weekly, daily and so on.

(a) True (b) false (c) both (d) none

21. Age of Applicants for life insurance and the premium of insurance – correlation is

(a) positive (b) negative (c) zero (d) none

22. "Unemployment index and the purchasing power of the common man"____ Correlation is

(a) positive (b) negative (c) zero (d) none

23. Production of pig iron and soot content in Durgapur – Correlations are

(a) positive (b) negative (c) zero (d) none

24. "Demand for goods and their prices under normal times" ____ Correlation is

(a) positive (b) negative (c) zero (d) none

25. ___________ is a relative measure of association between two or more variables.

(a) Coefficient of correlation (b) Coefficient of regression
(c) both (d) none

26. The lines of regression passes through the points, bearing _________ no. of points on both sides

(a) equal (b) unequal (c) zero (d) none

27. Under Algebraic Method we get ————— linear equations .

(a) one (b) two (c) three (d) none

28. In linear equations Y = a + bX and X= a + bY 'a' is the

(a) intercept of the line (b) slope
(c) both (d) none

29. In linear equations Y = a + bX and X = a + bY ' b ' is the

(a) intercept of the line (b) slope of the line
(c) both (d) none

30. The regression equations Y = a + bX and X = a + bY are based on the method of

(a) greatest squares (b) least squares (c) both (d) none

31. The line Y = a + bX represents the regression equation of

(a) Y on X (b) X on Y (c) both (d) none

32. The line X = a + bY represents the regression equation of

(a) Y on X (b) X onY (c) both (d) none

33. Two regression lines always intersect at the means.

(a) true (b) false (c) both (d) none

34. r, b_{xy}, b_{yx} all have ______ sign.

(a) different (b) same (c) both (d) none

35. The regression coefficients are zero if r is equal to

(a) 2 (b) –1 (c) 1 (d) 0

36. The regression lines are identical if r is equal to

(a) +1 (b) –1 (c) ± 1 (d) 0

37. The regression lines are perpendicular to each other if r is equal to
(a) 0 (b) +1 (c) –1 (d) ± 1

38. The sum of the deviations at the Y's or the X's from their regression lines are zero. This statements is __________

(a) true (b) false (c) both (d) none

39. The coefficient of determination is defined by the formula

(a) $r^2 = 1 - \frac{\text{unexplained variance}}{\text{total variance}}$ (b) $r^2 = \frac{\text{explained variance}}{\text{total variance}}$

(c) both (d) none

40. If the line Y = 13 –3X /2 is the regression equation of y on x then byx is

(a) $\frac{2}{3}$ (b) $\frac{-2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

41. In the line Y = 19 – 5X/2 is the regresson equation x on y then bxy is,
(a) 19/2 (b) 5/2 (c) –5/2 (d) –2/5

42. The line X = 31/6 — Y/6 is the regression equation of
(a) Y on X (b) X on Y (c) both (d) we can not say

43. In the regression equation x on y, X = 35/8 – 2Y /5, b_{xy} is equal to
(a) –2/5 (b) 35/8 (c) 2/5 (d) 5/2

44. The square of coefficient of correlation 'r' is called the coefficient of
(a) determination (b) regression (c) both (d) none

45. A relationship $r^2 = 1 - \frac{500}{300}$ is not possible
(a) true (b) false (c) both (d) none

46. Whatever may be the value of r, positive or negative, its square will be
(a) negative only (b) positive only (c) zero only (d) none only

47. Simple correlation is called
(a) linear correlation (b) nonlinear correlation
(c) both (d) none

48. A scatter diagram indicates the type of correlation between two variables.
(a) true (b) false (c) both (d) none

49. If the pattern of points (or dots) on the scatter diagram shows a linear path diagonally across the graph paper from the bottom left- hand corner to the top right, correlation will be
(a) negative (b) zero (c) positive (d) none

50. The correlation coefficient being +1 if the slope of the straight line in a scatter diagram is
(a) positive (b) negative (c) zero (d) none

51. The correlation coefficient being –1 if the slope of the straight line in a scatter diagram is
(a) positive (b) negative (c) zero (d) none

52. The more scattered the points are around a straight line in a scattered diagram the ________ is the correlation coefficient.
(a) zero (b) more (c) less (d) none

53. If the values of y are not affected by changes in the values of x, the variables are said to be
(a) correlated (b) uncorrelated (c) both (d) zero

54. If the amount of change in one variable tends to bear a constant ratio to the amount of change in the other variable, then correlation is said to be
(a) non linear (b) linear (c) both (d) none

55. Variance may be positive, negative or zero.
(a) true (b) false (c) both (d) none

56. Covariance may be positive, negative or zero.

(a) true (b) false (c) both (d) none

57. Correlation coefficient between x and y = correlation coefficient between u and v

(a) true (b) false (c) both (d) none

58. In case ' The ages of husbands and wives' _______________ correlation is

(a) positive (b) negative (c) zero (d) none

59. In case 'Shoe size and intelligence'

(a) positive correlation (b) negative correlation
(c) no correlation (d) none

60. In case 'Insurance companies' profits and the no of claims they have to pay "___________

(a) positive correlation (b) negative correlation
(c) no correlation (d) none

61. In case 'Years of education and income' ____________

(a) positive correlation (b) negative correlation
c) no correlation (d) none

62. In case 'Amount of rainfall and yield of crop'____________

(a) positive correlation (b) negative correlation
(c) no correlation (d) none

63. For calculation of correlation coefficient, a change of origin is

(a) not possible (b) possible (c) both (d) none

64. The relation $r_{xy} = cov\ (x,y)/\ \sigma_x.\sigma_y$ is

(a) true (b) false (c) both (d) none

65. A small value of r indicates only a _________ linear type of relationship between the variables.

(a) good (b) poor (c) maximum (d) highest

66. Two regression lines coincide when

(a) r = 0 (b) r = 2 (c) r = $\pm$ 1 (d) none

67. Neither y nor x can be estimated by a linear function of the other variable when r is equal to

(a) + 1 (b) – 1 (c) 0 (d) none

68. When r = 0 then cov (x,y) is equal to

(a) + 1 (b) – 1 (c) 0 (d) none

69. When the variables are not independent, the correlation coefficient may be zero

(a) true (b) false (c) both (d) none

70. b_{xy} is called regression coefficient of

(a) x on y (b) y on x (c) both (d) none

71. b_{yx} is called regression coefficient of

(a) x on y (b) y on x (c) both (d) none

72. The slopes of the regression line of y on x is

(a) b_{yx} (b) b_{xy} (c) b_{xx} (d) b_{yy}

73. The slopes of the regression line of x on y is

(a) b_{yx} (b) b_{xy} (c) $1/b_{xy}$ (d) $1/b_{yx}$

74. The angle between the regression lines depends on

(a) correlation coefficient (b) regression coefficient
(c) both (d) none

75. If x and y satisfy the relationship y = –5 + 7x, the value of r is

(a) 0 (b) – 1 (c) + 1 (d) none

76. If b_{yx} and b_{xy} are negative, r is

(a) positive (b) negative (c) zero (d) none

77. Correlation coefficient r lie between the regression coefficients b_{yx} and b_{xy}

(a) true (b) false (c) both (d) none

78. Since the correlation coefficient r cannot be greater than 1 numerically, the product of the regression must

(a) not exceed 1 (b) exceed 1 (c) be zero (d) none

79. The correlation coefficient r is the __________ of the two regression coefficients b_{yx} and b_{xy}

(a) A.M (b) G.M (c) H.M (d) none

80. Which is true?

(a) $b_{yx} = r\frac{\sigma_x}{\sigma_y}$ (b) $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

(c) $b_{yx} = r\frac{\sigma_{xy}}{\sigma_x}$ (d) $b_{yx} = r\frac{\sigma_{yy}}{\sigma_x}$

81. Maximum value of Rank Correlation coefficient is

(a) –1 (b) + 1 (c) 0 (d) none

82. The partial correlation coefficient lies between

(a) –1 and +1 inclusive of these two value (b) 0 and + 1
(c) –1 and (d) none

83. r_{12} is the correlation coefficient between

(a) x_1 and x_2 (b) x_2 and x_1 (c) x_1 and x_3 (d) x_2 and x_3

84. r_{12} is the same as r_{21}

(a) true (b) false (c) both (d) none

85. In case of employed persons 'Age and income' correlation is

(a) positive (b) negative (c) zero (d) none

86. In case 'Speed of an automobile and the distance required to stop the car after applying brakes' – correlation is

(a) positive (b) negative (c) zero (d) none

87. In case 'Sale of woolen garments and day temperature' ____________ correlation is

(a) positive (b) negative (c) zero (d) none

88. In case 'Sale of cold drinks and day temperature' ____________ correlation is

(a) positive (b) negative (c) zero (d) none

89. In case of 'Production and price per unit' – correlation is

(a) positive (b) negative (c) zero (d) none

90. If slopes at two regression lines are equal then r is equal to

(a) 1 (b) ± 1 (c) 0 (d) none

91. Co–variance measures the joint variations of two variables.

(a) true (b) false (c) both (d) none

92. The minimum value of correlation coefficient is

(a) 0 (b) –2 (c) 1 (d) –1

93. The maximum value of correlation coefficient is

(a) 0 (b) 2 (c) 1 (d) –1

94. When r = 0 , the regression coefficients are

(a) 0 (b) 1 (c) –1 (d) none

95. The regression equation of Y on X is, 2x + 3Y + 50 = 0. The value of b_{YX} is

(a) 2/3 (b) – 2/3 (c) –3/2 (d) none

96. In Method of Concurrent Deviations, only the directions of change (Positive direction / Negative direction) in the variables are taken into account for calculation of

(a) coefficient of S.D (b) coefficient of regression.
(c) coefficient of correlation (d) none

ANSWERS

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(c)	**5.**	(a)
6.	(b)	**7.**	(a)	**8.**	(b)	**9.**	(a)	**10.**	(b)
11.	(a)	**12.**	(a)	**13.**	(b)	**14.**	(a)	**15.**	(a)
16.	(b)	**17.**	(c)	**18.**	(c)	**19.**	(b)	**20.**	(a)
21.	(a)	**22.**	(b)	**23.**	(a)	**24.**	(b)	**25.**	(a)
26.	(d)	**27.**	(b)	**28.**	(a)	**29.**	(b)	**30.**	(b)
31.	(a)	**32.**	(b)	**33.**	(a)	**34.**	(b)	**35.**	(d)
36.	(c)	**37.**	(a)	**38.**	(a)	**39.**	(c)	**40.**	(d)
41.	(d)	**42.**	(d)	**43.**	(a)	**44.**	(a)	**45.**	(a)
46.	(b)	**47.**	(a)	**48.**	(a)	**49.**	(c)	**50.**	(a)
51.	(b)	**52.**	(c)	**53.**	(b)	**54.**	(b)	**55.**	(b)
56.	(a)	**57.**	(b)	**58.**	(a)	**59.**	(c)	**60.**	(b)
61.	(a)	**62.**	(a)	**63.**	(b)	**64.**	(a)	**65.**	(b)
66.	(c)	**67.**	(c)	**68.**	(c)	**69.**	(a)	**70.**	(a)
71.	(b)	**72.**	(a)	**73.**	(b)	**74.**	(a)	**75.**	(c)
76.	(b)	**77.**	(a)	**78.**	(a)	**79.**	(b)	**80.**	(b)
81.	(b)	**82.**	(a)	**83.**	(a) & (b)	**84.**	(a)	**85.**	(a)
86.	(b)	**87.**	(b)	**88.**	(a)	**89.**	(b)	**90.**	(b)
91.	(a)	**92.**	(d)	**93.**	(c)	**94.**	(a)	**95.**	(b)
96.	(c)								

NOTES

CHAPTER BN

INDEX NUMBERS

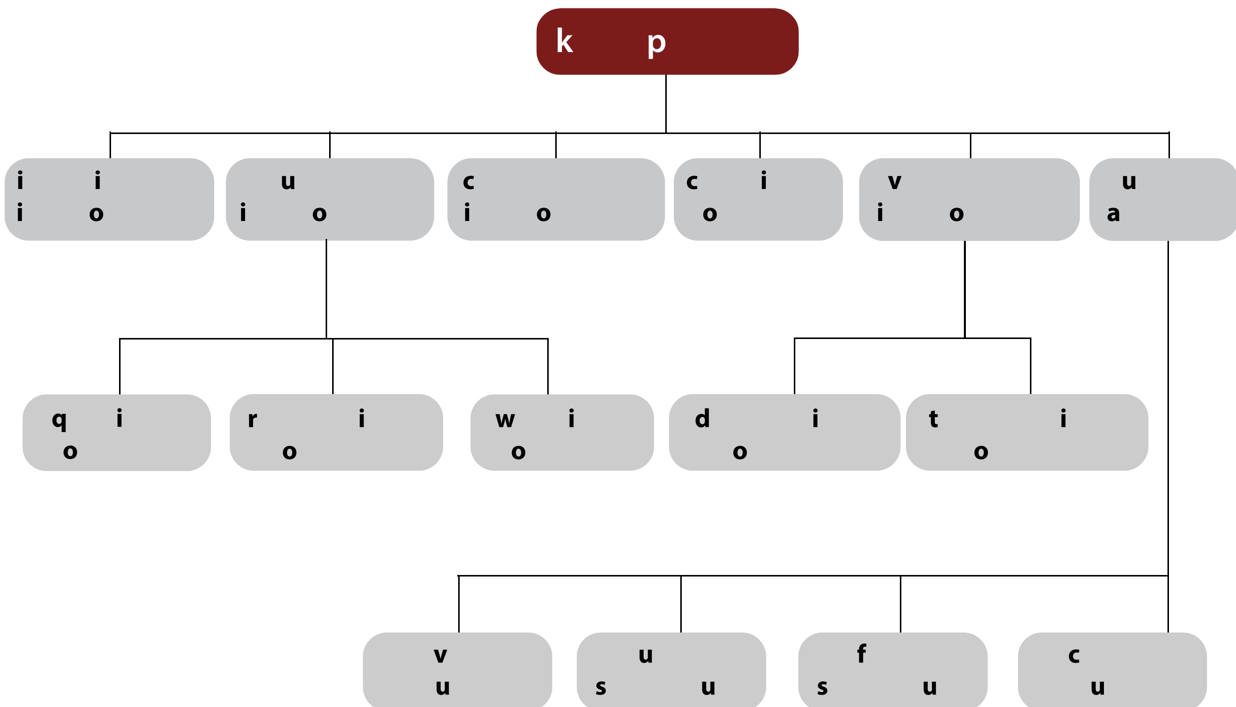
LEARNING OBJECTIVES

Often we encounter news of price rise, GDP growth, production growth, etc. It is important for students of Chartered Accountancy to learn techniques of measuring growth/rise or decline of various economic and business data and how to report them objectively.

After reading the chapter, students will be able to understand:

- Purpose of constructing index number and its important applications in understanding rise or decline of production, prices, etc.
- Different methods of computing index number.

CHAPTER OVERVIEW



18.1 INTRODUCTION

Index numbers are convenient devices for measuring relative changes of differences from time to time or from place to place. Just as the arithmetic mean is used to represent a set of values, an index number is used to represent a set of values over two or more different periods or localities.

The basic device used in all methods of index number construction is to average the relative change in either quantities or prices since relatives are comparable and can be added even though the data from which they were derived cannot themselves be added. For example, if wheat production has gone up to 110% of the previous year's producton and cotton production has gone up to 105%, it is possible to average the two percentages as they have gone up by 107.5%. This assumes that both have equal weight; but if wheat production is twice as important as cotton, percentage should be weighted 2 and 1. The average relatives obtained through this process are called the index numbers.

Definition: An index number is a ratio of two or more time periods are involved, one of which is the base time period. The value at the base time period serves as the standard point of comparison.

Example: NSE, BSE, WPI, CPI etc.

An index time series is a list of index numbers for two or more periods of time, where each index number employs the same base year.

Relatives are derived because absolute numbers measured in some appropriate unit, are often of little importance and meaningless in themselves. If the meaning of a relative figure remains ambiguous, it is necessary to know the absolute as well as the relative number.

Our discussion of index numbers is confined to various types of index numbers, their uses, the mathematical tests and the principles involved in the construction of index numbers.

Index numbers are studied here because some techniques for making forecasts or inferences about the figures are applied in terms of index number. In regression analysis, either the independent or dependent variable or both may be in the form of index numbers. They are less unwieldy than large numbers and are readily understandable.

These are of two broad types: simple and composite. The simple index is computed for one variable whereas the composite is calculated from two or more variables. Most index numbers are composite in nature.

18.2 ISSUES INVOLVED

Following are some of the important criteria/problems which have to be faced in the construction of index Numbers.

Selection of data: It is important to understand the purpose for which the index is used. If it is used for purposes of knowing the cost of living, there is no need of including the prices of capital goods which do not directly influence the living.

Index numbers are often constructed from the sample. It is necessary to ensure that it is representative. Random sampling, and if need be, a stratified random sampling can ensure this.

It is also necessary to ensure comparability of data. This can be ensured by consistency in the method of selection of the units for compilation of index numbers.

However, difficulties arise in the selection of commodities because the relative importance of commodities keep on changing with the advancement of the society. More so, if the period is quite long, these changes are quite significant both in the basket of production and the uses made by people.

Base Period: It should be carefully selected because it is a point of reference in comparing various data describing individual behaviour. The period should be normal i.e., one of the relative stability, not affected by extraordinary events like war, famine, etc. It should be relatively recent because we are more concerned with the changes with reference to the present and not with the distant past. There are three variants of the base fixed, chain, and the average.

Selection of Weights: It is necessary to point out that each variable involved in composite index should have a reasonable influence on the index, i.e., due consideration should be given to the relative importance of each variable which relates to the purpose for which the index is to be used. For example, in the computation of cost of living index, sugar cannot be given the same importance as the cereals.

Use of Averages: Since we have to arrive at a single index number summarising a large amount of information, it is easy to realise that average plays an important role in computing index numbers. The geometric mean is better in averaging relatives, but for most of the indices arithmetic mean is used because of its simplicity.

Choice of Variables: Index numbers are constructed with regard to price or quantity or any other measure. We have to decide about the unit. For example, in price index numbers it is necessary to decide whether to have wholesale or the retail prices. The choice would depend on the purpose. Further, it is necessary to decide about the period to which such prices will be related. There may be an average of price for certain time-period or the end of the period. The former is normally preferred.

Selection of Formula: The question of selection of an appropriate formula arises, since different types of indices give different values when applied to the same data. We will see different types of indices to be used for construction succeedingly.

18.3 CONSTRUCTION OF INDEX NUMBER

Notations: It is customary to let $P_n(^1)$, $P_n(^2)$, $P_n(^3)$ denote the prices during n^{th} period for the first, second and third commodity. The corresponding price during a base period are denoted by $P_o(^1)$, $P_o(^2)$, $P_o(^3)$, etc. With these notations the price of commodity *j* during period *n* can be indicated by $P_n(^j)$. We can use the summation notation by summing over the superscripts *j* as follows:

$$\sum_{j=1}^{k} P_n(j) \quad \text{or} \quad \sum P_n(j)$$

We can omit the superscript altogether and write as ΣP_n etc.

Relatives: One of the simplest examples of an index number is a price relative, which is the ratio of the price of single commodity in a given period to its price in another period called the base period or the reference period. It can be indicated as follows:

Price relative = $\frac{P_n}{P_o}$

It has to be expressed as a percentage, it is multiplied by 100

Price relative = $\frac{P_n}{P_o} \times 100$

There can be other relatives such as of quantities, volume of consumption, exports, etc. The relatives in that case will be:

Quantity relative = $\frac{Q_n}{Q_o}$

Similarly, there are value relatives:

Value relative = $\frac{V_n}{V_o} = \frac{P_n Q_n}{P_o Q_o} = \left(\frac{P_n}{P_o} \times \frac{Q_n}{Q_o}\right)$

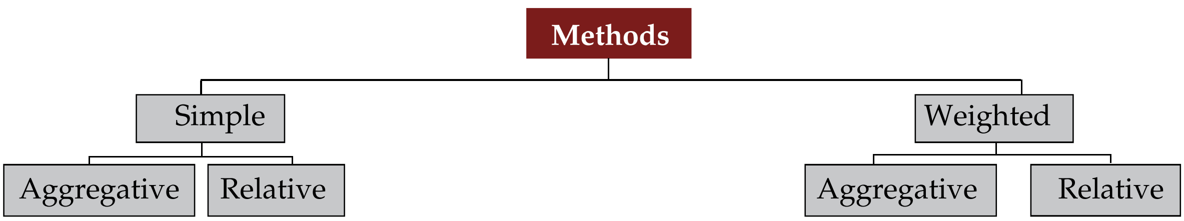
When successive prices or quantities are taken, the relatives are called the link relative,

$$\frac{P_1}{P_o}, \frac{P_2}{P_1}, \frac{P_3}{P_2}, \frac{P_n}{P_{n-1}}$$

When the above relatives are in respect to a fixed base period these are also called the chain relatives with respect to this base or the relatives chained to the fixed base. They are in the form of :

$$\frac{P_1}{P_o}, \frac{P_2}{P_o}, \frac{P_3}{P_o}, \frac{P_n}{P_o}$$

Methods: We can state the broad heads as follows:



18.3.1 SIMPLE AGGREGATIVE METHOD

In this method of computing a price index, we express the total of commodity prices in a given year as a percentage of total commodity price in the base year. In symbols, we have

Simple aggregative price index = $\frac{\Sigma P_n}{\Sigma P_o} \times 100$

where ΣP_n is the sum of all commodity prices in the current year and ΣP_o is the sum of all commodity prices in the base year.

ILLUSTRATIONS:

Commodities	1998	1999	2000
Cheese (per 100 gms)	12.00	15.00	15.60
Egg (per piece)	3.00	3.60	3.30
Potato (per kg)	5.00	6.00	5.70
Aggregrate	20.00	24.60	24.60
Index	100	123	123

Simple Aggregative Index for 1999 over 1998 = $\frac{\Sigma P_n}{\Sigma P_o} = \frac{24.60}{20.00} \times 100 = 123$

and for 2000 over 1998 = $\frac{\Sigma P_n}{\Sigma P_o} \times 100 = \frac{24.60}{20.00} \times 100 = 123$

The above method is easy to understand but it has a serious defect. It shows that the first commodity exerts greater influence than the other two because the price of the first commodity is higher than that of the other two. Further, if units are changed then the Index numbers will also change. Students should independently calculate the Index number taking the price of eggs per dozen i.e., ₹ 36, ₹ 43.20, ₹ 39.60 for the three years respectively. This is the major flaw in using absolute quantities and not the relatives. Such price quotations become the concealed weights which have no logical significance.

18.3.2 SIMPLE AVERAGE OF RELATIVES

One way to rectify the drawbacks of a simple aggregative index is to construct a simple average of relatives. Under it we invert the actual price for each variable into percentage of the base period. These percentages are called relatives because they are relative to the value for the base period. The index number is the average of all such relatives. One big advantage of price relatives is that they are pure numbers. Price index number computed from relatives will remain the same regardless of the units by which the prices are quoted. This method thus meets criterion of unit test (discussed later). Also quantity index can be constructed for a group of variables that are expressed in divergent units.

ILLUSTRATIONS:

In the proceeding example we will calculate relatives as follows:

Commodities	1998	1999	2000
A	100.0	125.0	130.0
B	100.0	120.0	110.0
C	100.0	120.0	114.0
Aggregate	300.0	365.0	354.0
Index	100.0	121.67	118.0

Inspite of some improvement, the above method has a flaw that it gives equal importance to each of the relatives. This amounts to giving undue weight to a commodity which is used in a small quantity because the relatives which have no regard to the absolute quantity will give weight more than what is due from the quantity used. This defect can be remedied by the introduction of an appropriate weighing system.

18.3.3 WEIGHTED METHOD

To meet the weakness of the simple or unweighted methods, we weigh the price of each commodity by a suitable factor often taken as the quantity or the volume of the commodity sold during the base year or some typical year. These indices can be classfied into broad groups:

(i) Weighted Aggregative Index.

(ii) Weighted Average of Relatives.

(i) Weighted Aggregative Index: Under this method we weigh the price of each commodity by a suitable factor often taken as the quantity or value weight sold during the base year or the given year or an average of some years. The choice of one or the other will depend on the importance we want to give to a period besides the quantity used. The indices are usually calculated in percentages. The various alternatives formulae in use are:

(The example has been given after the tests).

(a) Laspeyres' Index: In this Index base year quantities are used as weights:

$$\text{Laspeyres Index} = \frac{\Sigma P_n Q_0}{\Sigma P_0 Q_0} \times 100$$

(b) Paasche's Index: In this Index current year quantities are used as weights:

$$\text{Passche's Index} = \frac{\Sigma P_n Q_n}{\Sigma P_o Q_n} \times 100$$

(c) Methods based on some typical Period:

Index $\frac{\Sigma P_n Q_t}{\Sigma P_o Q_t} \times 100$ the subscript t stands for some typical period of years, the quantities of which are used as weight

Note: * Indices are usually calculated as percentages using the given formulae

The Marshall-Edgeworth index uses this method by taking the average of the base year and the current year

$$\text{Marshall-Edgeworth Index} = \frac{\Sigma P_n (Q_o + Q_n)}{\Sigma P_o (Q_o + Q_n)} \times 100$$

(d) Fisher's ideal Price Index: This index is the geometric mean of Laspeyres' and Paasche's.

$$\text{Fisher's Index} = \sqrt{\frac{\Sigma P_n Q_o}{\Sigma P_o Q_o} \times \frac{\Sigma P_n Q_n}{\Sigma P_o Q_n}} \times 100$$

(ii) **Weighted Average of Relative Method:** To overcome the disadvantage of a simple average of relative method, we can use weighted average of relative method. Generally weighted arithmetic mean is used although the weighted geometric mean can also be used. The weighted arithmetic mean of price relatives using base year value weights is represented by

$$\frac{\sum \frac{P_n}{P_o} \times (P_o Q_o)}{\sum P_o Q_o} \times 100 = \frac{\sum P_n Q_o}{\sum P_o Q_o} \times 100$$

Example:

		Price Relatives			Value Weights	Weighted Price Relatives	
Commodity	Q.	1998	1999	2000	1998	1999	2000
		$\frac{P_n}{P_0}$	$\frac{P_n}{P_0}$	$\frac{P_n}{P_0}$	P_0Q_0	$\frac{P_n}{P_0}P_0Q_0$	$\frac{P_n}{P_0}P_0Q_0$
Butter	0.7239	100	101.1	118.7	72.39	73.19	85.93
Milk	0.2711	100	101.7	126.7	27.11	27.57	34.35
Eggs	0.7703	100	100.9	117.8	77.03	77.72	90.74
Fruits	4.6077	100	96.0	114.7	460.77	442.34	528.50
Vegetables	1.9500	100	84.0	93.6	195.00	163.80	182.52
					832.30	784.62	922.04

Weighted Price Relative

For 1999 : $\frac{784.62}{832.30} \times 100 = 94.3$

For 2000 : $\frac{922.04}{832.30} \times 100 = 110.8$

18.3.4 THE CHAIN INDEX NUMBERS

So far we concentrated on a fixed base but it does not suit when conditions change quite fast. In such a case the changing base for example, 1998 for 1999, and 1999 for 2000, and so on, may be more suitable. If, however, it is desired to associate these relatives to a common base the results may be chained. Thus, under this method the relatives of each year are first related to the preceding year called the link relatives and then they are chained together by successive multiplication to form a chain index.

The formula is:

$$\text{Chain Index} = \frac{\text{Link relative of current year} \times \text{Chain Index of the previous year}}{100}$$

Example:

The following are the index numbers by a chain base method:

Year (1)	*Price* (2)	*Link Relatives* (3)	*Chain Indices* (4)
1991	50	100	100
1992	60	$\frac{60}{50} \times 100 = 120.0$	$\frac{120}{100} \times 100 = 120.0$
1993	62	$\frac{62}{60} \times 100 = 103.3$	$\frac{103.3}{100} \times 120 = 124.0$
1994	65	$\frac{65}{62} \times 100 = 104.8$	$\frac{104.8}{100} \times 124 = 129.9$
1995	70	$\frac{70}{65} \times 100 = 107.7$	$\frac{107.7}{100} \times 129.9 = 139.9$
1996	78	$\frac{78}{70} \times 100 = 111.4$	$\frac{111.4}{100} \times 139.9 = 155.8$
1997	82	$\frac{82}{78} \times 100 = 105.1$	$\frac{105.1}{100} \times 155.8 = 163.7$
1998	84	$\frac{84}{82} \times 100 = 102.4$	$\frac{102.4}{100} \times 163.7 = 167.7$
1999	88	$\frac{88}{84} \times 100 = 104.8$	$\frac{104.8}{100} \times 167.7 = 175.7$
2000	90	$\frac{90}{88} \times 100 = 102.3$	$\frac{102.3}{100} \times 175.7 = 179.7$

You will notice that link relatives reveal annual changes with reference to the previous year. But when they are chained, they change over to a fixed base from which they are chained, which in the above example is the year 1991. The chain index is an unnecessary complication unless of course where data for the whole period are not available or where commodity basket or the weights have to be changed. The link relatives of the current year and chain index from a given base will give also a fixed base index with the given base year as shown in the column 4 above.

18.3.5 QUANTITY INDEX NUMBERS

To measure and compare prices, we use price index numbers. When we want to measure and compare quantities, we resort to Quantity Index Numbers. Though price indices are widely used to measure the economic strength, Quantity indices are used as indicators of the level of output in economy. To construct Quantity indices, we measure changes in quantities and weight them using prices or values as weights. The various types of Quantity indices are:

1. Simple aggregate of quantities:

   This has the formula $\frac{\Sigma Q_n}{\Sigma Q_o}$

2. The simple average of quantity relatives:

   This can be expressed by the formula $\frac{\frac{\Sigma Q_n}{\Sigma Q_o}}{N}$

3. Weighted aggregate Quantity indices:

   (i) With base year weight : $\frac{\Sigma Q_n P_o}{\Sigma Q_o P_o}$ (Laspeyre's index)

   (ii) With current year weight : $\frac{\Sigma Q_n P_n}{\Sigma Q_o P_n}$ (Paasche's index)

   (iii) Geometric mean of (i) and (ii) : $\sqrt{\frac{\Sigma Q_n P_o}{\Sigma Q_o P_o} \times \frac{\Sigma Q_n P_n}{\Sigma Q_o P_n}}$ (Fisher's Ideal)

4. Base-year weighted average of quantity relatives. This has the formula $\frac{\Sigma\left(\frac{Q_n}{Q_o} P_o Q_o\right)}{\Sigma P_o Q_o}$

Note : Indices are usually calculated as percentages using the given formulae.

18.1.3.6 VALUE INDICES

Value equals price multiplied by quantity. Thus a value index equals the total sum of the values of a given year divided by the sum of the values of the base year, i.e.,

$$\frac{\Sigma V_n}{\Sigma V_o} = \frac{\Sigma P_n Q_n}{\Sigma P_0 Q_0}$$

18.4 USEFULNESS OF INDEX NUMBERS

So far we have studied various types of index numbers. However, they have certain limitations. They are :

1. As the indices are constructed mostly from deliberate samples, chances of errors creeping in cannot be always avoided.
2. Since index numbers are based on some selected items, they simply depict the broad trend and not the real picture.
3. Since many methods are employed for constructing index numbers, the result gives different values and this at times create confusion.

In spite of its limitations, index numbers are useful in the following areas :

1. Framing suitable policies in economics and business. They provide guidelines to make decisions in measuring intelligence quotients, research etc.
2. They reveal trends and tendencies in making important conclusions in cyclical forces, irregular forces, etc.
3. They are important in forecasting future economic activity. They are used in time series analysis to study long-term trend, seasonal variations and cyclical developments.
4. Index numbers are very useful in deflating i.e., they are used to adjust the original data for price changes and thus transform nominal wages into real wages.
5. Cost of living index numbers measure changes in the cost of living over a given period.

18.5 DEFLATING TIME SERIES USING INDEX NUMBERS

Sometimes a price index is used to measure the real values in economic time series data expressed in monetary units. For example, GNP initially is calculated in current price so that the effect of price changes over a period of time gets reflected in the data collected. Thereafter, to determine how much the physical goods and services have grown over time, the effect of changes in price over different values of GNP is excluded. The real economic growth in terms of constant prices of the base year therefore is determined by deflating GNP values using price index.

Year	*Wholesale Price Index*	*GNP at Current Prices*	*Real GNP*
1970	113.1	7499	6630
1971	116.3	7935	6823
1972	121.2	8657	7143
1973	127.7	9323	7301

The formula for conversion can be stated as

$$\text{Deflated Value} = \frac{\text{Current Value}}{\text{Price Index of the current year}}$$

$$\text{or Current Value} \times \frac{\text{Base Price } (P_0)}{\text{Current Price } (P_n)}$$

18.6 SHIFTING AND SPLICING OF INDEX NUMBERS

These refer to two technical points: (i) how the base period of the index may be shifted, (ii) how two index covering different bases may be combined into single series by splicing.

Shifted Price Index

Year	*Original Price Index*	*Shifted Price Index to base 1990*
1980	100	71.4
1981	104	74.3
1982	106	75.7
1983	107	76.4
1984	110	78.6
1985	112	80.0
1986	115	82.1
1987	117	83.6
1988	125	89.3
1989	131	93.6
1990	140	100.0
1991	147	105.0

The formula used is,

$$\text{Shifted Price Index} = \frac{\text{Original Price Index}}{\text{Price Index of the year on which it has to be shifted}} \times 100$$

Splicing two sets of price index numbers covering different periods of time is usually required when there is a major change in quantity weights. It may also be necessary on account of a new method of calculation or the inclusion of new commodity in the index.

Splicing Two Index Number Series

Year	*Old Price Index [1990 = 100]*	*Revised Price Index [1995 = 100]*	*Spliced Price Index [1995 = 100]*
1990	100.0		87.6
1991	102.3		89.6
1992	105.3		92.2
1993	107.6		94.2
1994	111.9		98.0
1995	114.2	100.0	100.0
1996		102.5	102.5
1997		106.4	106.4
1998		108.3	108.3
1999		111.7	111.7
2000		117.8	117.8

You will notice that the old series upto 1994 has to be converted shifting to the base. 1995 i.e, 114.2 to have a continuous series, even when the two parts have different weights

18.7 TEST OF ADEQUACY

There are four tests:

(i) **Unit Test:** This test requires that the formula should be independent of the unit in which or for which prices and quantities are quoted. Except for the simple (unweighted) aggregative index all other formulae satisfy this test.

(ii) **Time Reversal Test:** It is a test to determine whether a given method will work both ways in time, forward and backward. The test provides that the formula for calculating the index number should be such that two ratios, the current on the base and the base on the current should multiply into unity. In other words, the two indices should be reciprocals of each other. Symbolically,

$$P_{01} \times P_{10} = 1$$

where P_{01} is the index for time 1 on 0 and P_{10} is the index for time 0 on 1.

You will notice that Laspeyres' method and Paasche's method do not satisfy this test, but Fisher's Ideal Formula does.

While selecting an appropriate index formula, the Time Reversal Test and the Factor Reversal test are considered necessary in testing the consistency.

Laspeyres:

$$P_{01} = \frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \quad , \quad P_{10} = \frac{\Sigma P_0 Q_1}{\Sigma P_1 Q_1}$$

$$P_{01} \times P_{10} = \frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \times \frac{\Sigma P_0 Q_1}{\Sigma P_1 Q_1} \neq 1$$

Paasche's

$$P_{01} = \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1} \quad , \quad P_{10} = \frac{\Sigma P_0 Q_0}{\Sigma P_1 Q_0}$$

$$\therefore \quad P_{01} \times P_{10} = \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1} \times \frac{\Sigma P_0 Q_0}{\Sigma P_1 Q_0} \neq 1$$

Fisher's:

$$P_{01} = \sqrt{\frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \text{ x } \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1}} \qquad P_{10} = \sqrt{\frac{\Sigma P_0 Q_1}{\Sigma P_1 Q_1} \times \frac{\Sigma P_0 Q_0}{\Sigma P_1 Q_0}}$$

$$\therefore \quad P_{01} \times P_{10} = \sqrt{\frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1} \text{ x } \frac{\Sigma P_0 Q_1}{\Sigma P_1 Q_1} \times \frac{\Sigma P_0 Q_0}{\Sigma P_1 Q_0}} = 1$$

(iii) Factor Reversal Test: This holds when the product of price index and the quantity index should be equal to the corresponding value index, i.e., $\frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_0}$

Symbolically: $P_{01} \times Q_{01} = V_{01}$

Fishers'

$$P_{01} = \sqrt{\frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1}} \qquad Q_{01} = \sqrt{\frac{\Sigma Q_1 P_0}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_0 P_1}}$$

$$P_{01} \times Q_{01} = \sqrt{\frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1} \times \frac{\Sigma Q_1 P_0}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_0 P_1}} = \sqrt{\frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_0}}$$

$$= \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_0}$$

Thus Fisher's Index satisfies Factor Reversal test. Because Fisher's Index number satisfies both the tests in (ii) and (iii), it is called an Ideal Index Number.

(iv) Circular Test: It is concerned with the measurement of price changes over a period of years, when it is desirable to shift the base. For example, if the 1970 index with base 1965 is 200 and 1965 index with base 1960 is 150, the index 1970 on base 1960 will be 300. This property therefore enables us to adjust the index values from period to period without referring each time to the original base. The test of this shiftability of base is called the circular test.

This test is not met by Laspeyres, or Paasche's or the Fisher's ideal index. **The simple geometric mean of price relatives and the weighted aggregative with fixed weights meet this test.**

Example: Compute Fisher's Ideal Index from the following data:

	Base Year		Current Year	
Commodities	*Price*	*Quantity*	*Price*	*Quantity*
A	4	3	6	2
B	5	4	6	4
C	7	2	9	2
D	2	3	1	5

Show how it satisfies the time and factor reversal tests.

Solution:

Commodities	P_0	Q_0	P_1	Q_1	P_0Q_0	P_1Q_0	P_0Q_1	P_1Q_1
A	4	3	6	2	12	18	8	12
B	5	4	6	4	20	24	20	24
C	7	2	9	2	14	18	14	18
D	2	3	1	5	6	3	10	5
					52	63	52	59

Fisher's Ideal Index: $P_{01} = \sqrt{\dfrac{\Sigma P_1Q_0}{\Sigma P_0Q_0} \times \dfrac{\Sigma P_1Q_1}{\Sigma P_0Q_1}} \times 100 = \sqrt{\dfrac{63}{52} \times \dfrac{59}{52}} \times 100$

$$= \sqrt{1.375} \times 100 = 1.172 \times 100 = 117.3$$

Time Reversal Test:

$$P_{01} \times P_{10} = \sqrt{\frac{63}{52} \times \frac{59}{52} \times \frac{52}{59} \times \frac{52}{63}} = \sqrt{1} = 1$$

$\therefore$ Time Reversal Test is satisfied.

Factor Reversal Test:

$$P_{01} \times Q_{01} = \sqrt{\frac{63}{52} \times \frac{59}{52} \times \frac{52}{59} \times \frac{52}{63}} = \sqrt{\frac{59}{52} \times \frac{59}{52}} = \frac{59}{52}$$

Since, $\dfrac{\Sigma P_1Q_1}{\Sigma P_0Q_0}$ is also equal to $\dfrac{59}{52}$, the Factor Reversal Test is satisfied.

Concept Insight

Stock Market Index: It represents the entire stock market. It shows the changes taking place in the stock market. Movement of index is also an indication of average returns received by the investors. With the help of an index, it is easy for an investor to compare performance as it can be used as a benchmark, for e.g. a simple comparison of the stock and the index can be undertaken to find out the feasibility of holding a particular stock.

Each stock exchange has an index. For instance, in India, it is Sensex of BSE and Nifty of NSE. On the other hand, in outside India, popular indexes are Dow Jones, NASDAQ, FTSE etc.

(a) **Bombay Stock Exchange Limited:** It is the oldest stock exchange in Asia and was established as "The Native Share & Stock Brokers Association" in 1875. The Securities Contract (Regulation) Act, 1956 gives permanent recognition to Bombay Stock Exchange in 1956. BSE became the first stock exchange in India to obtain such permission from the Government under the Act. One of the Index as BSE Sensex which is basket of 30 constituent stocks. The base year of BSE SENSEX is 1978-79 and the base value is 100 which has grown over the years and quoted at about 592 times of base index as on date. As the oldest Index in the country, it provides the time series data over a fairly long period of time (from 1979 onward).

(b) **National Stock Exchange:** NSE was incorporated in 1992. It was recognized as a stock exchange by SEBI in April 1993 and commenced operations in 1994.NIFTY50 is a diversified 50 stocks Index of 13 sectors of the economy. The base period of NIFTY 50 Index is 3 November 1995 and base value is 1000 which has grown over years and quoted at 177 times as on date.

Computation of Index

Following steps are involved in calculation of index on a particular date:

- Calculate market capitalization of each individual company comprising the index.
- Calculate the total market capitalization by adding the individual market capitalization of all companies in the index.
- Computing index of next day requires the index value and the total market capitalization of the previous day and is computed as follows:

$$\text{IndexValue} = \text{Index on Previous Day} \times \frac{\text{Total market capitalisation for current day}}{\text{Total market capitalisation for previous day}}$$

- It should also be noted that Indices may also be calculated using the price weighted method. Here, the share price of the constituent companies forms the weight. However, almost all equity indices worldwide are calculated using the market capitalization weighted method.

- It is very important to note that constituents' companies does not remain the same. Hence , it may be possible the stocks of the company constituting index at the time of index inspection , may not be aprt of index as on date and new companies stock may have replaced them.

CPI- Consumer Price Index/ Cost of living Index or Retail Price Index is the Index which measures the effect of change in prices of basket of goods and services on the purchasing power of specific class of consumer during any current period w.r.t to some base period.

WPI- Whole Sale Price Index - The WPI measures the relative changes in prices of commodities traded in wholesale market.

SUMMARY

- An index number is a ratio or an average of ratios expressed as a percentage, Two or more time periods are involved, one of which is the base time period.
- Issues Involved in index numbers

  (a) Selection of Data
  (b) Base period
  (c) Selection of Weights:
  (d) Use of Averages:
  (e) Choice of Variables
- Construction of Index Number

  Price Index numbers

  (a) Simple aggregative price index = $\frac{\sum P_n}{\sum P_o} \times 100$

  (b) Laspeyres' Index: In this Index base year quantities are used as weights:

  Laspeyres Index = $\frac{\sum P_n Q_o}{\sum P_o Q_o} \times 100$

  (c) Paasche's Index: In this Index current year quantities are used as weights:

  Passche's Index = $\frac{\sum P_n Q_n}{\sum P_o Q_n} \times 100$

  (d) The Marshall-Edgeworth index uses this method by taking the average of the base year and the current year

  Marshall-Edgeworth Index $= \frac{\sum P_n (Q_o + Q_n)}{\sum P_o (Q_o + Q_n)} \times 100$

  (e) Fisher's ideal Price Index: This index is the geometric mean of Laspeyres' and Paasche's.

  Fisher's Index = $\sqrt{\frac{\sum P_n Q_o}{\sum P_o Q_o} \times \frac{\sum P_n Q_n}{\sum P_o Q_n}} \times 100$

  (g) Weighted Average of Relative Method: $\frac{\sum \frac{P_n}{P_o} \times (P_o Q_o)}{\sum P_o Q_o} \times 100 = \frac{\sum P_n Q_o}{\sum P_o Q_o} \times 100$

  (h) Chain Index = $\frac{\text{Link relative of current year} \times \text{Chain Index of the previous year}}{100}$

Quantity Index Numbers

- Simple aggregate of quantities: $\frac{\sum Q_n}{\sum Q_o}$

- The simple average of quantity relatives: $\frac{\frac{\sum Q_n}{\sum Q_o}}{N}$

- Weighted aggregate quantity indices:

  (i) With base year weight : $\frac{\sum Q_n P_o}{\sum Q_o P_o}$ (Laspeyre's index)

  (ii) With current year weight : $\frac{\sum Q_n P_n}{\sum Q_o P_n}$ (Paasche's index)

  (iii) Geometric mean of (i) and (ii) : $\sqrt{\frac{\sum Q_n P_o}{\sum Q_o P_o} \times \frac{\sum Q_n P_n}{\sum Q_o P_n}}$ (Fisher's Ideal)

- Base-year weighted average of quantity relatives. This has the formula $\frac{\sum\left(\frac{Q_n}{Q_o} P_o Q_o\right)}{\sum P_o Q_o}$

- Value Indices

  $$\frac{V_n}{V_o} = \frac{\sum P_n Q_n}{\sum P_o Q_o}$$

- Deflated Value = $\frac{\text{Current Value}}{\text{Price Index of the current year}}$

  or Current Value x $\frac{\text{Base Price } (P_0)}{\text{Current Price } (P_n)}$ Base Price (P_0)

- Shifted Price Index = $\frac{\text{Original Price Index}}{\text{Price Index of the year on which it has to be shifted}} \times 100$

- Test of Adequacy

  (1) Unit test
  (2) Time reversal Test
  (3) Factor reversal test
  (4) Circular Test

EXERCISE

Choose the most appropriate option (a) (b) (c) or (d).

1. A series of numerical figures which show the relative position is called
   a) index number b) relative number c) absolute number d) none
2. Index number for the base period is always taken as
   a) 200 b) 50 c) 1 d) 100
3. _________ play a very important part in the construction of index numbers.
   a) weights b) classes c) estimations d) none
4. ________ is particularly suitable for the construction of index numbers.
   a) H.M. b) A.M. c) G.M. d) none
5. Index numbers show _________ changes rather than absolute amounts of change.
   a) relative b) percentage c) both d) none
6. The ________ makes index numbers time-reversible.
   a) A.M. b) G.M. c) H.M. d) none
7. Price relative is equal to
   a) $\dfrac{\text{Price in the given year} \times 100}{\text{Price in the base year}}$ b) $\dfrac{\text{Price in the year base year} \times 100}{\text{Price in the given year}}$
   c) Price in the given year × 100 d) Price in the base year × 100
8. Index number is equal to
   a) sum of price relatives b) average of the price relatives
   c) product of price relative d) none
9. The ________ of group indices gives the General Index
   a) H.M. b) G.M. c) A.M. d) none
10. Circular Test is one of the tests of
    a) index numbers b) hypothesis c) both d) none
11. ___________ is an extension of time reversal test
    a) Factor Reversal test b) Circular test
    c) both d) none
12. Weighted G.M. of relative formula satisfy ________test
    a) Time Reversal Test b) Circular test
    c) Factor Reversal Test d) none
13. Factor Reversal test is satisfied by
    a) Fisher's Ideal Index b) Laspeyres Index
    c) Paasches Index d) none

14. Laspeyre's formula does not satisfy
    a) Factor Reversal Test   b) Time Reversal Test
    c) Circular Test   d) all the above
15. A ratio or an average of ratios expressed as a percentage is called
    a) a relative number   b) an absolute number
    c) an index number   d) none
16. The value at the base time period serves as the standard point of comparison
    a) false   b) true   c) both   d) none
17. An index time series is a list of _______ numbers for two or more periods of time
    a) index   b) absolute   c) relative   d) none
18. Index numbers are often constructed from the
    a) frequency   b) class   c) sample   d) none
19. ________ is a point of reference in comparing various data describing individual behaviour.
    a) Sample   b) Base period   c) Estimation   d) none
20. The ratio of price of single commodity in a given period to its price in the preceding year price is called the
    (a) base period   (b) price ratio   (c) relative price   (d) none
21. $\dfrac{\text{Sum of all commodity prices in the current year} \times 100}{\text{Sum of all commodity prices in the base year}}$ is
    (a) Relative Price Index   (b) Simple Aggregative Price Index
    (c) both   (d) none
22. Chain index is equal to
    (a) $\dfrac{\text{link relative of current year} \times \text{chain index of the current year}}{100}$
    (b) $\dfrac{\text{link relative of previous year} \times \text{chain index of the current year}}{100}$
    (c) $\dfrac{\text{link relative of current year} \times \text{chain index of the previous year}}{100}$
    (d) $\dfrac{\text{link relative of previous year} \times \text{chain index of the previous year}}{100}$
23. P_{01} is the index for time
    (a) 1 on 0   (b) 0 on 1   (c) 1 on 1   (d) 0 on 0
24. P_{10} is the index for time
    (a) 1 on 0   (b) 0 on 1   (c) 1 on 1   (d) 0 on 0

25. When the product of price index and the quantity index is equal to the corresponding value index then the test that holds is

(a) Unit Test (b) Time Reversal Test
(c) Factor Reversal Test (d) none holds

26. The formula should be independent of the unit in which or for which price and quantities are quoted in

(a) Unit Test (b) Time Reversal Test
(c) Factor Reversal Test (d) none

27. Laspeyre's method and Paasche's method do not satisfy

(a) Unit Test (b) Time Reversal Test
(c) Factor Reversal Test (d) b & c

28. The purpose determines the type of index number to use

(a) yes (b) no (c) may be (d) may not be

29. The index number is a special type of average

(a) false (b) true (c) both (d) none

30. The choice of suitable base period is at best temporary solution

(a) true (b) false (c) both (d) none

31. Fisher's Ideal Formula for calculating index numbers satisfies the _______ tests

(a) Unit Test (b) Factor Reversal Test
(c) both (d) none

32. Fisher's Ideal Formula dose not satisfy _________ test

(a) Unit Test (b) Circular Test (c) Time Reversal Test (d) none

33. _________________ satisfies circular test

a) G.M. of price relatives or the weighted aggregate with fixed weights

b) A.M. of price relatives or the weighted aggregate with fixed weights

c) H.M. of price relatives or the weighted aggregate with fixed weights

d) none

34. Laspeyre's and Paasche's method _________ time reversal test

(a) satisfy (b) do not satisfy (c) are (d) are not

35. There is no such thing as unweighted index numbers

(a) false (b) true (c) both (d) none

36. Theoretically, G.M. is the best average in the construction of index numbers but in practice, mostly the A.M. is used

(a) false (b) true (c) both (d) none

37. Laspeyre's or Paasche's or the Fisher's ideal index do not satisfy

(a) Time Reversal Test (b) Unit Test
(c) Circular Test (d) none

38. ___________ is concerned with the measurement of price changes over a period of years, when it is desirable to shift the base

(a) Unit Test (b) Circular Test
(c) Time Reversal Test (d) none

39. The test of shifting the base is called

(a) Unit Test (b) Time Reversal Test
(c) Circular Test (d) none

40. The formula for conversion to current value

(a) Deflated value = $\frac{\text{Price Index of the current year}}{\text{previous value}}$

(b) Deflated value = $\frac{\text{current value}}{\text{Price Index of the current year}}$

(c) Deflated value = $\frac{\text{Price Index of the previous year}}{\text{previous value}}$

(d) Deflated value = $\frac{\text{Price Index of the previous year}}{\text{previous value}}$

41. Shifted price Index = $\frac{\text{Original Price} \times 100}{\text{Price Index of the year on which it has to be shifted}}$

(a) True (b) false (c) both (d) none

42. The number of test of Adequacy is

(a) 2 (b) 5 (c) 3 (d) 4

43. We use price index numbers

(a) To measure and compare prices (b) to measure prices
(c) to compare prices (d) none

44. Simple aggregate of quantities is a type of

(a) Quantity control (b) Quantity indices
(c) both (d) none

ANSWERS

Exercise

1. (a)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (a)	**8.** (b)
9. (c)	**10.** (a)	**11.** (b)	**12.** (a)	**13.** (a)	**14.** (d)	**15.** (c)	**16.** (b)
17. (a)	**18.** (c)	**19.** (b)	**20.** (c)	**21.** (b)	**22.** (c)	**23.** (a)	**24.** (b)
25. (c)	**26.** (a)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (a)	**31.** (c)	**32.** (b)
33. (a)	**34.** (b)	**35.** (a)	**36.** (b)	**37.** (c)	**38.** (b)	**39.** (c)	**40.** (b)
41. (a)	**42.** (d)	**43.** (a)	**44.** (b)				

ADDITIONAL QUESTION BANK

1. Each of the following statements is either True or False write your choice of the answer by writing T for True

   (a) Index Numbers are the signs and guideposts along the business highway that indicate to the businessman how he should drive or manage.

   (b) "For Construction index number, the best method on theoretical ground is not the best method from practical point of view".

   (c) Weighting index numbers makes them less representative.

   (d) Fisher's index number is not an ideal index number.

2. Each of the following statements is either True or False. Write your choice of the answer by writing F for false.

   (a) Geometric mean is the most appropriate average to be used for constructing an index number.

   (b) Weighted average of relatives and weighted aggregative methods render the same result.

   (c) "Fisher's Ideal Index Number is a compromise between two well known indices – not a right compromise, economically speaking".

   (d) "Like all statistical tools, index numbers must be used with great caution".

3. The best average for constructing an index numbers is

   (a) Arithmetic Mean (b) Harmonic Mean

   (c) Geometric Mean (d) None of these.

4. The time reversal test is satisfied by

   (a) Fisher's index number. (b) Paasche's index number.

   (c) Laspeyre's index number. (d) None of these.

5. The factor reversal test is satisfied by

(a) Simple aggregative index number. (b) Paasche's index number.

(c) Laspeyre's index number. (d) Fisher's index

6. The circular test is satisfied by

(a) Fisher's index number. (b) Paasche's index number.

(c) Laspeyre's index number. (d) Simple GM price relative.

7. Fisher's index number is based on

(a) The Arithmetic mean of Laspeyre's and Paasche's index numbers.

(b) The Median of Laspeyre's and Paasche's index numbers.

(c) the Mode of Laspeyre's and Paasche's index numbers.

(d) None of these.

8. Paasche index is based on

(a) Base year quantities. (b) Current year quantities.

(c) Average of current and base year. (d) None of these.

9. Fisher's ideal index number is

(a) The Median of Laspeyre's and Paasche's index numbers

(b) The Arithmetic Mean of Laspeyre's and Paasche's index numbers

(c) The Geometric Mean of Laspeyre's and Paasche's index numbers

(d) None of these.

10. Price-relative is expressed in term of

(a) $P = \frac{P_n}{P_o}$ (b) $P = \frac{P_o}{P_n}$

(c) $P = \frac{P_n}{P_o} \times 100$ (d) $P = \frac{P_o}{P_n} \times 100$

11. Paasehe's index number is expressed in terms of :

(a) $\frac{\sum P_n q_n}{\sum P_o q_n}$ (b) $\frac{\sum P_o q_o}{\sum P_n q_n}$

(c) $\frac{\sum P_n q_n}{\sum P_o q_n} \times 100$ (d) $\frac{\sum P_n q_o}{\sum P_o q_o} \times 100$

12. Time reversal Test is satisfied by following index number formula is

(a) Laspeyre's Index number.

(b) Simple Arithmetic Mean of price relative formula

(c) Marshall-Edge worth formula.

(d) None of these.

13. Cost of Living Index number (C. L. I.) is expressed in terms of :

(a) $\frac{\sum P_n q_o}{\sum P_o q_o} \times 100$ (b) $\frac{\sum P_n q_n}{\sum P_o q_o}$

(c) $\frac{\sum P_o q_n}{\sum P_n q_n} \times 100$ (d) None of these.

14. If the ratio between Laspeyre's index number and Paasche's Index number is 28 : 27. Then the missing figure in the following table P is :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
X	L	10	2	5
Y	L	5	P	2

(a) 7 (b) 4 (c) 3 (d) 9

15. If the prices of all commodities in a place have increased 1.25 times in comparison to the base period, the index number of prices of that place now is

(a) 125 (b) 150 (c) 225 (d) None of these.

16. If the index number of prices at a place in 1994 is 250 with 1984 as base year, then the prices have increased on average by

(a) 250% (b) 150% (c) 350% (d) None of these.

17. If the prices of all commodities in a place have decreased 35% over the base period prices, then the index number of prices of that place is now

(a) 35 (b) 135 (c) 65 (d) None of these.

18. Link relative index number is expressed for period n is

(a) $\frac{P_n}{P_{n+1}}$ (b) $\frac{P_0}{P_{n-1}}$

(c) $\frac{P_n}{P_{n-1}} \times 100$ (d) None of these.

19. Fisher's Ideal Index number is expressed in terms of :

(a) $(P_{on})^F = \sqrt{\text{Laspeyre's Index} \times (\text{Paasche's Index})}$

(b) $(P_{on})^F$ = Laspeyre's Index X Paasche's Index

(c) $(P_{on})^F = \sqrt{\text{Marshall Edge worth Index} \times \text{Paasche's}}$

(d) None of these.

20. Factor Reversal Test According to Fisher is $P_{01} \times Q_{01} =$

(a) $\dfrac{\sum P_o q_o}{\sum P_n q_n}$ (b) $\dfrac{\sum P_n q_n}{\sum P_o q_o}$

(c) $\dfrac{\sum P_o q_n}{\sum P_n q_n}$ (d) None of these.

21. Marshall-edge worth Index formula after interchange of p and q is expressed in terms of :

(a) $\dfrac{\sum q_n (p_0 + p_n)}{\sum q_0 (p_0 + p_n)}$ (b) $\dfrac{\sum P_n (q_0 + q_n)}{\sum q P_0 (q_0 + q_n)}$

(c) $\dfrac{\sum P_0 (q_0 + q_n)}{\sum P_n (P_0 + P_n)}$ (d) None of these.

22. If $\sum P_n q_n = 249$, $\sum P_o q_o = 150$, Paasche's Index Number = 150 and Drobiseh and Bowely's Index number = 145, then the Fisher's Ideal Index Number is

(a) 75 (b) 60 (c) 145.97 (d) None of these.

23. Consumer Price index number for the year 1957 was 313 with 1940 as the base year. The Average Monthly wages in 1957 of the workers into factory be ₹ 160/- their real wages is

(a) ₹ 48.40 (b) ₹ 51.12 (c) ₹ 40.30 (d) None of these.

24. If $\sum P_o q_o = 3500$, $\sum P_n q_o = 3850$, then the Cost of living Index (C.L.I.) for 1950 w.r. to base 1960 is

(a) 110 (b) 90 (c) 100 (d) None of these.

25. From the following table by the method of relatives using Arithmetic mean the price Index number is

Commodity	Wheat	Milk	Fish	Sugar
Base Price	5	8	25	6
Current Price	7	10	32	12

(a) 140.35 (b) 148.25 (c) 140.75 (d) None of these.

From the Q.No. 26 to 29 each of the following statements is either True or False with your choice of the answer by writing F for False.

26. (a) Base year quantities are taken as weights in Laspeyre's price Index number.

(b) Fisher's ideal index is equal to the Arithmetic mean of Laspeyre's and Paasche's index numbers.

(c) Laspeyre's index number formula does not satisfy time reversal test.

(d) None of these.

27. (a) Current year quantities are taken as weights in Paasche's price index number.

(b) Edge worth Marshall's index number formula satisfies Time, Reversal Test.

(c) The Arithmetic mean of Laspeyre's and Paasche's index numbers is called Bowely's index numbers.

(d) None of these.

28. (a) Current year prices are taken as weights in Paasche's quantity index number.

(b) Fisher's Ideal Index formula satisfies factor Reversal Test.

(c) The sum of the quantities of the base period and current period is taken as weights in Laspeyre's index number.

(d) None of these.

29. (a) Simple Aggregative and simple Geometric mean of price relatives formula satisfy circular Test.

(b) Base year prices are taken as weights in Laspeyre's quantity index numbers.

(c) Fisher's Ideal Index formula obeys time reversal and factor reversal tests.

(d) None of these.

30. In 1980, the net monthly income of the employee was ₹ 800/- p. m. The consumer price index number was 160 in 1980. It rises to 200 in 1984. If he has to be rightly compensated. The additional D. A. to be paid to the employee is

(a) ₹ 175/- (b) ₹ 185/- (c) ₹ 200/- (d) ₹ 125.

31. The simple Aggregative formula and weighted aggregative formula satisfy is

(a) Factor Reversal Test (b) Circular Test

(c) Unit Test (d) None of these.

32. "Fisher's Ideal Index is the only formula which satisfies"

(a) Time Reversal Test (b) Circular Test

(c) Factor Reversal Test (d) a & c.

33. "Neither Laspeyre's formula nor Paasche's formula obeys" :

(a) Time Reversal and factor Reversal Tests of index numbers.

(b) Unit Test and circular Tests of index number.

(c) Time Reversal and Unit Test of index number.

(d) None of these.

34. Bowley's index number is 150. Fisher's index number is 149.95. Paasche's index number is

(a) 146.13 (b) 154 (c) 148 (d) 156

35. With the base year 1960 the C. L. I. in 1972 stood at 250. x was getting a monthly Salary of ₹ 500 in 1960 and ₹ 750 in 1972. In 1972 to maintain his standard of living in 1960 x has to receive as extra allowances of

(a) ₹ 600/- (b) ₹ 500/- (c) ₹ 300/- (d) none of these.

36. From the following data base year :-

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	4	3	6	2
B	5	4	6	4
C	7	2	9	2
D	2	3	1	5

Fisher's Ideal Index is

(a) 117.3 (b) 115.43 (c) 118.35 (d) 116.48

37. Which statement is False?

(a) The choice of suitable base period is at best a temporary solution.

(b) The index number is a special type of average.

(c) Those is no such thing as unweighted index numbers.

(d) Theoretically, geometric mean is the best average in the construction of index numbers but in practice, mostly the arithmetic mean is used.

38. Factor Reversal Test is expressed in terms of

(a) $\frac{\Sigma P_1Q_1}{\Sigma P_0Q_0}$ (b) $\frac{\Sigma P_1Q_1}{\Sigma P_0Q_0} \times \frac{\Sigma P_1Q_1}{\Sigma P_0Q_1}$

(c) $\frac{\Sigma P_1Q_1}{\Sigma Q_0P_1}$ (d) $\frac{\Sigma Q_1P_0}{\Sigma Q_0P_0} \times \frac{\Sigma P_1Q_1}{\Sigma Q_0P_1}$

39. Circular Test is satisfied by

(a) Laspeyre's Index number.

(b) Paasche's Index number

(c) The simple geometric mean of price relatives and the weighted aggregative with fixed weights.

(d) None of these.

40. From the following data for the 5 groups combined

Group	Weight	Index Number
Food	35	425
Cloth	15	235
Power & Fuel	20	215
Rent & Rates	8	115
Miscellaneous	22	150

The general Index number is

(a) 270 (b) 269.2 (c) 268.5 (d) 272.5

41. From the following data with 1966 as base year

Commodity	Quantity Units	Values (₹)
A	100	500
B	80	320
C	60	150
D	30	360

The price per unit of commodity A in 1966 is

(a) ₹ 5 (b) ₹ 6 (c) ₹ 4 (d) ₹ 12

42. The index number in whole sale prices is 152 for August 1999 compared to August 1998. During the year there is net increase in prices of whole sale commodities to the extent of

(a) 45% (b) 35% (c) 52% (d) 48%

43. The value Index is expressed in terms of

(a) $\frac{\sum P_1Q_1}{\sum P_0Q_0} \times 100$ (b) $\frac{\sum P_1Q_1}{\sum P_0Q_0}$

(c) $\frac{\sum P_0Q_0}{\sum P_1Q_1} \times 100$ (d) $\frac{\sum P_0Q_1 \times \sum P_1Q_1}{\sum P_0Q_0 \times \sum P_1Q_0}$

44. Purchasing Power of Money is

(a) Reciprocal of price index number. (b) Equal to price index number.

(c) Unequal to price index number. (d) None of these.

45. The price level of a country in a certain year has increased 25% over the base period. The index number is

(a) 25 (b) 125 (c) 225 (d) 2500

46. The index number of prices at a place in 1998 is 355 with 1991 as base. This means

(a) There has been on the average a 255% increase in prices.

(b) There has been on the average a 355% increase in price.

(c) There has been on the average a 250% increase in price.

(d) None of these.

47. If the price of all commodities in a place have increased 1.25 times in comparison to the base period prices, then the index number of prices for the place is now

(a) 100 (b) 125 (c) 225 (d) None of the above.

48. The wholesale price index number or agricultural commodities in a given region at a given date is 280. The percentage increase in prices of agricultural commodities over the base year is :

(a) 380 (b) 280 (c) 180 (d) 80

49. If now the prices of all the commodities in a place have been decreased by 35% over the base period prices, then the index number of prices for the place is now (index number of prices of base period = 100)

(a) 100 (b) 135 (c) 65 (d) None of these.

50. From the data given below

Commodity	Price Relative	Weight
A	125	5
B	67	2
C	250	3

Then the suitable index number is

(a) 150.9 (b) 155.8 (c) 145.8 (d) None of these.

51. Bowley's Index number is expressed in the form of :

(a) $\frac{\text{Laspeyre's index} + \text{Paasche's index}}{2}$ (b) $\frac{\text{Laspeyre's index} \times \text{Paasche's index}}{2}$

(c) $\frac{\text{Laspeyre's index} - \text{Paasche's index}}{2}$ (d) None of these.

52. From the following data

Commodity	Base Price	Current Price
Rice	35	42
Wheat	30	35
Pulse	40	38
Fish	107	120

The simple Aggregative Index is

(a) 115.8 (b) 110.8 (c) 112.5 (d) 113.4

53. With regard to Laspeyre's and Paasche's price index numbers, it is maintained that "If the prices of all the goods change in the same ratio, the two indices will be equal for them the weighting system is irrelevant; or if the quantities of all the goods change in the same ratio, they will be equal, for them the two weighting systems are the same relatively". Then the above statements satisfy.

(a) Laspeyre's Price index ≠ Paasche's Price Index.

(b) Laspeyre's Price Index = Paasche's Price Index.

(c) Laspeyre's Price Index may be equal Paasche's Price Index.

(d) None of these.

54. The quantity Index number using Fisher's formula satisfies :

(a) Unit Test (b) Factor Reversal Test.

(c) Time Reversal Test. (d) All the three

55. For constructing consumer price Index is used :

(a) Marshall Edge worth Method. (b) Paasche's Method.

(c) Dorbish and Bowley's Method. (d) Laspeyre's Method.

56. The cost of living Index (C.L.I.) is always :

(a) Weighted index (b) Price Index.

(c) Quantity Index. (d) None of these.

57. The Time Reversal Test is not satisfied to :

(a) Fisher's ideal Index. (b) Marshall Edge worth Method.

(c) Laspeyre's and Paasche Method. (d) None of these.

58. Given below are the data on prices of some consumer goods and the weights attached to the various items. Compute price index number for the year 1985 (Base 1984 = 100)

Items	Unit	1984	1985	Weight
Wheat	Kg.	0.50	0.75	2
Milk	Litre	0.60	0.75	5
Egg	Dozen	2.00	2.40	4
Sugar	Kg.	1.80	2.10	8
Shoes	Pair	8.00	10.00	1

Then weighted average of price Relative Index is :

(a) 125.43 (b) 123.3 (c) 124.53 (d) 124.52

59. The Factor Reversal Test is as represented symbolically is :

(a) $P_{01} \times Q_{01} = \frac{\sum P_1Q_1}{\sum P_0Q_0}$ (b) $I_{01} \times I_{10}$

(c) $\frac{\sum P_0Q_0}{\sum P_1Q_1}$ (d) $\sqrt{\frac{\sum P_1Q_1}{\sum P_0Q_0} \times \frac{\sum P_0Q_1}{\sum Q_{10}P_0}}$

60. If the 1970 index with base 1965 is 200 and 1965 index with base 1960 is 150, the index 1970 on base 1960 will be :

(a) 700 (b) 300 (c) 500 (d) 600

61. Circular Test is not met by :

(a) The simple Geometric mean of price relatives.

(b) The weighted aggregative with fixed weights.

(c) Laspeyre's or Paasche's or the fisher's Ideal index.

(d) None of these.

62. From the following data

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	4	3	6	2
B	5	4	6	4
C	7	2	9	2
D	2	3	1	5

Then the value ratio is:

(a) $\frac{59}{52}$ (b) $\frac{49}{47}$ (c) $\frac{41}{53}$ (d) $\frac{47}{53}$

63. The value index is equal to :

(a) The total sum of the values of a given year multiplied by the sum of the values of the base year.

(b) The total sum of the values of a given year Divided by the sum of the values of the base year.

(c) The total sum of the values of a given year plus by the sum of the values of the base year.

(d) None of these.

64. Time Reversal Test is represented symbolically by :

(a) $P_{01} \times P_{10}$ (b) $P_{01} \times P_{10} = 1$

(c) $P_{01} \times P_{10} \neq 1$ (d) None of these.

65. In 1996 the average price of a commodity was 20% more than in 1995 but 20% less than in 1994; and more over it was 50% more than in 1997 to price relatives using 1995 as base (1995 price relative 100). Reduce the data is :

(a) 150, 100, 120, 80 for (1994–97) (b) 135, 100, 125, 87 for (1994–97)

(c) 140, 100, 120, 80 for (1994–97) (d) None of these.

66. From the following data

Commodities	Base Year 1922 Price (₹)	Current Year 1934 Price
A	6	10
B	2	2
C	4	6
D	11	12
E	8	12

The price index number for the year 1934 is :

(a) 140 (b) 145 (c) 147 (d) None of these.

67. From the following data

Commodities	Base Price 1964	Current Price 1968
Rice	36	54
Pulse	30	50
Fish	130	155
Potato	40	35
Oil	110	110

The index number by unweighted methods :

(a) 116.8 (b) 117.25 (c) 115.35 (d) 119.37

68. The Bowley's Price index number is represented in terms of :

(a) A.M. of Laspeyre's and Paasche's Price index number.

(b) G.M. of Laspeyre's and Paasche's Price index number.

(c) A.M. of Laspeyre's and Walsh's price index number.

(d) None of these.

69. Fisher's price index number equal is :

(a) G.M. of Kelly's price index number and Paasche's price index number.

(b) G.M. of Laspeyre's and Paasche's Price index number.

(c) G.M. of Bowley's price index number and Paasche's price index number.

(d) None of these.

70. The price index number using simple G.M. of the n relatives is given by :

(a) $\log I_{on} = 2 - \frac{1}{n}\sum \log \frac{P_n}{P_o}$

(b) $\log I_{on} = 2 + \frac{1}{n}\sum \log \frac{P_n}{P_o}$

(c) $\log I_{on} = \frac{1}{2n}\sum \log \frac{P_n}{P_o}$

(d) None of these.

71. The price of a number of commodities are given below in the current year 1975 and base year 1970.

Commodities	A	B	C	D	E	F
Base Price	45	60	20	50	85	120
Current Price	55	70	30	75	90	130

For 1975 with base 1970 by the Method of price relatives using Geometrical mean, the price index is :

(a) 125.3 (b) 124.3 (c) 128.8 (d) None of these.

72. From the following data

Group	A	B	C	D	E	F
Group Index	120	132	98	115	108	95
Weight	6	3	4	2	1	4

The general Index I is given by :

(a) 111.3 (b) 113.45 (c) 117.25 (d) 114.75

73. The price of a commodity increases from ₹ 5 per unit in 1990 to ₹ 7.50 per unit in 1995 and the quantity consumed decreases from 120 units in 1990 to 90 units in 1995. The price and quantity in 1995 are 150% and 75% respectively of the corresponding price and quantity in 1990. Therefore, the product of the price ratio and quantity ratio is :

(a) 1.8 (b) 1.125 (c) 1.75 (d) None of these.

74. Test whether the index number due to Walsh given by :

$$I = \frac{\sum P_1\sqrt{Q_0 Q_1}}{\sum P_0\sqrt{Q_0 Q_1}} \times 100$$ Satisfies is :-

(a) Time reversal Test.

(b) Factor reversal Test.

(c) Circular Test.

(d) None of these.

75. From the following data

Group	Weight	Index Number Base : 1952–53 = 100
Food	50	241
Clothing	2	21
Fuel and Light	3	204
Rent	16	256
Miscellaneous	29	179

The Cost of living index numbers is :

(a) 224.5 (b) 223.91 (c) 225.32 (d) None of these.

76. Consumer price index number goes up from 110 to 200 and the Salary of a worker is also raised from ₹ 325 to ₹ 500. Therefore, in real terms, to maintain his previous standard of living he should get an additional amount of :

(a) ₹ 85 (b) ₹ 90.91 (c) ₹ 98.25 (d) None of these.

77. The prices of a commodity in the year 1975 and 1980 were 25 and 30 respectively taking 1980 as base year the price relative is :

(a) 109.78 (b) 110.25 (c) 113.25 (d) None of these.

78. The average price of certain commodities in 1980 was ₹ 60 and the average price of the same commodities in 1982 was ₹ 120. Therefore, the increase in 1982 on the basis of 1980 was 100%. 80. The decrease in 1980 with 1982 as base is: using 1982, comment on the above statement is :

(a) The price in 1980 decreases by 60% using 1982 as base.

(b) The price in 1980 decreases by 50% using 1982 as base.

(c) The price in 1980 decreases by 90% using 1982 as base.

(d) None of these.

79. Cost of Living Index (C.L.I.) numbers are also used to find real wages by the process of

(a) Deflating of Index number. (b) Splicing of Index number.

(c) Base shifting. (d) None of these.

80. From the following data

Commodities		A	B	C	D
1992 Base	Price	3	5	4	1
	Quantity	18	6	20	14
1993 Current Year	Price	4	5	6	3
	Quantity	15	9	26	15

The Passche price Index number is :

(a) 146.41 (b) 148.25 (c) 144.25 (d) None of these.

81. From the following data

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	7	17	13	25
B	6	23	7	25
C	11	14	13	15
D	4	10	8	8

The Marshall Edge Worth Index number is :

(a) 148.25 (b) 144.19 (c) 147.25 (d) 143.78

82. The circular Test is an extension of

(a) The time reversal Test. (b) The factor reversal Test.

(c) The unit Test. (d) None of these.

83. Circular test, an index constructed for the year 'x' on the base year 'y' and for the year 'y' on the base year 'z' should yield the same result as an index constructed for 'x' on base year 'z' i.e. $I_{01} \times I_{12} \times I_{20}$ equal is :

(a) 3 (b) 2 (c) 1 (d) None of these.

84. In 1976 the average price of a commodity was 20% more than that in 1975 but 20% less than that in 1974 and more over it was 50% more than that in 1977. The price relatives using 1975 as base year (1975 price relative = 100) then the reduce data is :

(a) 8,.75 (b) 150,80 (c) 75,125 (d) None of these.

85. Time Reversal Test is represented by symbolically is :

(a) $P_{01} \times Q_{01} = 1$ (b) $I_{01} \times I_{10} = 1$

(b) $I_{01} \times I_{12} \times I_{23} \times \ldots I_{(n-1)n} \times I_{n0} = 1$ (d) None of these.

86. The prices of a commodity in the years 1975 and 1980 were 25 and 30 respectively, taking 1975 as base year the price relative is :

(a) 120 (b) 135 (c) 122 (d) None of these.

87. From the following data

Year	1992	1993	1995	1996	1997
Link Index	100	103	105	112	108

(Base 1992 = 100) for the years 1993–97. The construction of chain index is :

(a) 103, 100.94, 107, 118.72 (b) 103, 108.15, 121.3, 130.82

(c) 107, 100.25, 104, 118.72 (d) None of these.

88. During a certain period the cost of living index number goes up from 110 to 200 and the salary of a worker is also raised from ₹ 330 to ₹ 500. The worker does not get really gain. Then the real wages decreased by :

(a) ₹ 45.45 (b) ₹ 43.25 (c) ₹ 100 (d) None of these.

89. Net monthly salary of an employee was ₹ 3000 in 1980. The consumer price index number in 1985 is 250 with 1980 as base year. If the has to be rightly compensated then, 7^{th} dearness allowances to be paid to the employee is :

(a) ₹ 4.800.00 (b) ₹ 4,700.00 (c) ₹ 4,500.0 (d) None of these.

90. Net Monthly income of an employee was ₹ 800 in 1980. The consumer price Index number was 160 in 1980. It is rises to 200 in 1984. If he has to be rightly compensated. The additional dearness allowance to be paid to the employee is :

(a) ₹ 240 (b) ₹ 275 (c) ₹ 250 (d) None of these.

91. When the cost of Tobacco was increased by 50%, a certain hardened smoker, who maintained his formal scale of consumption, said that the rise had increased his cost of living by 5%. Before the change in price, the percentage of his cost of living was due to buying Tobacco is

(a) 15% (b) 8% (c) 10% (d) None of these.

92. If the price index for the year, say 1960 be 110.3 and the price index for the year, say 1950 be 98.4, then the purchasing power of money (Rupees) of 1950 in 1960 is

(a) ₹ 1.12 (b) ₹ 1.25 (c) ₹ 1.37 (d) None of these.

93. If $\sum P_0Q_0 = 1360$, $\sum PnQ_0 = 1900$, $\sum P_0Q_n = 1344$, $\sum P_nQ_n = 1880$ then the Laspeyre's Index number is

(a) 0.71 (b) 1.39 (c) 1.75 (d) None of these.

94. The consumer price Index for April 1985 was 125. The food price index was 120 and other items index was 135. The percentage of the total weight index given to food is

(a) 66.67 (b) 68.28 (c) 90.25 (d) None of these.

95. The total value of retained imports into India in 1960 was ₹ 71.5 million per month. The corresponding total for 1967 was ₹ 87.6 million per month. The index of volume of retained imports in 1967 composed with 1960 (= 100) was 62.0. The price index for retained inputs for 1967 our 1960 as base is

(a) 198.61 (b) 197.61 (c) 198.25 (d) None of these.

96. During the certain period the C.L.I. goes up from 110 to 200 and the Salary of a worker is also raised from 330 to 500, then the real terms is

(a) Loss by ₹ 50 (b) Loss by 75 (c) Loss by ₹ 90 (d) None of these.

[Hint : Real Wage = (Actual wage/Cost of Living Index) * 100]

97. From the following data

Commodities	Q_0	P_0	Q_1	P_1
A	2	2	6	18
B	5	5	2	2
C	7	7	4	24

Then the fisher's quantity index number is

(a) 87.34 (b) 85.24 (c) 87.25 (d) 78.93

98. From the following data

Commodities	Base year	Current year
A	25	55
B	30	45

Then index numbers from G. M. Method is :

(a) 181.66 (b) 185.25 (c) 181.75 (d) None of these.

99. Using the following data

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
X	4	10	6	15
Y	6	15	4	20
Z	8	5	10	4

the Paasche's formula for index is :

(a) 125.38 (b) 147.25 (c) 129.8 (d) 99.06

100. Group index number is represented by

(a) $\frac{\text{Price Relative for the year}}{\text{Price Relative for the previous year}} \times 100$ (b) $\frac{\sum(\text{Price Relative} \times w)}{\sum w}$

(c) $\frac{\sum(\text{Price Relative} \times w)}{\sum w} \times 100$ (d) None of these.

ANSWERS

1.	(a)	**2.**	(c)	**3.**	(c)	**4.**	(a)	**5.**	(d)
6.	(d)	**7.**	(d)	**8.**	(b)	**9.**	(c)	**10.**	(c)
11.	(c)	**12.**	(c)	**13.**	(a)	**14.**	(b)	**15.**	(c)
16.	(b)	**17.**	(c)	**18.**	(c)	**19.**	(a)	**20.**	(b)
21.	(a)	**22.**	(d)	**23.**	(b)	**24.**	(a)	**25.**	(b)
26.	(b)	**27.**	(d)	**28.**	(c)	**29.**	(d)	**30.**	(c)
31.	(b)	**32.**	(d)	**33.**	(a)	**34.**	(a)	**35.**	(b)
36.	(a)	**37.**	(c)	**38.**	(a)	**39.**	(c)	**40.**	(b)
41.	(a)	**42.**	(c)	**43.**	(a)	**44.**	(a)	**45.**	(b)
46.	(a)	**47.**	(c)	**48.**	(c)	**49.**	(c)	**50.**	(a)
51.	(a)	**52.**	(b)	**53.**	(b)	**54.**	(d)	**55.**	(d)
56.	(a)	**57.**	(c)	**58.**	(b)	**59.**	(a)	**60.**	(b)
61.	(c)	**62.**	(a)	**63.**	(b)	**64.**	(b)	**65.**	(a)
66.	(d)	**67.**	(a)	**68.**	(a)	**69.**	(b)	**70.**	(b)
71.	(b)	**72.**	(a)	**73.**	(b)	**74.**	(a)	**75.**	(d)
76.	(b)	**77.**	(d)	**78.**	(b)	**79.**	(a)	**80.**	(a)
81.	(b)	**82.**	(a)	**83.**	(c)	**84.**	(b)	**85.**	(b)
86.	(a)	**87.**	(b)	**88.**	(c)	**89.**	(c)	**90.**	(d)
91.	(c)	**92.**	(a)	**93.**	(b)	**94.**	(a)	**95.**	(b)
96.	(a)	**97.**	(d)	**98.**	(a)	**99.**	(d)	**100.**	(b)

NOTES

TABLE 1(a)

Compound Interest

Annual Compounding

No. of Periods n	$(1 + i)^n$		
	10% per Annum $i = 0.10$	14% per Annum $i = 0.14$	18% per Annum $i = 0.18$
1	1.1	1.14	1.18
2	1.21	1.2996	1.3924
3	1.331	1.48154	1.64303
4	1.4641	1.68896	1.93878
5	1.61051	1.92541	2.28776
6	1.77156	2.19497	2.69955
7	1.94872	2.50227	3.18547
8	2.14359	2.85258	3.75886
9	2.35795	3.25194	4.43546
10	2.59374	3.70722	5.23384
11	2.85312	4.22622	6.17593
12	3.13843	4.8179	7.28759
13	3.45227	5.4924	8.59936
14	3.7975	6.26,133	10.1472
15	4.17725	7.13792	11.9738
16	4.59497	8.13723	14.129
17	5.05447	9.27644	16.6723
18	5.55992	10.5751	19.6733
19	6.11591	12.0557	23.2144
20	6.7275	12.7435	27.393

TABLE 1(b)

Present Value of Re. 1

Annual Compounding

No. of Periods n	10% per Annum	$(1 + i)^{-n}$ 14% per Annum	18% per Annum
1	.909091	.877193	.847458
2	.826446	.769468	.718184
3	.751315	.674972	.608631
4	.683014	.592081	.515789
5	.620921	.519369	.437109
6	.564474	.455587	.370432
7	.513158	.399638	.313925
8	.466507	.35056	.266038
9	.424098	.307508	.225456
10	.385543	.269744	.191064
11	.350494	.236618	.161919
12	.318631	.20756	.137219
13	.289664	.18207	.116288
14	.263331	.15971	.0985489
15	.239392	.140097	.083516
16	.217629	.122892	.0707763
17	.197845	.1078	.0599799
18	.179859	.0945614	.0508304
19	.163508	.0829486	.0430766
20	.148644	0.72762	.0365056

TABLE 2(a)

Present Value of an Annuity

Annual Compounding

No. of Periods n	10% per Annum		14% per Annum		18% per Annum	
	P(n, i)	1/P(n, i)	P(n, i)	1/P(n, i)	P(n, i)	1/P(n, i)
1	.909091	1.1	.877192	1.14	.847458	1.18
2	1.73554	.576191	1.64666	.60729	1.56564	.638716
3	2.48685	.402115	2.32163	.430732	2.17427	.459924
4	3.16987	.315471	2.91371	.343205	2.69006	.371739
5	3.79079	.263798	3.43308	.291284	3.12717	.319778
6	4.35526	.229607	3.88867	.257158	3.4976	.28591
7	4.86842	.205406	4.2883	.233193	3.81153	.262362
8	5.33493	.187444	4.63886	.21557	4.07757	.245244
9	5.75902	.173641	4.94637	.202169	4.30302	.232395
10	6.14457	.162745	5.21611	.191714	4.49409	.222515
11	6.49506	.153963	5.45273	.183394	4.65601	.214776
12	6.81369	.146763	5.66029	.176669	4.79323	.208628
13	7.10336	.140779	5.84236	.171164	4.90951	.203686
14	7.36669	.135746	6.00207	.166609	5.00806	.199678
15	7.60608	.131474	6.14217	.162809	5.09158	.196403
16	7.82371	.127817	6.26506	.159615	5.16236	.19371
17	8.02155	.124664	6.37286	.156915	5.22233	.191485
18	8.20141	.12193	6.46742	.154621	5.27316	.189639
19	8.36492	.119547	6.55037	.152663	5.31624	.188103
20	8.51356	.11746	6.62313	.150986	5.35275	.18682

TABLE 2(b)

Amount of an Annuity

Annual Computing

No. of Periods n	10% per Annum		14% per Annum		18% per Annum	
	A(n, i)	1/A(n, i)	A(n, i)	1/A(n, i)	A(n, i)	1/A(n, i)
1	1,000000	.999999994	1.00000001	999999993	1	.999999996
2	2.100000	.476190473	2.14000001	.467289717	2.18000001	.458715595
3	3.310000	.302114802	3.43960003	.290731478	3.57240001	.27992386
4	4,641000	.215470802	4.92114404	.203204782	5.21543202	.19173867
5	6.105100	.16379748	6.61010421	.151283545	7.15420979	.139777841
6	7.71561006	.129607379	8.53551881	.117157495	9.44196755	.105910129
7	9,48717108	.105405499	10.7304915	.0931923765	12.1415217	.082361999
8	11.4358882	.0874440168	13.2327603	0.755700232	15.3269956	.0652443586
9	13.579477	0.736405385	16.0853467	.0621683833	19.0858549	.0523948237
10	15.9374248	.0627453949	19.3372953	0.517135403	23.5213088	.0425146411
11	18.5311672	0.539631415	23.0445166	.043394271	28.7551443	.034776386
12	21.384284	.0467633146	27.270749	.0366693265	34.9310704	.0286278087
13	24.5227124	.0407785234	32.0886539	.0311636631	42.218663	.0236862072
14	27.9749837	.0357462229	37.5810655	.0266091445	50.8180224	0.196780582
15	31.772482	.0314737765	43.8424147	.0228089627	60.9652664	.0164027824
16	35.9497303	.0278166204	50.9803528	.0196153998	72.9390144	.0137100838
17	40.5447033	.0246641341	59.1176022	.0169154357	87.0680371	.011485271
18	45.5991737	.021930222	68.3940666	.0146211514	103.740284	.00963945696
19	51.1590911	.019546868	78.969236	.0126631591	123.413535	.00810283897
20	57.274999	.0174596250	91.0249291	.0109860014	146.627971	.00681998115

TABLE 3
Future Value and Present Value

i = rate of interest per period, n = number of periods

n	$i=\frac{1}{4}\%$ $(1+i)^n$	$i=\frac{1}{4}\%$ $(1+i)^{-n}$	$i=\frac{1}{2}\%$ $(1+i)^n$	$i=\frac{1}{2}\%$ $(1+i)^{-n}$	$i=\frac{3}{4}\%$ $(1+i)^n$	$i=\frac{3}{4}\%$ $(1+i)^{-n}$
1	1.0025 0000	0.9975 0623	1.0050 0000	0.9950 2488	1.0075 0000	0.9925 5583
2	1.0050 0625	0.9950 1869	1.0100 2500	0.9900 7450	1.0150 5625	0.9851 6708
3	1.0075 1877	0.9925 3734	1.0150 7513	0.9851 4876	1.0226 6917	0.9778 3333
4	1.0100 3756	0.9900 6219	1.0201 5050	0.9802 4752	1.0303 3919	0.9705 5417
5	1.0125 6266	0.9875 9321	1.0252 5125	0.9753 7067	1.0380 6673	0.9633 2920
6	1.0150 9406	0.9851 3038	1.0303 7751	0.9705 1808	1.0458 5224	0.9561 5802
7	1.0176 3180	0.9826 7370	1.0355 2940	0.9656 8963	1.0536 9613	0.9490 4022
8	1.0201 7588	0.9802 2314	1.0407 0704	0.9608 8520	1.0615 9885	0.9419 7540
9	1.0227 2632	0.9777 7869	1.0459 1058	0.9561 0468	1.0695 6084	0.9349 6318
10	1.0252 8313	0.9753 4034	1.0511 4013	0.9513 4794	1.0775 8255	0.9280 0315
11	1.0278 4634	0.9729 0807	1.0563 9583	0.9466 1487	1.0856 6441	0.9210 9494
12	1.0304 1596	0.9704 8187	1.0616 7781	0.9419 0534	1.0938 0690	0.9142 3815
13	1.0329 9200	0.9680 6171	1.0669 8620	0.9372 1924	1.1020 1045	0.9074 3241
14	1.0355 7448	0.9656 4759	1.0723 2113	0.9325 5646	1.1102 7553	0.9006 7733
15	1.0381 6341	0.9632 3949	1.0776 8274	0.9279 1688	1.1186 0259	0.8939 7254
16	1.0407 5882	0.9608 3740	1.0830 7115	0.9233 0037	1.1269 9211	0.8873 1766
17	1.0433 6072	0.9584 4130	1.0884 8651	0.9187 0684	1.1354 4455	0.8307 1231
18	1.0459 6912	0.9560 5117	1.0939 2894	0.9141 3616	1.1439 6039	0.8741 5614
19	1.0485 8404	0.9536 6700	1.0993 9858	0.9095 8822	1.1525 4009	0.8676 4878
20	1.0512 0550	0.9512 8878	1.1048 9558	0.9050 6290	1.1611 8414	0.8611 8985
21	1.0538 3352	0.9489 1649	1.1104 2006	0.9005 6010	1.1698 9302	0.8547 7901
22	1.0564 6810	0.9465 5011	1.1159 7216	0.8960 7971	1.1786 6722	0.8484 1589
23	1.0591 0927	0.9441 8964	1.1215 5202	0.8916 2160	1.1875 0723	0.8421 0014
24	1.0617 5704	0.9418 3505	1.1271 5978	0.8871 8567	1.1964 1353	0.8358 3140
25	1.0644 1144	0.9394 8634	1.1327 9558	0.8827 7181	1.2053 8663	0.8296 0933
26	1.0670 7247	0.9371 4348	1.1384 5955	0.8783 7991	1.2144 2703	0.8234 3358
27	1.0697 4015	0.9348 0646	1.1441 5185	0.8740 0986	1.2235 3523	0.8173 0380
28	1.0724 1450	0.9324 7527	1.1498 7261	0.8696 6155	1.2327 1175	0.8112 1966
29	1.0750 9553	0.9301 4990	1.1556 2197	0.8653 3488	1.2419 5709	0.8051 8080
30	1.0777 8327	0.9278 3032	1.1614 0008	0.8610 2973	1.2512 7176	0.7991 8690
31	1.0804 7773	0.9255 1653	1.1672 0708	0.8567 4600	1.2606 5630	0.7932 3762
32	1.0831 7892	0.9232 0851	1.1730 4312	0.8524 8358	1.2701 1122	0.7873 3262
33	1.0858 8687	0.9209 0624	1.1789 0833	0.8482 4237	1.2796 3706	0.7814 7158
34	1.0886 0159	0.9186 0972	1.1848 0288	0.8440 2226	1.2892 3434	0.7756 5418
35	1.0913 2309	0.9163 1892	1.1907 2689	0.8398 2314	1.2989 0359	0.7698 8008
36	1.0940 5140	0.9140 3384	1.1966 8052	0.8356 4492	1.3086 4537	0.7641 4896
37	1.0967 8653	0.9117 5445	1.2026 6393	0.8314 8748	1.3184 6021	0.7584 6051
38	1.0995 2850	0.9094 8075	1.2086 7725	0.8273 5073	1.3283 4866	0.7528 1440
39	1.1022 7732	0.9072 1272	1.2147 2063	0.8232 3455	1.3383 1128	0.7472 1032
40	1.1050 3301	0.9049 5034	1.2207 9424	0.8191 3886	1.3483 4861	0.7416 4796
41	1.1077 9559	0.9026 9361	1.2268 9821	0.8150 6354	1.3584 6123	0.7361 2701
42	1.1105 6508	0.9004 4250	1.2330 3270	0.8110 0850	1.3686 4969	0.7306 4716
43	1.1133 4149	0.8981 9701	1.2391 9786	0.8069 7363	1.3789 1456	0.7252 0809
44	1.1161 2485	0.8959 5712	1.2453 9385	0.8029 5884	1.3892 5642	0.7198 0952
45	1.1189 1516	0.8937 2281	1.2516 2082	0.7989 6402	1.3996 7584	0.7144 5114
46	1.1217 1245	0.8914 9407	1.2578 7892	0.7949 8907	1.4101 7341	0.7091 3264
47	1.1245 1673	0.8892 7090	1.2641 6832	0.7910 3390	1.4207 4971	0.7038 5374
48	1.1273 2802	0.8870 5326	1.2704 8916	0.7870 9841	1.4314 0533	0.6986 1414
49	1.1301 4634	0.8848 4116	1.2768 4161	0.7831 8250	1.4421 4087	0.6934 1353
50	1.1329 7171	0.8826 3457	1.2832 2581	0.7792 8607	1.4529 5693	0.6882 5165

	$i = 1\%$		$i = 1\frac{1}{4}\%$		$i = 1\frac{1}{2}\%$	
n	$(1+i)^n$	$(1+i)^{-n}$	$(1+i)^n$	$(1+i)^{-n}$	$(1+i)^n$	$(1+i)^{-n}$
1	1.0100 0000	0.9900 9901	1.0125 0000	0.9876 5432	1.0150 0000	0.9852 2167
2	1.0201 0000	0.9802 9605	1.0251 5625	0.9754 6106	1.0302 2500	0.9706 6175
3	1.0303 0100	0.9705 9015	1.0379 7070	0.9634 1833	1.0456 7838	0.9563 1699
4	1.0406 0401	0.9609 8034	1.0509 4534	0.9515 2428	1.0613 6355	0.9421 8423
5	1.0510 1005	0.9514 6569	1.0640 8215	0.9397 7706	1.0772 8400	0.9282 6033
6	1.0615 2015	0.9420 4524	1.0773 8318	0.9281 7488	1.0934 4326	0.9145 4219
7	1.0721 3535	0.9327 1805	1.0908 5047	0.9167 1593	1.1098 4491	0.9010 2679
8	1.0828 5671	0.9234 8322	1.1044 8610	0.9053 9845	1.1264 9259	0.8877 1112
9	1.0936 8527	0.9143 3982	1.1182 9218	0.8942 2069	1.1433 8998	0.8745 9224
10	1.1046 2213	0.9052 8695	1.1322 7083	0.8831 8093	1.1605 4083	0.8616 6723
11	1,1156 6835	0.8963 2372	1,1464 2422	0.8722 7746	1.1779 4894	0.8489 3323
12	1.1268 2503	0.8874 4923	1.1607 5452	0.8615 0860	1.1956 1817	0.8363 8742
13	1.1380 9328	0.8786 6260	1.1752 6395	0.8508 7269	1.2135 5244	0.8240 2702
14	1.1494 7421	0.8699 6297	1.1899 5475	0.8403 6809	1.2317 5573	0.8118 4928
15	1.1609 6896	0.8613 4947	1.2048 2918	0.8299 9318	1.2502 3207	0.7998 5150
16	1.1725 7864	0.8528 2126	1.2198 8955	0.8197 4635	1.2689 8555	0.7880 3104
17	1.1843 0443	0.8443 7749	1.2351 3817	0.8096 2602	1.2880 2033	0.7763 8526
18	1.1961 4748	0.8360 1731	1.2505 7739	0.7996 3064	1.3073 4064	0.7649 1159
19	1.2081 0895	0.8277 3992	1.2662 0961	0.7897 5866	1.3269 5075	0.7536 0747
20	1.2201 9004	0.8195 4447	1.2820 3723	0.7800 0855	1.3468 5501	0.7424 7042
21	1.2323 9194	0.8114 3017	1.2980 6270	0.7703 7881	1.3670 5783	0.7314 9795
22	1.2447 1586	0.8033 9621	1.3142 8848	0.7608 6796	1.3875 6370	0.7206 8763
23	1.2571 6302	0.7954 4179	1.3307 1709	0.7514 7453	1.4083 7715	0.7100 3708
24	1.2697 3465	0.7875 6613	1.3473 5105	0.7421 9707	1.4295 0281	0.6995 4392
25	1.2824 3200	0.7797 6844	1.3641 9294	0.7330 3414	1.4509 4535	0.6892 0583
26	1.2952 5631	0.7720 4796	1.3812 4535	0.7239 8434	1.4727 0953	0.6790 2052
27	1.3082 0888	0.7644 0392	1.3985 1092	0.7150 4626	1.4948 0018	0.6689 8574
28	1.3212 9097	0.7568 3557	1.4159 9230	0.7062 1853	1.5172 2218	0.6590 9925
29	1.3345 0388	0.7493 4215	1.4336 9221	0.6974 9978	1.5399 8051	0.6493 5887
30	1.3478 4892	0.7419 2292	1.4516 1336	0.6888 8867	1.5630 8022	0.6397 6243
31	1.3613 2740	0.7345 7715	1.4697 5853	0.6803 8387	1.5865 2642	0.6303 0781
32	1.3749 4068	0.7273 0411	1.4881 3051	0.6719 8407	1.6103 2432	0.6209 9292
33	1.3886 9009	0.7201 0307	1.5067 3214	0.6636 8797	1.6344 7918	0.6118 1568
34	1.4025 7699	0.7129 7334	1.5255 6629	0.6554 9429	1.6589 9637	0.6027 7407
35	1.4166 0276	0.7059 1420	1.5446 3587	0.6474 0177	1.6838 8132	0.5938 6608
36	1.4307 6878	0.6989 2495	1.5639 4382	0.6394 0916	1.7091 3954	0.5850 8974
37	1.4450 7647	0.6920 0490	1.5834 9312	0.6315 1522	1.7347 7663	0.5764 4309
38	1.4595 2724	0.6851 5337	1.6032 8678	0.6237 1873	1.7607 9828	0.5679 2423
39	1.4741 2251	0.6783 6967	16233 2787	06160 1850	1.7872 1025	0.5595 3126
40	1.4888 6373	0.6716 5314	1.6436 1946	0.6084 1334	1.8140 1841	0.5512 6232
41	1.5037 5237	0.6650 0311	1.6641 6471	0.6009 0206	1.8412 2868	0.5431 1559
42	1.5187 8989	0.6584 1892	1.6849 6677	0.5934 8352	1.8688 4712	0.5350 8925
43	1.5339 7779	0.6518 9992	1.7060 2885	0.5861 5656	1.8968 7982	0.5271 8153
44	1.5493 1757	0.6454 4546	1.7273 5421	0.5789 2006	1.9253 3302	0.5193 9067
45	1.5648 1075	0.6390 5492	1.7489 4614	0.5717 7290	1.9542 1301	0.5117 1494
46	1.5804 5885	0.6327 2764	1.7708 0797	0.5647 1397	1.9835 2621	0.5041 5265
47	1.5962 6344	0.6264 6301	1.7929 4306	0.5577 4219	2.0132 7910	0.4967 0212
48	1.6122 2608	0.6202 6041	1.8153 5485	0.5508 5649	2.0434 7829	0.4893 6170
49	1.6283 4834	0.6141 1921	1.8380 4679	0.5440 5579	2.0741 3046	0.4821 2975
50	1.6446 3182	0.6080 3882	1.8610 2237	0.5373 3905	2.1052 4242	0.4750 0468

n	$i = 1\frac{3}{4}\%$ (1 + i)^n	$i = 1\frac{3}{4}\%$ (1 + i)^-n	$i = 2\%$ (1 + i)^n	$i = 2\%$ (1 + i)^-n	$i = 2\frac{1}{4}\%$ (1 + i)^n	$i = 2\frac{1}{4}\%$ (1 + i)^-n
1	1.0175 0000	0.9828 0098	1.0200 0000	0.9803 9216	1.0225 0000	0.9779 9511
2	1.0353 0625	0.9658 9777	1.0404 0000	0.9611 6878	1.0455 0625	0.9564 7444
3	1.0534 2411	0.9492 8528	1.0612 0800	0.9423 2233	1.0690 3014	0.9354 2732
4	1.0718 5903	0.9329 5851	1.0824 3216	0.9238 4543	1.0930 8332	0.9148 4335
5	1.0906 1656	0.9169 1254	1.1040 8080	0.9057 3081	1.1176 7769	0.8947 1232
6	1.1097 0235	0.9011 4254	1.1261 6242	0.8879 7138	1.1428 2544	0.8750 2427
7	1.1291 2215	0.8856 4378	1.1486 8567	0.8705 6018	1.1685 3901	0.8557 6946
8	1.1488 8178	0.8704 1157	1.1716 5938	0.8534 9037	1.1948 3114	0.8369 3835
9	1.1689 8721	0.8554 4135	1.1950 9257	0.8367 5527	1.2217 1484	0.8185 2161
10	1.1894 4449	0.8407 2860	1.2189 9442	0.8203 4830	1.2492 0343	0.8005 1013
11	1.2102 5977	0.8262 6889	1.2433 7431	0.8042 6304	1.2773 1050	0.7828 9499
12	1.2314 3931	0.8120 5788	1.2682 4179	0.7884 9318	1.3060 4999	0.7656 6748
13	1.2529 8950	0.7980 9128	1.2936 0663	0.7730 3253	1.3354 3611	0.7488 1905
14	1.2749 1682	0.7843 6490	1.3194 7876	0.7578 7502	1.3654 8343	0.7323 4137
15	1.2972 2786	0.7708 7459	1.3458 6834	0.7430 1473	1.3962 0680	0.7162 2628
16	1.3199 2935	0.7576 1631	1.3727 8571	0.7284 4581	1.4276 2146	0.7004 6580
17	1.3430 2811	0.7445 8605	1.4002 4142	0.7141 6256	1.4597 4294	0.6850 5212
18	1.3665 3111	0.7317 7990	1.4282 4625	0.7001 5937	1.4925 8716	0.6699 7763
19	1.3904 4540	0.7191 9401	1.4568 1117	0.6864 3076	1.5261 7037	0.6552 3484
20	1.4147 7820	0.7068 2458	1.4859 4740	0.6729 7133	1.5605 0920	0.6408 1647
21	1.4395 3681	0.6946 6789	1.5156 6634	0.6597 7582	1.5956 2066	0.6267 1538
22	1.4647 2871	0.6827 2028	1.5459 7967	0.6468 3904	1.6315 2212	0.6129 2457
23	1.4903 6146	0.6709 7817	1.5768 9926	0.6341 5592	1.6682 3137	0.5994 3724
24	1.5164 4279	0.6594 3800	1.6084 3725	0.6217 2149	1.7057 6658	0.5862 4668
25	1.5429 8054	0.6480 9632	1.6406 0599	0.6095 3087	1.7441 4632	0.5733 4639
26	1.5699 8269	0.6369 4970	1.6734 1811	0.5975 7928	1.7833 8962	0.5607 2997
27	1.5974 5739	0.6259 9479	1.7068 8648	0.5858 6204	1.8235 1588	0.5483 9117
28	1.6254 1290	0.6152 2829	1.7410 2421	0.5743 7455	1.8645 4499	0.5363 2388
29	1.6538 5762	0.6046 4697	1.7758 4469	0.5631 1231	1.9064 9725	0.5245 2213
30	1.6828 0013	0.5942 4764	1.8113 6158	0.5520 7089	1.9493 9344	0.5129 8008
31	1.7122 4913	0.5840 2716	1.8475 8882	0.5412 4597	1.9932 5479	0.5016 9201
32	1.7422 1349	0.5739 8247	1.8845 4059	0.5306 3330	2.0381 0303	0.4906 5233
33	1.7727 0223	0.5641 1053	1.9222 3140	0.5205 2873	2.0839 6034	0.4798 5558
34	1,8037 2452	0.5544 0839	1.9606 7603	0.5100 2817	2.1308 4945	0.4692 9641
35	1.8352 8970	0.5448 7311	1.9998 8955	0.5000 2761	2.1787 9356	0.4589 6960
36	1.8674 0727	0.5355 0183	2.0398 8734	0.4902 2315	2.2278 1642	0.4488 7002
37	1.9000 8689	0.5262 9172	2.0806 8509	0.4806 1093	2.2779 4229	0.4389 9268
38	1.9333 3841	0.5172 4002	2.1222 9879	0.4711 8719	2.3291 9599	0.4293 3270
39	1.9671 7184	0.5083 4400	2.1647 4477	0.4619 4822	2.3816 0290	0.4198 8528
40	2.0015 9734	0.4996 0098	2.2080 3966	0.4528 9042	2.4351 8897	0.4106 4575
41	2.0366 2530	0.4910 0834	2.2522 0046	0.4440 1021	2.4899 8072	0.4016 0954
42	2.0722 6624	0.4825 6348	2.2972 4447	0.4353 0413	2.5460 0528	0.3927 7216
43	2.1085 3090	0.4742 6386	2.3431 8936	0.4267 6875	2.6032 9040	0.3841 2925
44	2.1454 3019	0.4661 0699	2.3900 5314	0.4184 0074	2.6618 6444	0.3756 7653
45	2.1829 7522	0.4580 9040	2.4378 5421	0.4101 9680	2.7217 5639	0.3674 0981
46	2.2211 7728	0.4502 1170	2.4866 1129	0.4021 5373	2.7829 9590	0.3593 2500
47	2.2600 4789	0.4424 6850	2.5363 4352	0.3942 6836	2.8456 1331	0.3514 1809
48	2.2995 9872	0.4348 5848	2.5870 7039	0.3865 3761	2.9096 3961	0.3436 8518
49	2.3398 4170	0.4273 7934	2.6388 1179	0.3789 5844	2.9751 0650	0.3361 2242
50	2.3807 8893	0.4200 2883	2.6915 8803	0.3715 2788	3.0420 4640	0.3287 2608

n	$i = 2\frac{1}{2}\%$ (1 + i)^n	$i = 2\frac{1}{2}\%$ $(1+i)^{-n}$	$i = 3\%$ $(1+i)^n$	$i = 3\%$ $(1+i)^{-n}$	$i = 3\frac{1}{2}\%$ $(1+i)^n$	$i = 3\frac{1}{2}\%$ $(1+i)^{-n}$
1	1.0250 0000	0.9756 0976	1.0300 0000	0.9708 7379	1.0350 0000	0.9661 8357
2	1.0506 2500	0.9518 1440	1.0609 0000	0.9425 9591	1.0712 2500	0.9335 1070
3	1.0768 9063	0.9285 9941	1.0927 2700	0.9151 4166	1.1087 1788	0.9019 4271
4	1.1038 1289	0.9059 5064	1.1255 0881	0.8884 8705	1.1475 2300	0.8714 4223
5	1.1314 0821	0.8838 5429	1.1592 7407	0.8626 0878	1.1876 8631	0.8419 7317
6	1.1596 9342	0.8622 9687	1.1940 5230	0.8374 8426	1.2292 5533	0.8135 0064
7	1.1886 8575	0.8412 6524	1.2298 7387	0.8130 9151	1.2722 7926	0.7859 9096
8	1.2184 0290	0.8207 4657	1.2667 7008	0.7894 0923	1.3168 0904	0.7594 1156
9	1.2488 6297	0.8007 2836	1.3047 7318	0.7664 1673	1.3628 9735	0.7337 3097
10	1.2800 8454	0.7811 9840	1.3439 1638	0.7440 9391	1.4105 9876	0.7089 1881
11	1.3120 8666	0.7621 4478	1.3842 3387	0.7224 2128	1.4599 6972	0.6849 4571
12	1.3448 8882	0.7435 5589	1.4257 6089	0.7013 7988	1.5110 6866	0.6617 8330
13	1.3785 1104	0.7254 2038	1.4685 3371	0.6809 5134	1.5639 5606	0.6394 0415
14	1.4129 7382	0.7077 2720	1.5125 8972	0.6611 1781	1.6186 9452	0.6177 8179
15	1.4482 9817	0.6904 6556	1.5579 6742	0.6418 6195	1.6753 4883	0.5968 9062
16	1.4845 0562	0.6736 2493	1.6047 0644	0.6231 6694	1.7339 8604	0.5767 0591
17	1.5216 1826	0.6571 9506	1.6528 4763	0.6050 1645	1.7946 7555	0.5572 0378
18	1.5596 5872	0.6411 6591	1.7024 3306	0.5873 9461	1.8574 8920	0.5383 6114
19	1.5986 5019	0.6255 2772	1.7535 0605	0.5702 8603	1.9225 0132	0.5201 5569
20	1.6386 1644	0.6102 7094	1.8061 1123	0.5536 7575	1.9897 8886	0.5025 6588
21	1.6795 8185	0.5953 8629	1.8602 9457	0.5375 4928	2.0594 3147	0.4855 7090
22	1,7215 7140	0.5808 6467	1,9161 0341	0.5218 9250	2,1315 1158	0.4691 5063
23	1.7646 1068	0.5666 9724	1.9735 8651	0.5066 9175	2.2061 1448	0.4532 8563
24	1.8087 2595	0.5528 7535	2.0327 9411	0.4919 3374	2.2833 2849	0.4379 5713
25	1.8539 4410	0.5393 9059	2.0937 7793	0.4776 0557	2.3632 4498	0.4231 4699
26	1.9002 9270	0.5262 3472	2.1565 9127	0.4636 9473	2.4459 5856	0.4088 3767
27	1.9478 0002	0.5133 9973	2.2212 8901	0.4501 8906	2.5315 6711	0.3950 1224
28	1.9964 9502	0.5008 7778	2.2879 2768	0.4370 7675	2.6201 7196	0.3816 5434
29	2.0464 0739	0.4886 6125	2.3565 6551	0.4243 4636	2.7118 7798	0.3687 4815
30	2.0975 6758	0.4767 4269	2.4272 6247	0.4119 8676	2.8067 9370	0.3562 7841
31	2.1500 0677	0.4651 1481	2.5000 8035	0.3999 8715	2.9050 3148	0.3442 3035
32	2.2037 5694	0.4537 7055	2.5750 8276	0.3883 3703	3.0067 0759	0.3325 8971
33	2.2588 5086	0.4427 0298	2.6523 3524	0.3770 2625	3.1119 4235	0.3213 4271
34	2.3153 2213	0.4319 0534	2.7319 0530	0.3660 4490	3.2208 6033	0.3104 7605
35	2.3732 0519	0.4213 7107	2.8138 6245	0.3553 8340	3.3335 9045	0.2999 7686
36	2.4325 3532	0.4110 9372	2.8982 7833	0.3450 3243	3.4502 6611	0.2898 3272
37	2.4933 4870	0.4010 6705	2.9852 2668	0.3349 8294	3.5710 2543	0.2800 3161
38	2.5556 8242	0.3912 8492	3.0747 8348	0.3252 2615	3.6960 1132	0.2705 6194
39	2.6195 7448	0.3817 4139	3.1670 2698	0.3157 5355	3.8253 7171	0.2614 1250
40	2.6850 6384	0.3724 3062	3.2620 3779	0.3065 5684	3.9592 5972	0.2525 7247
41	2.7521 9043	0.3633 4695	3.3598 9893	0.2976 2800	4.0978 3381	0.2440 3137
42	2.8209 9520	0.3544 8483	3.4606 9589	0.2889 5922	4.2412 5799	0.2357 7910
43	2.8915 2008	0.3458 3886	3.5645 1677	0.2805 4294	4.3897 0202	0.2278 0590
44	2.9638 0808	03374 0376	3.6714 5227	0.2723 7178	4.5433 4160	0.2201 0231
45	3.0379 0328	0.3291 7440	3.7815 9584	0.2644 3862	4.7023 5855	0.2126 5924
46	3.1138 5086	0.3211 4576	3.8950 4372	0.2567 3653	4.8669 4110	0.2054 6787
47	3.1916 9713	0.3133 1294	4.0118 9503	0.2492 5876	5.0372 8404	0.1985 1968
48	3.2714 8956	0.3056 7116	4.1322 5188	0.2419 9880	5.2135 8898	0.1918 0645
49	3.3532 7680	0.2982 1576	4.2562 1944	0.2349 5029	5.3960 6459	0.1853 2024
50	3.4371 0872	0.2909 4221	4.3839 0602	0.2281 0708	5.5849 2686	0.1790 5337

	$i = 4\%$		$i = 4\frac{1}{2}\%$		$i = 5\%$	
n	$(1+i)^n$	$(1+i)^{-n}$	$(1+i)^n$	$(1+i)^{-n}$	$(1+i)^n$	$(1+i)^{-n}$
1	1.0400 0000	0.9615 3846	1.0450 0000	0.9569 3780	1.0500 0000	0.9523 8095
2	1.0816 0000	0.9245 5621	1.0920 2500	0.9157 2995	1.1025 0000	0.9070 2948
3	1.1248 6400	0.8889 9636	1.1411 6613	0.8762 9660	1.1576 2500	0.8638 3760
4	1.1698 5856	0.8548 0419	1.1925 1860	0.8382 6134	1.2155 0625	0.8227 0247
5	1.2166 5290	0.8219 2711	1.2461 8194	0.8024 5105	1.2762 8156	0.7835 2617
6	1.2653 1902	0.7903 1453	1.3022 6012	0.7678 9574	1.3400 9564	0.7462 1540
7	1.3159 3178	0.7599 1781	1.3608 6183	0.7348 2846	1.4071 0042	0.7106 8133
8	1.3685 6905	0.7306 9021	1.4221 0061	0.7031 8513	1.4774 5544	0.6768 3936
9	1.4233 1181	0.7025 8674	1.4860 9514	0.6729 0443	1.5513 2822	0.6446 0892
10	1.4802 4428	0.6755 6417	1.5529 6942	0.6439 2768	1.6288 9463	0.6139 1325
11	1.5394 5406	0.6495 8093	1.6228 5305	0.6161 9874	1.7103 3936	0.5846 7929
12	1.6010 3222	0.6245 9705	1.6958 8143	0.5896 6386	1.7958 5633	0.5568 3742
13	1.6650 7351	0.6005 7409	1.7721 9610	0.5642 7164	1.8856 4914	0.5303 2135
14	1.7316 7645	0.5774 7508	1.8519 4492	0.5399 7286	1.9799 3160	0.5050 6795
15	1.8009 4351	0.5552 6450	1.9352 8244	0.5167 2044	2.0789 2818	0.4810 1710
16	1.8729 8125	0.5339 0818	2.0223 7015	0.4944 6932	2.1828 7459	0.4581 1152
17	1.9479 0050	0.5133 7325	2.1133 7681	0.4731 7639	2.2920 1832	0.4362 9669
18	2.0258 1652	0.4936 2812	2.2084 7877	0.4528 0037	2.4066 1923	0.4155 2065
19	2.1068 4918	0.4746 4242	2.3078 6031	0.4333 0179	2.5269 5020	0.3957 3396
20	2.1911 2314	0.4563 8695	2.4117 1402	0.4146 4286	2.6532 9771	0.3768 8948
21	2.2787 6807	0.4388 3360	2.5202 4116	0.3967 8743	2.7859 6259	0.3589 4236
22	2.3699 1879	0.4219 5539	2.6336 5201	0.3797 0089	2.9252 6072	0.3418 4987
23	2.4647 1554	0.4057 2633	2.7521 6635	0.3633 5013	3.0715 2376	0.3255 7131
24	2.5633 0416	0.3901 2147	2.8760 1383	0.3477 0347	3.2250 9994	0.3100 6791
25	2.6658 3633	0.3751 1680	3.0054 3446	0.3327 3060	3.3863 5494	0.2953 0277
26	2.7724 6978	0.3606 8923	3.1406 7901	0.3184 0248	3.5556 7269	0.2812 4073
27	2.8833 6858	0.3468 1657	3.2820 0956	0.3046 9137	3.7334 5632	0.2678 4832
28	2.9987 0332	0.3334 7747	3.4296 9999	0.2915 7069	3.9201 2914	0.2550 9364
29	3.1186 5145	0.3206 5141	3.5840 3649	0.2790 1502	4.1161 3560	0.2429 4632
30	3.2433 9751	0.3083 1867	3.7453 1813	0.2670 0002	4.3219 4238	0.2313 7745
31	3.3731 3341	0.2964 6026	3.9138 5745	0.2555 0241	4.5380 3949	0.2203 5947
32	3.5080 5875	0.2850 5794	4.0899 8104	0.2444 9991	4.7649 4147	0.2098 6617
33	3.6483 8110	0.2740 9417	4.2740 3018	0.2339 7121	5.0031 8854	0.1998 7254
34	3.7943 1634	0.2635 5209	4.4663 6154	0.2238 9589	5.2533 4797	0.1903 5480
35	3.9460 8899	0.2534 1547	4.6673 4781	0.2142 5444	5.5160 1537	0.1812 9029
36	4.1039 3255	0.2436 6872	4.8773 7846	0.2050 2817	5.7918 1614	0.1726 5741
37	4.2680 8986	0.2342 9685	5.0968 6049	0.1961 9921	6.0814 0694	0.1644 3563
38	4.4388 1345	0.2252 8543	5.3262 1921	0.1877 5044	6.3854 7729	0.1566 0536
39	4.6163 6599	0.2166 2061	5.5658 9908	0.1796 6549	6.7047 5115	0.1491 4797
40	4.8010 2063	0.2082 8904	5.8163 6454	0.1719 2870	7.0399 8871	0.1420 4568
41	4.9930 6145	0.2002 7793	6.0781 0094	0.1645 2507	7.3919 8815	0.1352 8160
42	5.1927 8391	0.1925 7493	6.3516 1548	0.1574 4026	7.7615 8756	0.1288 3962
43	5.4004 9527	0.1851 6820	6.6374 3818	0.1506 6054	8.1496 6693	0.1227 0440
44	5.6165 1508	0.1780 4635	6.9361 2290	0.1441 7276	8.5571 5028	0.1168 6133
45	5.8411 7568	0.1711 9841	7.2482 4843	0.1379 6437	8.9850 0779	0.1112 9651
46	6.0748 2271	0.1646 1386	7.5744 1961	0.1320 2332	9.4342 5818	0.1059 9668
47	6.3178 1562	0.1582 8256	7.9152 6849	0.1263 3810	9.9059 7109	0.1009 4921
48	6.5705 2824	0.1521 9476	8.2714 5557	0.1208 9771	10.4012 6965	0.0961 4211
49	6.8333 4937	0.1463 4112	8.6436 7107	0.1156 9158	10.9213 3313	0.0915 6391
50	7.1066 8335	0.1407 1262	9.0326 3627	0.1107 0965	11.4673 9979	0.0872 0373

	i = 6%		i = 7%		i = 8%	
n	$(1+i)^n$	$(1+i)^{-n}$	$(1+i)^n$	$(1+i)^{-n}$	$(1+i)^n$	$(1+i)^{-n}$
1	1.0600 0000	0.9433 9623	1.0700 0000	0.9345 7944	1.0800 0000	0.9259 2593
2	1.1236 0000	0.8899 9644	1.1449 0000	0.8734 3873	1.1664 0000	0.8573 3882
3	1.1910 1600	0.8396 1928	1.2250 4300	0.8162 9788	1.2597 1200	0.7938 3224
4	1.2624 7696	0.7920 9366	1.3107 9601	0.7628 9521	1.3604 8896	0.7350 2985
5	1.3382 2558	0.7472 5817	1.4025 5173	0.7129 8618	1.4693 2808	0.6805 8320
6	1.4185 1911	0.7049 6054	1.5007 3035	0.6663 4222	1.5868 7432	0.6301 6963
7	1.5036 3026	0.6650 5711	1.6057 8148	0.6227 4974	1.7138 2427	0.5834 9040
8	1.5938 4807	0.6274 1237	1.7181 8618	0.5820 0910	1.8509 3021	0.5402 6888
9	1.6894 7896	0.5918 9846	1.8384 5921	0.5439 3374	1.9990 0463	0.5002 4897
10	1.7908 4770	0.5583 9478	1.9671 5136	0.5083 4929	2.1589 2500	0.4631 9349
11	1.8982 9856	0.5267 8753	2.1048 5195	0.4750 9280	2.3316 3900	0.4288 8286
12	2.0121 9647	0.4969 6936	2.2521 9159	0.4440 1196	2.5181 7012	0.3971 1376
13	2.1329 2826	0.4688 3902	2.4098 4500	0.4149 6445	2.7196 2373	0.3676 9792
14	2.2609 0396	04423 0096	2.5785 3415	0.3878 1724	2.9371 9362	0.3404 6104
15	2.3965 5819	0.4172 6506	2.7590 3154	0.3624 4602	3.1721 6911	0.3152 4170
16	2.5403 5168	0.3936 4628	2.9521 6375	0.3387 3460	3.4259 4264	0.2918 9047
17	2.6927 7279	0.3713 6442	3.1588 1521	0.3165 7439	3.7000 1805	0.2702 6895
18	2.8543 3915	0.3503 4379	3.3799 3228	0.2958 6392	3.9960 1950	0.2502 4903
19	3.0255 9950	0.3305 1301	3.6165 2754	0.2765 0833	4.3157 0106	0.2317 1206
20	3.2071 3547	0.3118 0473	3.8696 8446	0.2584 1900	4.6609 5714	0.2145 4821
21	3.3995 6360	0.2941 5540	4.1405 6237	0.2415 1309	5.0338 3372	0.1986 5575
22	3.6035 3742	0.2775 0510	4.4304 0174	0.2257 1317	5.4365 4041	0.1839 4051
23	3.8197 4966	0.2617 9726	4.7405 2986	0.2109 4688	5.8714 6365	0.1703 1528
24	4.0489 3464	0.2469 7855	5.0723 6695	0.1971 4662	6.3411 8074	0.1576 9934
25	4.2918 7072	0.2329 9863	5.4274 3264	0.1842 4918	6.8484 7520	0.1460 1790
26	4.5493 8296	0.2198 1003	5.8073 5292	0.1721 9549	7.3963 5321	0.1352 0176
27	4.8223 4594	0.2073 6795	6.2138 6763	0.1609 3037	7.9880 6147	0.1251 8682
28	5.1116 8670	0.1956 3014	6.6488 3836	0.1504 0221	8.6271 0639	0.1159 1372
29	5.4183 8790	0.1845 5674	7.1142 5705	0.1405 6282	9.3172 7490	0.1073 2752
30	5.7434 9117	0.1741 1013	7.6122 5504	0.1313 6712	10.0626 5689	0.0993 7733
31	6.0881 0064	0.1642 5484	8.1451 1290	0.1227 7301	10.8676 6944	0.0920 1605
32	6.4533 8668	0.1549 5740	8.7152 7080	0.1147 4113	11.7370 8300	0.0852 0005
33	6.8405 8988	0.1461 8622	9.3253 3975	0.1072 3470	12.6760 4964	0.0788 8893
34	7.2510 2528	0.1379 1153	9.9781 1354	0.1002 1934	13.6901 3361	0.0730 4531
35	7.6860 8679	0.1301 0522	10.6765 8148	0.0936 6294	14.7853 4429	0.0676 3454
36	8.1472 5200	0.1227 4077	11.4239 4219	0.0875 3546	15.9681 7184	0.0626 2458
37	8.6360 8712	0.1157 9318	12.2236 1814	0.0818 0884	17.2456 2558	0.0579 8572
38	9.1542 5235	0.1092 3885	13.0792 7141	0.0764 5686	18.6252 7563	0.0536 9048
39	9.7035 0749	0.1030 5552	13.9948 2041	0.0714 5501	20.1152 9768	0.0497 1341
40	10.2857 1794	0.0972 2219	14.9744 5784	0.0667 8038	21.7245 2150	0.0460 3093
41	10.9028 6101	0.0917 1905	16.0226 6989	0.0624 1157	23.4624 8322	0.0426 2123
42	11.5570 3267	0.0865 2740	17.1442 5678	0.0583 2857	25.3394 8187	0.0394 6411
43	12.2504 5463	0.0816 2962	18.3443 5475	0.0545 1268	27.3666 4042	0.0365 4084
44	12.9854 8191	0.0770 0908	19.6284 5959	0.0509 4643	29.5559 7166	0.0338 3411
45	13.7646 1083	0.0726 5007	21.0024 5176	0.0476 1349	31.9204 4939	0.0313 2788
46	14.5904 8748	0.0685 3781	22.4726 2338	0.0444 9859	34.4740 8534	0.0290 0730
47	15.4659 1673	0.0646 5831	24.0457 0702	0.0415 8747	37.2320 1217	0.0268 5861
48	16.3938 7173	0.0609 9840	25.7289 0651	0.0388 6679	40.2105 7314	0.0248 6908
49	17.3775 0403	0.0575 4566	27.5299 2997	0.0363 2410	43.4274 1899	0.0230 2693
50	18.4201 5427	0.0542 8836	29.4570 2506	0.0339 4776	46.9016 1251	0.0213 2123

TABLE 4
Log-Tables

LOGARITHAMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5566	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6345	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

LOG-TABLES

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7769	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9843	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

TABLE 5
ANTILOGARITHAMS

LOG-TABLES

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1289	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.46	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

LOG-TABLES

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6715	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	12	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

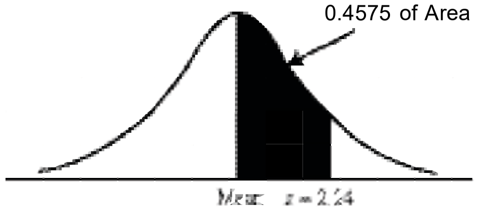


TABLE 6

Areas under the Standard Normal Probability Distribution between the Mean and Positive Values of z

Example:

To find the area under the curve between the mean and a point 2.24 standard deviations to the right of the mean, look up the value opposite 2.2 and under 0.04 in the table; 0.4875 of the area under the curve lies between the mean and a z Value of 2.24

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	04956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

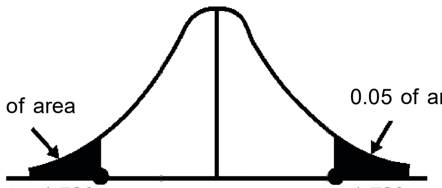


TABLE 7

Areas in Both Tails Combined for Student's t Distribution

Example:

To find the value of *t* that corresponds to an area of 0.10 in both tails of the distribution combined, when there are 19 degress of freedom, look under the 0.10 column, and proceed down to the 19 degrees of freedom row; the appropriate *t* value there is 1.729

Degree of Freedom	Area in Both Tails Combined 0.10	0.05	0.02	0.01
1	6.314	12.706	31.821	63.657
2	2.920	4.303	6.965	9.925
3	2.353	3.182	4.541	5.841
4	2.132	2.776	3.747	4.604
5	2.015	2.571	3.365	4.032
6	1.943	2.447	3.143	3.707
7	1.895	2.365	2.998	3.499
8	1.860	2.306	2.896	3.355
9	1.833	2.262	2.821	3.250
10	1.812	2.228	2.764	3.169
11	1.796	2.201	2.718	3.106
12	1.782	2.179	2.681	3.055
13	1.771	2.160	2.650	3.012
14	1.761	2.145	2.624	2.977
15	1.753	2.131	2.602	2.947
16	1.746	2.120	2.583	2.921
17	1.740	2.110	2.567	2.898
18	1.734	2.101	2.552	2.878
19	1.729	2.093	2.539	2.861
20	1.725	2.086	2.528	2.845
21	1.721	2.080	2.518	2.831
22	1.717	2.074	2.508	2.819
23	1.714	2.069	2.500	2.807
24	1.711	2.064	2.492	2.797
25	1.708	2.060	2.485	2.787
26	1.706	2.056	2.479	2.779
27	1.703	2.052	2.473	2.771
28	1.701	2.048	2.467	2.763
29	1.699	2.045	2.462	2.756
30	1.697	2.042	2.457	2.750
40	1.684	2.021	2.423	2.704
60	1.671	2.000	2.390	2.660
120	1.658	1.980	2.358	2.617
Normal Distribution	1.645	1.960	2.326	2.576